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ON THE MOORE–PENROSE INVERSE IN RINGS WITH INVOLUTION

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Abstract. We study equivalent conditions to those given in Penrose equations for an element to be the Moore–Penrose inverse of a given element in a ring with involution, using the concept of EP, normal, bi–EP, bi–normal, l–(or r–)quasi–EP, l–(or r–)quasi–normal and *-cancellable elements. The mentioned conditions are weaker that the one of being self–adjoint.

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1. INTRODUCTION

Let $\mathcal{R}$ be an associative ring with the unit 1. An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^*.$$

An element $a \in \mathcal{R}$ is self-adjoint if $a^* = a$. An element $a \in \mathcal{R}$ satisfying $aa^* = a^* a$ is called normal.

The Moore–Penrose inverse of $a \in \mathcal{R}$ is the element $x \in \mathcal{R}$, if the following Penrose equations hold [7]:

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (4)\ (xa)^* = xa.$$

There is at most one x such that above conditions hold, and such x is denoted by $a^\dagger$. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$. In the last decades, the Moore-Penrose inverse has found a wide range of applications in many areas of science and became a useful tool for dealing with optimization problems, data analysis, the solution of linear integral equations, etc.

If $\delta \subseteq \{1,2,3,4\}$ and x satisfies the equations (i) for all $i \in \delta$, then x is a δ–inverse of a. The set of all δ–inverse of a is denote by $a\{\delta\}$. Notice that $a\{1,2,3,4\} = \{a^\dagger\}$. Recall that $x \in a\{1,4\}$ if and only if $x^* \in a^*\{1,3\}$.

The following result is frequently used in the rest of the paper.

Theorem 1 ([1]). *For any $a \in \mathcal{R}^\dagger$, the following is satisfied:*

(a) $(a^\dagger)^\dagger = a$;
(b) $(a^*)^\dagger = (a^\dagger)^*$.

An element $a \in \mathcal{R}$ is said to be EP if $a \in \mathcal{R}^\dagger$ and $aa^\dagger = a^\dagger a$. Observe that a is EP if and only if a^* is EP.

We recall the reader that $a \in \mathcal{R}$ is called (i) bi–normal if $aa^*a^*a = a^*aaa^*$; (ii) bi–EP if $a \in \mathcal{R}^\dagger$ and $aa^\dagger a^\dagger a = a^\dagger aaa^\dagger$; (iii) l–quasi–normal if $aa^*a = a^*aa$; (iv) r–quasi–normal if $aaa^* = aa^*a$; (v) l–quasi–EP if $a \in \mathcal{R}^\dagger$ and $aa^\dagger a = a^\dagger aa$; (vi) r–quasi–EP if $a \in \mathcal{R}^\dagger$ and $aaa^\dagger = aa^\dagger a$ [2,4,6].

An element $a \in \mathcal{R}$ is: left *-cancellable if $a^*ax = a^*ay$ implies $ax = ay$; it is right *-cancellable if $xaa^* = yaa^*$ implies $xa = ya$; and it is *-cancellable if it is both left and right *-cancellable. Notice that a is left *-cancellable if and only if a^* is right *-cancellable. In C^*-algebras all elements are *-cancellable. A ring $\mathcal{R}$ is called *-reducing if every element of $\mathcal{R}$ is *-cancellable. This is equivalent to the implication $a^*a = 0 \Rightarrow a = 0$ for all $a \in \mathcal{R}$.

Malik and Thome [6] investigated weaker conditions than those given by Penrose for an operator to be the Moore–Penrose inverse of a given bounded linear operator between two Hilbert spaces, using operator matrices.

In this paper, applying a purely algebraic technique, we characterize the Moore–Penrose inverse of an element in a ring with involution in terms of EP, normal, bi–EP, bi–normal, l–quasi–EP, r–quasi–EP, l–quasi–normal, r–quasi–normal and *-cancellable elements, generalizing the results from [6]. Thus, we show that the properties of operator matrices are not necessary for the characterization of the Moore–Penrose inverse.

