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NEAR SEMIRINGS AND SEMIRINGS WITH INVOLUTION

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Abstract. We study so-called near semirings endowed with an antitone involution. Such a near semiring is in fact a bounded lattice which has one more binary operation, the multiplication. We classify several families of bounded lattices which can be organized in such near semirings, e.g. chains or orthomodular lattices. A particular case are the so-called balanced near semirings which form a variety which is congruence distributive, permutable and regular.

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1. Introduction

The concept of a near semiring was introduced by the authors in [3]. It seems to be useful in order to axiomatize certain so-called quantum structures, e.g. basic algebras, MV-algebras and orthomodular lattices. With respect to addition, near semirings used in these representations are in fact join semilattices with an additional operation, which is multiplication. Hence, in every such a near semiring there can be introduced a semilattice order and these near semirings can be considered to be ordered. Among other things this means that if an antitone involution is introduced, such a near semiring becomes a lattice with multiplication. The question arises which lattices can be equipped with a suitable multiplication in order to become a near semiring with antitone involution. In the present paper we will study this problem and provide partial solutions. Adding some natural identities to the axioms of a near semiring with involution we obtain so-called balanced near semirings. They form a variety having very strong congruence properties. Namely, this variety turns out to be congruence distributive, permutable and regular.

We start with the following definition:

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Definition 1. A *near semiring* is an algebra $\mathbf{R} = (R, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ such that $(R, +, 0)$ is a commutative monoid and $\mathbf{R}$ satisfies the following identities:

- $(x + y)z \approx xz + yz$ (so-called *right distributivity*),
- $x0 \approx 0x \approx 0$,
- $x1 \approx 1x \approx x$.

$\mathbf{R}$ is called

- *commutative* if it satisfies $xy \approx yx$,
- *idempotent* if it satisfies $x + x \approx x$,
- *multiplicatively idempotent* if it satisfies $xx \approx x$,
- a *semiring* if it satisfies $(xy)z \approx x(yz)$ and $z(x + y) \approx zx + zy$.

If $\mathbf{R}$ is idempotent then we define a partial order relation $\leq$ on R by

$$x \leq y \text{ if and only if } x + y = y.$$

Then $(R, \leq)$ is a poset with smallest element 0 since it corresponds to the join-semilattice $(R, +)$. In the following we will call $\leq$ the *induced order* of the near semiring $\mathbf{R}$.

If $\mathbf{R}$ is a commutative semiring which is multiplicatively idempotent then we can define a partial order relation $\leq_1$ on R by

$$x \leq_1 y \text{ if and only if } xy = x$$

$(x, y \in R)$. Of course, the partial order relations $\leq$ and $\leq_1$ may be different as we will see later.

Now we define our main concept.

Definition 2. A *near semiring with involution* is an algebra $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(R, +, \cdot, 0, 1)$ is an idempotent near semiring and the following conditions hold for all $x, y \in R$:

(a) If $x \leq y$ then $y' \leq x'$.
(b) $(x')' = x$.

Let $\mathcal{N}$ denote the class of near semirings with involution. Directly by definition, $\mathcal{N}$ is a variety. $\mathbf{R} \in \mathcal{N}$ is called a *semiring with involution* if $(R, +, \cdot, 0, 1)$ is a semiring. Let $\mathcal{S}$ denote the class of semirings with involution. Clearly, also $\mathcal{S}$ is a variety.

Lemma 1. *Every* $\mathbf{R} = (R, +, \cdot, ', 0, 1) \in \mathcal{N}$ *satisfies* $(x + y)' + x' \approx x'$.

Proof. For $a, b \in R$ we have $a \leq a + b$ and hence $(a + b)' \leq a'$ which shows $(a + b)' + a' = a'$. ☐

It is almost evident that condition (a) of Definition 2 can be replaced by the identity from Lemma 1.

