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Quenching time for a system of semilinear heat equations

*Theodore K. Boni, Halima Nachid, and Diabate
Nabongo*



QUENCHING TIME FOR A SYSTEM OF SEMILINEAR HEAT EQUATIONS

THEODORE K. BONI, HALIMA NACHID, AND DIABATE NABONGO

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Abstract. This paper concerns the study of the quenching time of the solution of the initial-boundary value problem for a system of reaction-diffusion equations.

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1. INTRODUCTION

Let Ω be a bounded domain in \mathbb{R}^N with the smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem for a system of reaction-diffusion equations of the form

$$u_t = \varepsilon \Delta u + f(v) \quad \text{in } \Omega \times (0, T), \quad (1.1)$$

$$v_t = \varepsilon \Delta v + g(u) \quad \text{in } \Omega \times (0, T), \quad (1.2)$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (1.3)$$

$$v = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (1.4)$$

$$u(x, 0) = u_0(x) \geq 0 \quad \text{in } \Omega, \quad (1.5)$$

$$v(x, 0) = v_0(x) \geq 0 \quad \text{in } \Omega, \quad (1.6)$$

where $f: (-\infty, b_1) \rightarrow (0, \infty)$ is C^1 convex, increasing function with $b_1 = \text{const} > 0$, $\lim_{s \rightarrow b_1} f(s) = +\infty$, $\int_0^{b_1} \frac{ds}{f(s)} < +\infty$, $g: (-\infty, b_2) \rightarrow (0, \infty)$ is C^1 convex, increasing function with $b_2 = \text{const} > 0$, $\lim_{s \rightarrow b_2} g(s) = +\infty$, $\int_0^{b_2} \frac{ds}{g(s)} < +\infty$. The initial data (u_0, v_0) is such that $u_0 \in C^1(\bar{\Omega})$, $v_0 \in C^1(\bar{\Omega})$, $u_0(x) \geq 0$ in Ω , $u_0(x) = 0$ on $\partial\Omega$, $v_0(x) \geq 0$ in Ω , $v_0(x) = 0$ on $\partial\Omega$, $\sup_{x \in \Omega} u_0(x) < b_2$, $\sup_{x \in \Omega} v_0(x) < b_1$. Here, $(0, T)$ is the maximal time interval of existence of the solution (u, v) . The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. When T is finite, then the solution (u, v) develops a singularity in a finite

time, namely

$$\left(\lim_{t \rightarrow T} \|u(\cdot, t)\|_\infty, \lim_{t \rightarrow T} \|v(\cdot, t)\|_\infty \right) \not\leq (b_2, b_1),$$

where $\|u(\cdot, t)\|_\infty = \max_{x \in \Omega} |u(x, t)|$. Here, $(a, b) \not\leq (c, d)$ means that $(a, b) \leq (c, d)$ and at least one of the equalities $a = c, b = d$ is valid.

Solutions for systems of semilinear heat equations which quench in a finite time have been the subject of investigation of several authors (see [3–5] and the references cited therein). By standard methods based on the maximum principle, the local existence, uniqueness, global existence, and quenching have been treated. One may also find in [1, 6, 7, 9, 10] some results about quenching for semilinear heat equations. In this paper, we are interested in the asymptotic behaviour of the quenching time as ε goes to zero. Our work is motivated by the paper [8] by Friedman and Lacey, where they consider the initial-boundary value problem

$$u_t = \varepsilon \Delta u + g(u) \quad \text{in } \Omega \times (0, T), \quad (1.7)$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (1.8)$$

$$u(x, 0) = u_0(x) \geq 0 \quad \text{in } \Omega, \quad (1.9)$$

where $g(s)$ is a positive, increasing, convex function for the nonnegative values of s and such that $\int_0^{+\infty} \frac{ds}{g(s)} < +\infty$. The initial data u_0 is a positive and continuous function in $\bar{\Omega}$. Under some additional conditions on the initial data, they showed that the solution u of (1.7)–(1.9) blows up in a finite time, and its blow-up time goes to that of the solution $\alpha(t)$ of the