2. RESULTS

In the first theorem, if x is an 1–inverse of a, we present necessary and sufficient conditions for x to be a 3–inverse of a.

Theorem 2. *Let $a \in \mathcal{R}$ such that $a\{1\} \neq \emptyset$. If $x \in a\{1\}$, then the following statements are equivalent:*

(i) *ax is self–adjoint;*
(ii) *ax is EP;*
(iii) *ax is right *-cancellable and normal;*
(iv) *ax is bi–EP;*
(v) *ax is *-cancellable and bi–normal;*
(vi) *ax is l–quasi–EP;*
(vii) *ax is r–quasi–EP;*
(viii) *ax is left *-cancellable and l–quasi–normal;*
(ix) *ax is right *-cancellable and r–quasi–normal.*

Proof. (i) $\Rightarrow$ (ii)–(ix): Because ax is self–adjoint idempotent, we deduce that $ax \in \mathcal{R}^\dagger$ and $(ax)^\dagger = ax$. So, the statements (ii)–(ix) hold.

(ii) $\Rightarrow$ (i): Since ax is EP, then $ax \in \mathcal{R}^\dagger$ and $(ax)^\dagger ax = ax(ax)^\dagger$. Thus,

$$ax = ax(ax)^\dagger ax = axax(ax)^\dagger = ax(ax)^\dagger$$

is self–adjoint.

(iii) $\Rightarrow$ (i): If ax is right *-cancellable and normal, then $(ax)^*$ is left *-cancellable and $ax(ax)^* = (ax)^* ax$. Multiplying this equality by ax from the left side, it follows

$$ax(ax)^* = ax(ax)^* ax.$$

Using *-cancellation, we get $(ax)^* = (ax)^* ax$, i.e. $ax = (ax)^* ax$ is self–adjoint.

(iv) $\Rightarrow$ (i): Suppose that ax is bi–EP, that is, $ax \in \mathcal{R}^\dagger$ and

$$ax(ax)^\dagger (ax)^\dagger ax = (ax)^\dagger axax(ax)^\dagger.$$

Now, we get

$$\begin{aligned} [(ax)^\dagger]^* &= [(ax)^\dagger ax(ax)^\dagger]^* = [(ax)^\dagger axax(ax)^\dagger]^* \\ &= [ax(ax)^\dagger (ax)^\dagger ax]^* = (ax)^\dagger axax(ax)^\dagger \\ &= (ax)^\dagger ax(ax)^\dagger = (ax)^\dagger, \end{aligned}$$

which implies $(ax)^* = \left([(ax)^\dagger]^*\right)^\dagger = [(ax)^\dagger]^\dagger = ax$.

(v) $\Rightarrow$ (i): Let ax be *-cancellable and bi–normal. Then

$$ax(ax)^*(ax)^* ax = (ax)^* axax(ax)^*$$

gives

$$ax(ax)^* ax = (ax)^* ax(ax)^*.$$

Multiplying the previous equality by ax from the left side, we obtain

$$ax(ax)^* ax = ax(ax)^* ax(ax)^*. \tag{2.1}$$

Since ax and $(ax)^*$ are *-cancellable, by (2.1), $ax = ax(ax)^*$ is self–adjoint.

(vi) $\Rightarrow$ (i): Because ax is l–quasi–EP, we have that $ax \in \mathcal{R}^\dagger$ and

$$ax(ax)^\dagger ax = (ax)^\dagger axax$$

implying $ax = (ax)^\dagger ax$ is self–adjoint.

(vii) $\Rightarrow$ (i): Similarly as (vi) $\Rightarrow$ (i).

(viii) $\Rightarrow$ (i): Assume that ax is left *-cancellable and l–quasi–normal. We deduce that $(ax)^*$ is right *-cancellable and

$$ax(ax)^* ax = (ax)^* axax = (ax)^* ax,$$

which yield $ax(ax)^* = (ax)^*$. Hence, $ax(ax)^* = ax$ is self–adjoint.