2. DUALS OF NEAR SEMIRINGS WITH INVOLUTION

In what follows, we use the antitone involution in order to show that the concept of a near semiring with involution can be dualized. Although $(R,+,\cdot)$ need not be a lattice, we can define new operations as follows by using the De Morgan laws.

Definition 3. For every $\mathbf{R} = (R,+,\cdot,',0,1) \in \mathcal{N}$ define an algebra $\mathbf{R}' := (R,+',\cdot',', 0',1')$ by $x+'y := (x'+y')'$ and $x\cdot' y := (x'y')'$ for all $x,y \in R$. The algebra $\mathbf{R}'$ will be called the *dual* of $\mathbf{R}$.

Of course, if $\mathbf{R} = (R,+,\cdot,',0,1) \in \mathcal{N}$ then the induced algebra $\mathbb{L}(\mathbf{R}) := (R,+,+',',0,0')$ is a bounded lattice with an antitone involution and hence it satisfies the De Morgan laws.

We can prove the following statement concerning the dual of $\mathbf{R}$.

Theorem 1. *If* $\mathbf{R} = (R,+,\cdot,',0,1) \in \mathcal{N}$ *then* $\mathbf{R}' := (R,+',\cdot',',0',1') \in \mathcal{N}$ *and* $(\mathbf{R}')' = \mathbf{R}$*. The mapping* $h(x) = x'$ *is an isomorphism from* $\mathbf{R}$ *onto* $\mathbf{R}'$*. Moreover,* $\mathbf{R}' \in \mathcal{S}$ *if and only if* $\mathbf{R} \in \mathcal{S}$*.*

Proof. It is easy to check that $(R,+',\cdot',0',1')$ is an idempotent near semiring whose induced order is $\geq$. Hence $\mathbf{R}'$ satisfies the corresponding conditions mentioned in Definition 2. Now $(\mathbf{R}')' = \mathbf{R}$ follows by a straightforward calculation. The remaining assertions are immediate. □

Let us recall that a ternary term $m(x,y,z)$ is called a *majority term* in a variety $\mathcal{V}$ if it satisfies the identities

$$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx x.$$

It is well-known that if $\mathcal{V}$ has a majority term then it is congruence distributive, see e.g. [2].

The following important result follows directly from the fact that $(R,+,+')$ is a lattice as mentioned above.

Theorem 2. *The variety* $\mathcal{N}$ *has the majority term*

$$m(x,y,z) = (x+y)+'(y+z)+'(z+x)$$

and hence is congruence distributive.

3. ORTHO NEAR SEMIRINGS

Up to now, we did not assume the involution to have some more important properties. This will be done now. In what follows we will ask the involution to be a complementation in the induced bounded lattice $(R,+,+',0,0')$. Hence we define

Definition 4. A member of $\mathcal{N}$ is called an *ortho near semiring* if it satisfies $x+x' \approx 0'$. If, in addition, it belongs to $\mathcal{S}$ then it is called an *orthosemiring*.

It is evident that in this case also the identity $x +' x' \approx 0$ holds and that the dual of an ortho near semiring is an ortho near semiring again. It is clear that the class of ortho near semirings forms a variety.

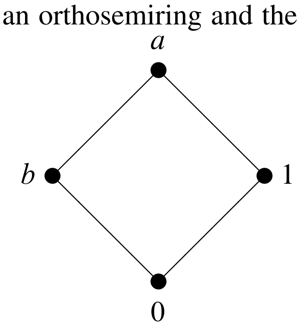
Example 1. Let $R = \{0, a, b, 1\}$ and define binary operations $+$ and $\cdot$ and a unary operation $'$ on R by

$+$	0	a	b	1
0	0	a	b	1
a	a	a	a	a
b	b	a	b	a
1	1	a	a	1

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	b	a
b	0	b	b	b
1	0	a	b	1

x	x'
0	a
a	0
b	1
1	b

Then $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ is an orthosemiring and the Hasse diagram of its induced order looks as follows:



Hence $(R, +, +')$ is in fact a distributive lattice which is not a chain. Since $\mathbf{R}$ is, moreover, multiplicatively idempotent, commutative and associative, we can introduce the order $\leq_1$ induced by multiplication. For this order we have $0 <_1 b <_1 a <_1 1$ which is a chain.