$$\alpha'(t) = g(\alpha(t)), \quad t > 0, \quad \alpha(0) = M, \quad (1.10)$$

when ε goes to zero, where $M = \sup_{x \in \Omega} u_0(x)$ (we say that a solution u *blows up* in a finite time if it reaches the value infinity in a finite time). Nabongo and Boni have obtained in [10] an analogous result in the case of the phenomenon of quenching for semilinear heat equations. In this article, we get a comparable result for the system described in (1.1)–(1.6). More precisely, under some hypotheses, we show that if ε is small enough, then the solution (u, v) of (1.1)–(1.6) quenches in a finite time, and its quenching time tends to that of the solution $(\alpha(t), \beta(t))$ of the differential system defined below

$$\alpha'(t) = f(\beta(t)), \quad t > 0, \quad (1.11)$$

$$\beta'(t) = g(\alpha(t)), \quad t > 0, \quad (1.12)$$

$$\alpha(0) = M_1, \quad (1.13)$$

$$\beta(0) = M_2, \quad (1.14)$$

as ε goes to zero, where $M_1 = \sup_{x \in \Omega} u_0(x)$, $M_2 = \sup_{x \in \Omega} v_0(x)$.

Our paper is written in the following manner. In the next section, under some conditions, we prove that if ε is small enough, then the solution (u, v) of (1.1)–(1.6)

quenches in a finite time, and its quenching time tends to that of the solution of the differential system defined in (1.11)–(1.14). Finally, in the last section, we give some numerical results to illustrate our analysis.

2. QUENCHING SOLUTIONS

In this section, under some assumptions, we prove that if ε is small enough, then the solution (u, v) of (1.1)–(1.6) quenches in a finite time, and its quenching time goes to that of the solution of the differential system defined in (1.11)–(1.14) as ε goes to zero. We start by recalling an important result.

Consider the following eigenvalue problem

$$-\Delta\varphi(x) = \lambda\varphi(x) \quad \text{in } \Omega, \quad (2.1)$$

$$\varphi(x) = 0 \quad \text{on } \partial\Omega, \quad (2.2)$$

$$\varphi(x) > 0 \quad \text{in } \Omega. \quad (2.3)$$

It is well known that the above eigenvalue problem admits a solution (φ, λ) such that $\lambda > 0$. We can normalize φ so that $\int_{\Omega} \varphi(x) dx = 1$. Now, let us give our first result about the quenching time.

Theorem 1. *Suppose that $u_0(x) = 0$ and $v_0(x) = 0$. Let*

$$A = \lambda \max \left\{ \frac{b_2}{f(0)}, \frac{b_1}{g(0)} \right\}.$$

If $\varepsilon < \frac{1}{A}$, then the solution (u, v) of (1.1)–(1.6) quenches in a finite time, and its quenching time T obeys the following estimates

$$0 \leq T - T_0 \leq \varepsilon T_0 A + o(\varepsilon),$$

where T_0 is the quenching time of the solution $(\alpha(t), \beta(t))$ of the differential system defined in (1.11)–(1.14).

Proof. Since $(0, T)$ is the maximal time interval of existence of the solution (u, v) , our aim is to show that T is finite and satisfies the above estimates. Introduce the functions $w(t)$ and $z(t)$ defined as follows

$$w(t) = \int_{\Omega} u(x, t) \varphi(x) dx, \quad z(t) = \int_{\Omega} v(x, t) \varphi(x) dx, \quad t \in [0, T).$$

Due to the fact that the initial data (u_0, v_0) is nonnegative in Ω , from the maximum principle, (u, v) is also nonnegative in $\Omega \times (0, T)$. Take the derivative of w in t and use (1.1) to obtain

$$w'(t) = \varepsilon \int_{\Omega} \varphi \Delta u dx + \int_{\Omega} f(v) \varphi dx \quad \text{for } t \in (0, T).$$