(ix) $\Rightarrow$ (i): In the same way as in the part (viii) $\Rightarrow$ (i). □

The following result can be checked similarly as Theorem 2.

Corollary 1. *Let $a \in \mathcal{R}$ such that $a\{1\} \neq \emptyset$. If $x \in a\{1\}$, then the following statements are equivalent:*

(i) *ax is self–adjoint;*
(ii) *ax is EP;*
(iii) *$ax \in \mathcal{R}^\dagger$ and ax is normal;*
(iv) *ax is bi–EP;*
(v) *$ax \in \mathcal{R}^\dagger$ and ax is bi–normal;*
(vi) *ax is l–quasi–EP;*
(vii) *ax is r–quasi–EP;*
(viii) *$ax \in \mathcal{R}^\dagger$ and ax is l–quasi–normal;*
(ix) *$ax \in \mathcal{R}^\dagger$ and ax is r–quasi–normal.*

Proof. Since $ax \in \mathcal{R}^\dagger$, by [5, Theorem 5.3], we deduce that ax is *-cancellable. The rest of this proof follows as in the proof of Theorem 2. □

For an 1–inverse x of a, several equivalent conditions which insure that x is 4–inverse of a are given now.

Theorem 3. *Let $a \in \mathcal{R}$ such that $a\{1\} \neq \emptyset$. If $x \in a\{1\}$, then the following statements are equivalent:*

(i) *xa is self–adjoint;*
(ii) *xa is EP;*
(iii) *xa is left *-cancellable and normal;*
(iv) *xa is bi–EP;*
(v) *xa is *-cancellable and bi–normal;*
(vi) *xa is r–quasi–EP;*
(vii) *xa is l–quasi–EP;*
(viii) *xa is right *-cancellable and r–quasi–normal;*
(ix) *xa is left *-cancellable and l–quasi–normal.*

Proof. For $x \in a\{1\}$ such that xa satisfies any of the conditions (i)–(ix), we have that $x^* \in a^*\{1\}$ such that a^*x^* satisfies any of the conditions (i)–(ix) of Theorem 2. Applying Theorem 2, we verify this result. □

In the main result of this article, weaker conditions than those given in Penrose equations for an element x to be the Moore–Penorose inverse of a are given.

Theorem 4. *Let $a \in \mathcal{R}$ such that $a\{1,2\} \neq \emptyset$ and $x \in a\{1,2\}$. If ax satisfies any of the conditions* (ii)–(ix) *of Theorem 2 and xa satisfies any of the conditions* (ii)–(ix) *of Theorem 3, then x is the Moore–Penrose inverse of a.*

Proof. This result is a consequence of Theorem 2 and Theorem 3. □

The preceding theorems hold in rings with involution assuming in some conditions that ax and xa are left or (and) right *-cancellable which is automatically satisfied in C^*–algebras and *-reducing rings. Thus, we recover the results in [6] for Hilbert space operators.

In a unital C^*–algebra $\mathcal{A}$, since $a \in \mathcal{A}$ is Moore–Penrose invertible if and only if $a\{1\} \neq \varnothing$ (see [3]), we get the following result as a consequence of Theorem 4.

Corollary 2. *Let $a \in \mathcal{A}$ such that $a\{1\} \neq \varnothing$ and $x \in a\{1,2\}$. If both ax and xa satisfy any of the statements:*

(i) *EP;*
(ii) *normal;*
(iv) *bi–EP;*
(v) *bi–normal;*
(vi) *l–quasi–EP;*
(vii) *r–quasi–EP;*
(viii) *l–quasi–normal;*
(ix) *r–quasi–normal,*

then x is the Moore–Penrose inverse of a.

By corresponding examples in [6], it is showed that conditions (i)–(ix) of Corollary 2 are weaker that one of being self–adjoint and can be adopted to define the Moore–Penrose inverse.

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