An example of an ortho near semiring whose induced lattice is not distributive is shown in the following

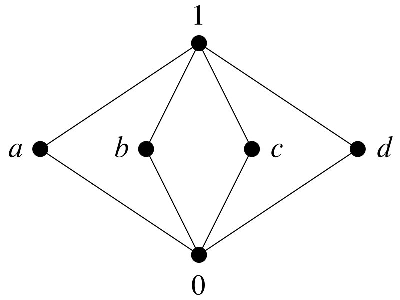
Example 2. Let $R = \{0, a, b, c, d, 1\}$ and define binary operations $+$ and $\cdot$ and a unary operation $'$ on R by

$+$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	1	1	1	1
b	b	1	b	1	1	1
c	c	1	1	c	1	1
d	d	1	1	1	d	1
1	1	1	1	1	1	1

$\cdot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	0	d	a
b	0	a	b	c	0	b
c	0	0	b	c	d	c
d	0	a	0	c	d	d
1	0	a	b	c	d	1

x	x'
0	1
a	c
b	d
c	a
d	b
1	0

Then $\mathbf{R} = (R, +, \cdot, ', 0, 1)$ is an ortho near semiring the Hasse diagram of its induced order looks as follows:



R is neither an orthosemiring since $(ab)c = bc = c \neq 0 = ac = a(bc)$ nor commutative since $cd = d \neq c = dc$.

4. CONSTRUCTIONS OF NEAR SEMIRINGS WITH INVOLUTION

Now we investigate the question which bounded lattices $(L, \vee, \wedge, ', 0, 1)$ can be equipped with a binary operation of multiplication in such a way that $(L, \vee, \cdot, ', 0, 1)$ becomes a near semiring with involution and $(L, \vee, \cdot)$ is not a lattice.

If $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and $a \in L$ we call the interval $[0, a]$ a *section*. We say that the mapping $x \mapsto x^a$ from $[0, a]$ to $[0, a]$ is a *sectional antitone involution* on $([0, a], \leq)$ if $(x^a)^a = x$ and $x \leq y$ implies $y^a \leq x^a$ for each $x, y \in [0, a]$.

Theorem 3. *Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice where for each $a \in L$ there exists a sectional antitone involution ^{a} on the section $[0, a]$. Define $xy := (x^1 \wedge y)^y$ for all $x, y \in L$. Then $(L, \vee, \cdot, ^1, 0, 1)$ is a near semiring with involution.*

Proof. It is evident that $(L, \vee, 0)$ is an idempotent commutative monoid. Moreover,

$$\begin{aligned}
(x \vee y)z &= ((x \vee y)^1 \wedge z)^z = (x^1 \wedge y^1 \wedge z)^z = ((x^1 \wedge z) \wedge (y^1 \wedge z))^z \\
&= (x^1 \wedge z)^z \vee (y^1 \wedge z)^z = (xz) \vee (yz), \\
x0 &= (x^1 \wedge 0)^0 = 0^0 = 0, \\
0x &= (0^1 \wedge x)^x = x^x = 0, \\
x1 &= (x^1 \wedge 1)^1 = (x^1)^1 = x \text{ and} \\
1x &= (1^1 \wedge x)^x = 0^x = x
\end{aligned}$$

for all $x, y, z \in L$. □

For the reader's convenience let us recall that an *orthocomplemented lattice* is a bounded lattice $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ with an antitone involution satisfying the identities $x \vee x' \approx 1$ and $x \wedge x' \approx 0$.