Applying Green's formula, we arrive at the equality

$$w'(t) = \varepsilon \int_{\Omega} u \Delta \varphi dx + \int_{\Omega} f(v) \varphi dx \quad \text{for } t \in (0, T).$$

It follows from (2.1) and Jensen's inequality that

$$w'(t) \geq -\varepsilon \lambda w(t) + f(z(t)) \quad \text{for } t \in (0, T).$$

In the same way, we also get

$$z'(t) \geq -\varepsilon \lambda z(t) + g(w(t)) \quad \text{for } t \in (0, T).$$

It is not difficult to see that $0 \leq w(t) \leq b_2$ and $0 \leq z(t) \leq b_1$. We deduce that

$$\begin{aligned} w'(t) &\geq -\varepsilon \lambda b_2 + f(z(t)) \quad \text{for } t \in (0, T), \\ z'(t) &\geq -\varepsilon \lambda b_1 + g(w(t)) \quad \text{for } t \in (0, T), \end{aligned}$$

whence

$$\begin{aligned} w'(t) &\geq f(z(t)) \left(1 - \frac{\varepsilon \lambda b_2}{f(z(t))}\right) \quad \text{for } t \in (0, T), \\ z'(t) &\geq g(w(t)) \left(1 - \frac{\varepsilon \lambda b_1}{g(w(t))}\right) \quad \text{for } t \in (0, T). \end{aligned}$$

Since $f(z(t)) \geq f(0)$ and $g(w(t)) \geq g(0)$, we arrive at the inequalities

$$\begin{aligned} w'(t) &\geq f(z(t)) \left(1 - \frac{\varepsilon \lambda b_2}{f(0)}\right) \quad \text{for } t \in (0, T), \\ z'(t) &\geq g(w(t)) \left(1 - \frac{\varepsilon \lambda b_1}{g(0)}\right) \quad \text{for } t \in (0, T). \end{aligned}$$

Taking into account the fact that $\frac{\lambda b_2}{f(0)} \leq A$ and $\frac{\lambda b_1}{g(0)} \leq A$, we find that

$$\begin{aligned} w'(t) &\geq f(z(t)) (1 - \varepsilon A) \quad \text{for } t \in (0, T), \\ z'(t) &\geq g(w(t)) (1 - \varepsilon A) \quad \text{for } t \in (0, T). \end{aligned}$$

Let us set

$$\begin{aligned} w_1(t) &= w \left(\frac{t}{1 - \varepsilon A} \right) \quad \text{for } t \in (0, (1 - \varepsilon A)T), \\ z_1(t) &= z \left(\frac{t}{1 - \varepsilon A} \right) \quad \text{for } t \in (0, (1 - \varepsilon A)T). \end{aligned}$$

A straightforward computation reveals that

$$\begin{aligned} w_1'(t) &\geq f(z_1(t)) \quad \text{for } t \in (0, (1 - \varepsilon A)T), & w_1(0) &= w(0) = 0, \\ z_1'(t) &\geq g(w_1(t)) \quad \text{for } t \in (0, (1 - \varepsilon A)T), & z_1(0) &= z(0) = 0. \end{aligned}$$

From the maximum principle, we get

$$w_1(t) \geq \alpha(t) \quad \text{for } t \in (0, T_*),$$

$$z_1(t) \geq \beta(t) \quad \text{for } t \in (0, T_*),$$

where $T_* = \min\{T_0, (1 - \varepsilon A)T\}$, which implies that

$$T \leq \frac{T_0}{1 - \varepsilon A}. \quad (2.4)$$

In fact, suppose that $T > \frac{T_0}{1 - \varepsilon A} = T'$. We deduce that

$$(w(T'), z(T')) = (w_1(T_0), z_1(T_0)) \geq (\alpha(T_0), \beta(T_0)) \not\leq (b_2, b_1).$$

Since

$$(\|u(\cdot, T')\|_\infty, \|v(\cdot, T')\|_\infty) \geq (w(T'), z(T')),$$

we have a contradiction because $(0, T)$ is the maximum time interval of existence of (u, v) . On the other hand, let $(u_1(x, t), v_1(x, t))$ be such that

$$\begin{aligned} u_1(x, t) &= \alpha(t) & \text{in } \bar{\Omega} \times [0, T_0], \\ v_1(x, t) &= \beta(t) & \text{in } \bar{\Omega} \times [0, T_0]. \end{aligned}$$

A routine computation reveals that

$$\begin{aligned} u_{1t} &= \varepsilon \Delta u_1 + f(v_1) & \text{in } \Omega \times (0, T_0), \\ v_{1t} &= \varepsilon \Delta v_1 + g(u_1) & \text{in } \Omega \times (0, T_0), \end{aligned}$$

$u_1 \geq 0$ on $\partial\Omega \times (0, T_0)$, $v_1 \geq 0$ on $\partial\Omega \times (0, T_0)$, and

$$\begin{aligned} u_1(x, 0) &\geq 0 & \text{in } \Omega, \\ v_1(x, 0) &\geq 0 & \text{in } \Omega. \end{aligned}$$

The maximum principle implies that

$$\begin{aligned} 0 &\leq u(x, t) \leq u_1(x, t) = \alpha(t) & \text{in } \Omega \times (0, T^0), \\ 0 &\leq v(x, t) \leq v_1(x, t) = \beta(t) & \text{in } \Omega \times (0, T^0), \end{aligned}$$

where $T^0 = \min\{T, T_0\}$. We deduce that

$$T \geq T_0. \quad (2.5)$$

Indeed, assume that $T < T_0$. We get

$$(\|u(\cdot, T)\|_\infty, \|v(\cdot, T)\|_\infty) \leq (\alpha(T), \beta(T)) < (b_2, b_1),$$

which is a contradiction because $(0, T)$ is the maximal time interval of existence of the solution (u, v) . Applying Taylor's expansion, we obtain

$$\frac{1}{1 - \varepsilon A} = 1 + \varepsilon A + o(\varepsilon).$$

Using (2.4), (2.5) and the relation above, we complete the proof. \square

Now, let us consider the case where the initial data is not null. In the sequel, we suppose that there exists $a \in \Omega$ such that

$$\sup_{x \in \Omega} u_0(x) = u_0(a) \quad \text{and} \quad \sup_{x \in \Omega} v_0(x) = v_0(a).$$

Consider the following eigenvalue problem

$$-\Delta \psi(x) = \lambda_\delta \psi(x) \quad \text{in} \quad B(a, \delta), \quad (2.6)$$

$$\psi(x) = 0 \quad \text{on} \quad \partial B(a, \delta), \quad (2.7)$$

$$\psi(x) > 0 \quad \text{in} \quad B(a, \delta), \quad (2.8)$$

where $\delta > 0$, such that, $B(a, \delta) = \{x \in \mathbb{R}^N; \|x - a\| < \delta\} \subset \Omega$.

It is well known that the above problem has a solution (ψ, λ_δ) such that $\lambda_\delta = \frac{\lambda_1}{\delta^2}$, where λ_1 is the eigenvalue corresponding to the above eigenvalue problem for $\delta = 1$.

We are in position to state our result in the case where the initial data is not null.