Not for every orthocomplemented lattice $(L, \vee, \wedge, ', 0, 1)$ there exists a multiplication $\cdot$ such that $(L, \vee, \cdot, ', 0, 1)$ becomes an ortho near semiring. A certain sufficient condition for this will be given in the next corollary.

Corollary 1. *If $(L,\vee,\wedge,',0,1)$ is an orthocomplemented lattice such that for every $a \in L$ there exists a sectional antitone involution ^{a} on $([0,a],\leq)$ such that $^1 =' $ and $xy := (x' \wedge y)^y$ for all $x,y \in L$ then $(L,\vee,\cdot,',0,1)$ is an ortho near semiring.*

For finite chains the antitone involution is determined uniquely and we can prove that there exists a multiplication such that the induced near semiring is even a semiring.

Theorem 4. *Let $\mathbf{L} = (L,\vee,\wedge,',0,1)$ be a finite chain with antitone involution containing more than two elements. Then there exists a binary operation $\cdot$ on L such that $\mathbf{R} = (L,\vee,\cdot,',0,1) \in \mathcal{S}$ and $(L,\vee,\cdot)$ is not a lattice.*

Proof. Without loss of generality we can assume $L = \{1,\dots,n\}$ with an integer $n > 2$ and we define an order on L as follows: $1 < 3 < \cdots < n-1 < n < \cdots < 4 < 2$ if n is even and $1 < 3 < \cdots < n < n-1 < \cdots < 4 < 2$ if n is odd and denote by 'the unique antitone involution on $(L,\leq)$. As proved in [5], if $\cdot$ denotes the minimum operation on L with respect to the natural ordering $\leq_1$ of integers then $(L,\vee,\cdot,0,1)$ is a commutative idempotent semiring and hence $\mathbf{R} = (L,\vee,\cdot,',0,1) \in \mathcal{S}$. Because of $\leq \neq \leq_1$, $(L,\vee,\cdot)$ is not a lattice. □

Theorem 5. *For every infinite cardinal k there exists a bounded chain $\mathbf{L} = (L,\vee,\wedge,',0,1)$ of cardinality k with an antitone involution and a binary operation $\cdot$ on L such that $\mathbf{R} = (L,\vee,\cdot,',0,1) \in \mathcal{S}$ and $(L,\vee,\cdot)$ is not a lattice.*

Proof. Let $(C,\leq_2,0,1)$ be a bounded chain of infinite cardinality k, put $L := C \times \{1,2\}$ and define binary relations $\leq$ and $\leq_1$ and a unary operation $'$ on L as follows:

- $(x,i) \leq (y,j)$ if either $((i,j) = (1,1)$ and $x \leq_2 y)$ or (i,j)=(1,2) or $((i,j) = (2,2)$ and $x \geq_2 y)$,
- $(x,i) \leq_1 (y,j)$ if either $x <_2 y$ or $(x = y$ and $i \leq j)$,
- $(x,i)' := (x,3-i)$

for $(x,i),(y,j) \in L$. Then $(L,\leq,(0,1),(0,2))$ and $(L,\leq_1,(0,1),(1,2))$ are bounded chains of cardinality k and $'$ is an antitone involution on $(L,\leq)$. Let $\mathbf{L} = (L,\vee,\wedge,',(0,1),(0,2))$ denote the bounded lattice with antitone involution corresponding to $(L,\leq)$. Then $\mathbf{L}$ is a chain of cardinality k with an antitone involution as proved in [5]. If $\cdot$ denotes the minimum operation on L with respect to $\leq_1$ then $(L,\vee,\cdot,0,1)$ is a commutative idempotent semiring and hence $\mathbf{R} = (L,\vee,\cdot,',0,1) \in \mathcal{S}$. Because of $\leq \neq \leq_1$, $(L,\vee,\cdot)$ is not a lattice. □

For the reader's convenience let us recall that an *orthomodular lattice* is an orthocomplemented lattice $\mathbf{L} = (L,\vee,\wedge,',0,1)$ satisfying the identity $x \vee y \approx x \vee ((x \vee y) \wedge x')$.

Now we can show that also every orthomodular lattice $\mathbf{L} = (L,\vee,\wedge,',0,1)$ can be equipped with a multiplication operation $\cdot$ in such a way that $(L,\vee,\cdot,',0,1)$ becomes an ortho near semiring such that $(L,\vee,\cdot)$ is not a lattice provided $\mathbf{L}$ is not a Boolean algebra.

As it was defined in [6] and [1], two elements a,b of an orthomodular lattice $\mathbf{L}=(L,\vee,\wedge,',0,1)$ are said to *commute with each other*, in symbols $a\,\mathrm{C}\,b$ if $a=(a\wedge b)\vee(a\wedge b')$. The following result can be found in [6] and [1]:

Proposition 1. *If* $\mathbf{L}=(L,\vee,\wedge,',0,1)$ *is an orthomodular lattice and* $a,b,c\in L$ *then the following hold:*

(1) *If* $a\,\mathrm{C}\,b$ *then* $b\,\mathrm{C}\,a$ *and* $a\,\mathrm{C}\,b'$.
(2) *If* $a\le b$ *then* $a\,\mathrm{C}\,b$.
(3) *If one of the three elements* a,b,c *commutes with the remaining two then the distributive laws hold for* a,b,c.
(4) $\mathbf{L}$ *is a Boolean algebra if and only if any two elements commute with each other.*

Now we are ready to get a positive answer to the previous question in the case of orthomodular lattices.

Theorem 6. *Let* $\mathbf{L}=(L,\vee,\wedge,',0,1)$ *be an orthomodular lattice which is not a Boolean algebra. Then there exists a binary operation* $\cdot$ *on* L *such that* $\mathbf{R}=(L,\vee,\cdot,',0,1)$ *is an ortho near semiring and* $(L,\vee,\cdot)$ *is not a lattice.*

Proof. Let $a,b,c\in L$. Define a binary operation $\cdot$ on L by $ab:=(a\vee b')\wedge b$. Since $c'\le a\vee c'$ and $c'\le b\vee c'$ we have by Proposition 1(2) that $c'\,\mathrm{C}\,a\vee c'$ and $c'\,\mathrm{C}\,b\vee c'$. Again according to Proposition 1(1) we have $c\,\mathrm{C}\,a\vee c'$ and $c\,\mathrm{C}\,b\vee c'$ and we can use distributivity for these elements (by Proposition 1(3)) and hence compute

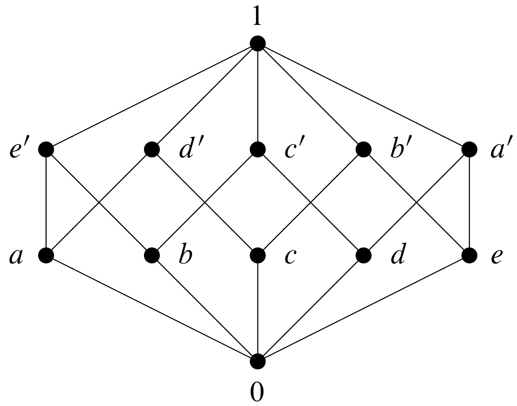
$$(a\vee b)c=((a\vee b)\vee c')\wedge c=((a\vee c')\vee(b\vee c'))\wedge c$$
$$=((a\vee c')\wedge c)\vee((b\vee c')\wedge c)=ac\vee bc.$$