Theorem 2. *Suppose that $M_1 = \sup_{x \in \Omega} u_0(x) > 0$, $M_2 = \sup_{x \in \Omega} v_0(x) > 0$, and let K be an upper bound of the first derivatives of u_0 and v_0 . Let*

$$A = \lambda_1 \max \left\{ \frac{K^2 b_2}{f(0)}, \frac{K^2 b_1}{g(0)} \right\}.$$

If

$$\varepsilon < \min \left\{ (M_1/2)^3, (M_2/2)^3, (2A)^{-3}, (K \operatorname{dist}(a, \partial\Omega))^3 \right\},$$

then the solution (u, v) of (1.1)–(1.6) quenches in a finite time, and its quenching time T obeys the estimates

$$0 \leq T - T_0 \leq (T_0 A + C) \varepsilon^{1/3} + o(\varepsilon^{1/3}),$$

where

$$C = \frac{1}{\min \left\{ \frac{1}{2} f\left(\frac{M_2}{2}\right), \frac{1}{2} g\left(\frac{M_1}{2}\right) \right\}}$$

and T_0 is the quenching time of the solution $(\alpha(t), \beta(t))$ of the differential system defined in (1.11)–(1.14).

Proof. Due to the fact that $u_0 \in C^1(\bar{\Omega})$ and $v_0 \in C^1(\bar{\Omega})$, using the mean value theorem and the triangle inequality, we get

$$u_0(x) \geq u_0(a) - \varepsilon^{1/3} \quad \text{for } x \in B(a, \delta) \subset \Omega,$$

$$v_0(x) \geq v_0(a) - \varepsilon^{1/3} \quad \text{for } x \in B(a, \delta) \subset \Omega,$$

where $\delta = \frac{\varepsilon^{1/3}}{K}$. Since the initial data (u_0, v_0) is nonnegative in Ω , from the maximum principle, (u, v) is also nonnegative in $\Omega \times (0, T)$. Introduce the functions $w(t)$ and $z(t)$ defined as follows

$$w(t) = \int_{B(a, \delta)} u(x, t) \psi(x) dx, \quad z(t) = \int_{B(a, \delta)} v(x, t) \psi(x) dx, \quad t \in [0, T).$$

Take the derivative of w in t and use (1.1) to obtain

$$w'(t) = \varepsilon \int_{B(a,\delta)} \psi \Delta u dx + \int_{B(a,\delta)} f(v) \psi dx \quad \text{for } t \in (0, T).$$

Applying Green's formula, we arrive at

$$\begin{aligned} w'(t) = & \varepsilon \int_{B(a,\delta)} u \Delta \psi dx - \varepsilon \int_{\partial B(a,\delta)} u \frac{\partial \psi}{\partial \nu} ds + \varepsilon \int_{\partial B(a,\delta)} \psi \frac{\partial u}{\partial \nu} ds \\ & + \int_{B(a,\delta)} f(v) \psi dx \quad \text{for } t \in (0, T), \end{aligned}$$

where ν is the exterior normal unit vector on $\partial B(a,\delta)$. Taking into account (2.6), (2.7) and the fact that $\frac{\partial \psi}{\partial \nu} \leq 0$, we arrive at the relation

$$w'(t) \geq -\varepsilon \lambda_\delta w(t) + \int_{B(a,\delta)} f(v) \psi dx \quad \text{for } t \in (0, T).$$

It follows from Jensen's inequality that

$$w'(t) \geq -\varepsilon \lambda_\delta w(t) + f(z(t)) \quad \text{for } t \in (0, T).$$

In the same way, we also prove that

$$z'(t) \geq -\varepsilon \lambda_\delta z(t) + g(w(t)) \quad \text{for } t \in (0, T).$$

As in the proof of Theorem 1, we get

$$\begin{aligned} w'(t) & \geq f(z(t)) \left(1 - \frac{\varepsilon \lambda_\delta b_2}{f(0)} \right) \quad \text{for } t \in (0, T), \\ z'(t) & \geq g(w(t)) \left(1 - \frac{\varepsilon \lambda_\delta b_1}{g(0)} \right) \quad \text{for } t \in (0, T), \end{aligned}$$