By straightforward calculations it follows that $\mathbf{R}=(L,\vee,\cdot,',0,1)$ is a ortho near semiring. If $(L,\vee,\cdot)$ would be a lattice then we would have

$$a=(a')'=(a'b'\vee a')'=(((a'\vee b)\wedge b')\vee a')'=(((a'\vee b)\vee a')\wedge(b'\vee a'))'$$
$$=((a'\vee b')\wedge(a'\vee b))'=(a\wedge b)\vee(a\wedge b'),$$

i.e. $a\,\mathrm{C}\,b$ and hence $\mathbf{L}$ would be a Boolean algebra (by Proposition 1(4)) contradicting our assumption. This shows that $(L,\vee,\cdot)$ is not a lattice. □

The following example shows that there exist orthocomplemented lattices $(L,\vee,\wedge,',0,1)$ which are not orthomodular but have an antitone involution on every section $([0,a],\le)$ and hence according to Corollary 1 can be converted into an ortho near semiring:



5. Varieties of near semirings with involution

In the remaining part of our paper we investigate some varieties of near semirings with involution. For this purpose we define the following:

Let $R = \{0, a, 1\}$ and define binary operations $+$ and $\cdot$ and a unary operation $'$ on R by

$+$	0	a	1
0	0	a	1
a	a	a	a
1	1	a	1

$\cdot$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

x	x'
0	a
a	0
1	1

Then $(R, +, \cdot, 0, 1)$ is a commutative idempotent semiring which was denoted by $\mathbf{S}_3$ in [5] and $\mathbf{S}_3' := (R, +, \cdot, ', 0, 1) \in \mathcal{S}$. The induced order of $\mathbf{S}_3$ is $0 < 1 < a$. However, the order induced by multiplication is as follows: $0 <_1 a <_1 1$. Thus $\leq \neq \leq_1$ and hence $(R, +, \cdot)$ is not a lattice.

As pointed out in [5], a prominent role plays the variety $\mathcal{V}(\mathbf{S}_3)$ generated by the near semiring $\mathbf{S}_3$. If we endow $\mathbf{S}_3$ with an antitone involution, we can ask about the properties of the variety $\mathcal{V}(\mathbf{S}_3')$ generated by $\mathbf{S}_3'$. It is rather surprising that contrary to $\mathcal{V}(\mathbf{S}_3)$, which is residually large, $\mathcal{V}(\mathbf{S}_3')$ has completely different properties.

It was shown in [5] that $\mathbf{S}_3$ is a subdirectly irreducible semiring and in [4] that $\mathcal{V}(\mathbf{S}_3)$ is the variety of commutative idempotent and multiplicatively idempotent semirings satisfying the identity $x + y + xy \approx x + y$ and that it has a proper class of subdirectly irreducible members. Contrary to this we can prove:

Theorem 7. *The only subdirectly irreducible member of* $\mathcal{V}(\mathbf{S}_3')$ *is* $\mathbf{S}_3'$.

Proof. Apparently, $\mathcal{V}(\mathbf{S}_3')$ is a subvariety of $\mathcal{S}$ which is in turn a subvariety of $\mathcal{N}$. By Theorem 2, $\mathcal{V}(\mathbf{S}_3')$ is congruence distributive. Since $\mathcal{V}(\mathbf{S}_3')$ is congruence distributive and $\mathbf{S}_3'$ is finite, every subdirectly irreducible member of $\mathcal{V}(\mathbf{S}_3')$ belongs to $\mathrm{HS}(\mathbf{S}_3')$ according to Jónsson's Lemma. However, as can be easily verified, $\mathbf{S}_3'$ has no proper subalgebras and it is simple and hence it has no non-trivial homomorphic image. This shows that $\mathbf{S}_3'$ is the only subdirectly irreducible member of $\mathcal{V}(\mathbf{S}_3')$. □

Very strong congruence properties have varieties of ortho near semirings satisfying a certain more or less natural property. Hence we define:

Definition 5. A *balanced near semiring* is an ortho near semiring $\mathbf{R} = (R,+,\cdot,',0,1)$ satisfying $((x+y)'+y)'+x \approx x$. Let $\mathcal{B}$ denote the class of balanced near semirings.