which implies that

$$\begin{aligned} w'(t) & \geq f(z(t)) \left(1 - \frac{\varepsilon^{1/3} \lambda_1 K^2 b_2}{f(0)} \right) \quad \text{for } t \in (0, T), \\ z'(t) & \geq g(w(t)) \left(1 - \frac{\varepsilon^{1/3} \lambda_1 K^2 b_1}{g(0)} \right) \quad \text{for } t \in (0, T), \end{aligned}$$

because $\lambda_\delta = \frac{\lambda_1}{\delta^2} = \frac{\lambda_1 K^2}{\varepsilon^{2/3}}$. Consequently,

$$\begin{aligned} w'(t) & \geq f(z(t)) \left(1 - \varepsilon^{1/3} A \right) \quad \text{for } t \in (0, T), \\ w(0) & \geq M_1 - \varepsilon^{1/3}, \end{aligned}$$

and

$$z'(t) \geq g(w(t)) \left(1 - \varepsilon^{1/3} A \right) \quad \text{for } t \in (0, T),$$

$$z(0) \geq M_2 - \varepsilon^{1/3}.$$

We have

$$w'(t) \geq f(z(0))(1 - \varepsilon^{1/3}A) \geq \frac{1}{2}f(M_2/2)$$

for $t \in (0, T)$. In the same way, we get $z'(t) \geq \frac{1}{2}g(M_1/2)$ for $t \in (0, T)$, which implies that

$$w'(t) \geq \frac{1}{C}, \quad z'(t) \geq \frac{1}{C}$$

for $t \in (0, T)$. Using the mean value theorem, we get

$$w(C\varepsilon^{1/3}) \geq M_1, \quad z(C\varepsilon^{1/3}) \geq M_2.$$

Set

$$w_1(t) = w\left(\frac{t}{1 - \varepsilon^{1/3}A} + C\varepsilon^{1/3}\right)$$

for $t \in (0, (T - C\varepsilon^{1/3})(1 - \varepsilon^{1/3}A))$, and

$$z_1(t) = z\left(\frac{t}{1 - \varepsilon^{1/3}A} + C\varepsilon^{1/3}\right)$$

for $t \in (0, (T - C\varepsilon^{1/3})(1 - \varepsilon^{1/3}A))$. A straightforward computation reveals that

$$\begin{aligned} w_1'(t) &\geq f(z_1(t)) \quad \text{for } t \in (0, (T - C\varepsilon^{1/3})(1 - \varepsilon^{1/3}A)), \\ w_1(0) &\geq M_1 \end{aligned}$$

and

$$\begin{aligned} z_1'(t) &\geq g(w_1(t)) \quad \text{for } t \in (0, (T - C\varepsilon^{1/3})(1 - \varepsilon^{1/3}A)), \\ z_1(0) &\geq M_2. \end{aligned}$$

It follows from the maximum principle that

$$\begin{aligned} w_1(t) &\geq \alpha(t) \quad \text{for } t \in (0, T^*), \\ z_1(t) &\geq \beta(t) \quad \text{for } t \in (0, T^*), \end{aligned}$$

where $T^* = \min\{T_0, (T - C\varepsilon^{1/3})(1 - \varepsilon^{1/3}A)\}$. We deduce that

$$T \leq \frac{T_0}{1 - \varepsilon^{1/3}A} + C\varepsilon^{1/3}. \quad (2.9)$$

Indeed, suppose that

$$T > \frac{T_0}{1 - \varepsilon^{1/3}A} + C\varepsilon^{1/3} = T'.$$

We get

$$(w(T'), z(T')) = (w_1(T_0), z_1(T_0)) \geq (\alpha(T_0), \beta(T_0)) \not\leq (b_2, b_1).$$