This identity is surely satisfied in an ortho near semiring provided the elements x and y are comparable. Namely, if $x \leq y$ then $x+y=y$ and thus $((x+y)'+y)'+x=(y'+y)'+x=(0')'+x=0+x=x$ and, if $y \leq x$ then $((x+y)'+y)'+x=(x'+y)'+x=(x+'y')+x=x$ according to the absorption law.

Remark 1.

(1) Of course, $\mathcal{B}$ is a variety.
(2) Every bounded distributive lattice with an antitone involution belongs to $\mathcal{B}$.
(3) We have $\mathcal{B} \subseteq \mathcal{N}$ and hence $\mathcal{B}$ is congruence distributive according to Theorem 2.

Theorem 8. *The variety $\mathcal{B}$ is congruence permutable and hence arithmetical.*

Proof. If $\mathbf{R}=(R,+,\cdot,',0,1) \in \mathcal{B}$, $a,b,c \in R$ and

$$p(x,y,z) := (((x'+y)'+z)'+((z'+y)'+x)')'$$

for all $x,y,z \in R$ then

$$\begin{aligned} p(a,a,b) &= (((a'+a)'+b)'+((b'+a)'+a)')' = (b'+((b'+a)'+a)')' \\ &= (((b'+a)'+a)'+b')' = (b')' = b \text{ and} \\ p(x,y,z) &= p(z,y,x) \text{ for all } x,y,z \in R \text{ and hence} \\ p(a,b,b) &= a \end{aligned}$$

which shows that p is a Malcev term proving congruence permutability of $\mathcal{B}$. According to Remark 1(3), $\mathcal{B}$ is arithmetical. □

Recall that a *variety* $\mathcal{V}$ is called *congruence regular* if for every $\mathbf{A}=(A,F) \in \mathcal{V}$ and each $a \in A$ and for every $\Theta, \Phi \in \operatorname{Con}\mathbf{A}$ the equality $[a]\Theta = [a]\Phi$ implies $\Theta = \Phi$. It is well-known (cf. [2]) that a variety $\mathcal{V}$ is congruence regular if and only if there exists a positive integer n and ternary terms $t_1,\ldots,t_n$ such that $t_1(x,y,z)=\cdots=t_n(x,y,z)=z$ is equivalent to $x=y$.

Theorem 9. *The variety $\mathcal{B}$ is congruence regular.*

Proof. Let $\mathbf{R}=(R,+,\cdot,',0,1) \in \mathcal{B}$ and $a,b,c \in R$ and put

$$\begin{aligned} t_1(x,y,z) &:= (x'+y)'+(x+y')'+z \text{ and} \\ t_2(x,y,z) &:= ((x+y')'+(x'+y)'+z')' \end{aligned}$$

for all $x,y,z \in R$. A straightforward calculation shows $t_1(a,a,b) = t_2(a,a,b) = b$. Conversely, assume $t_1(a,b,c) = t_2(a,b,c) = c$. Then $(a'+b)'+(a+b')' \leq c$ and $(a+b')'+(a'+b)' \leq c'$ whence $((a+b')'+(a'+b)')' \geq (c')' = c$ and $(a'+b)'+(a+b')' \leq ((a+b')'+(a'+b)')'$. Hence $(a'+b)' \leq ((a'+b)')' = a'+b$ and $(a+b')' \leq ((a+b')')' = a+b'$. From this we obtain $a+b' = a'+b = 0'$. Now we have

$$\begin{aligned} a &= (a')' = (((a'+b)'+b)'+a')' = ((0+b)'+a')' = (b'+a')' \leq (b')' \\ &= b \leq b+a = (0+b')'+a = ((a+b')'+b')'+a = a \end{aligned}$$

whence $a = b$. This shows that t_1 and t_2 are Csákány terms proving congruence regularity of $\mathcal{B}$ (cf. [2]). □

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