Since $(\|u(\cdot, T')\|_\infty, \|v(\cdot, T')\|_\infty) \geq (w(T'), z(T'))$, we have a contradiction because $(0, T)$ is the maximal time interval of existence of the solution (u, v) . On the other hand, setting

$$\begin{aligned} w_2(x, t) &= \alpha(t) \quad \text{in } \bar{\Omega} \times [0, T_0], \\ z_2(x, t) &= \beta(t) \quad \text{in } \bar{\Omega} \times [0, T_0], \end{aligned}$$

it is not difficult to see that

$$\begin{aligned} (w_2)_t &= \varepsilon \Delta w_2 + f(z_2) \quad \text{in } \Omega \times (0, T_0), \\ (z_2)_t &= \varepsilon \Delta z_2 + g(w_2) \quad \text{in } \Omega \times (0, T_0), \\ w_2 &\geq 0 \quad \text{on } \partial\Omega \times (0, T_0), \\ z_2 &\geq 0 \quad \text{on } \partial\Omega \times (0, T_0), \\ w_2(x, 0) &\geq u_0(x) \quad \text{in } \Omega, \\ z_2(x, 0) &\geq v_0(x) \quad \text{in } \Omega. \end{aligned}$$

It follows from the maximum principle that

$$\begin{aligned} \alpha(t) &= w_2(x, t) \geq u(x, t) \quad \text{in } \Omega \times (0, T_*^*), \\ \beta(t) &= z_2(x, t) \geq v(x, t) \quad \text{in } \Omega \times (0, T_*^*), \end{aligned}$$

where $T_*^* = \min\{T_0, T\}$. We deduce that

$$T \geq T_0. \quad (2.10)$$

Indeed, suppose that $T < T_0$. We get

$$(\|u(\cdot, T)\|_\infty, \|v(\cdot, T)\|_\infty) \leq (\alpha(T), \beta(T)) < (b_2, b_1),$$

which is a contradiction because $(0, T)$ is the maximal time interval of existence of the solution (u, v) . Applying Taylor's expansion, we obtain

$$\frac{1}{1 - \varepsilon^{1/3} A} = 1 + \varepsilon^{1/3} + o(\varepsilon^{1/3}).$$

Using (2.9), (2.10) and the above relation, we complete the proof. \square

3. NUMERICAL EXPERIMENTS

In this section, we give some computational results to confirm the theory established in the previous section. We consider the radial symmetric solution of the problem (1.1)–(1.6) in the case where $\Omega = B = \{x \in \mathbb{R}^N; \|x\| < 1\}$, $\partial\Omega = S = \{x \in \mathbb{R}^N; \|x\| = 1\}$, $f(v) = (1 - v)^{-p}$, $g(u) = (2 - u)^{-q}$ with $p > 0$, $q > 0$. The above problem may be rewritten in the following form

$$u_t = \varepsilon(u_{rr} + \frac{N-1}{r}u_r) + (1 - v)^{-p}, \quad r \in (0, 1), \quad t \in (0, T), \quad (3.1)$$

$$v_t = \varepsilon(v_{rr} + \frac{N-1}{r}v_r) + (2 - u)^{-q}, \quad r \in (0, 1), \quad t \in (0, T), \quad (3.2)$$

$$u_r(0, t) = 0, \quad u(1, t) = 0, \quad t \in (0, T), \quad (3.3)$$

$$v_r(0, t) = 0, \quad v(1, t) = 0, \quad t \in (0, T), \quad (3.4)$$

$$u(r, 0) = \varphi(r), \quad r \in (0, 1), \quad (3.5)$$

$$v(r, 0) = \psi(r), \quad r \in (0, 1). \quad (3.6)$$

Here, we take

$$\varphi(r) = a \sin(\pi r),$$

$$\psi(r) = b \sin(\pi r),$$

with $a \in [0, 2)$, $b \in [0, 1)$. We begin by the construction of some adaptive schemes as follows.

Let I be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \leq i \leq I$, and approximate the solution (u, v) of (3.1)–(3.6) by the solution $(U_h^{(n)}, V_h^{(n)})$ of the explicit scheme

$$\begin{aligned} \frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} &= \varepsilon N \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} + (1 - V_0^{(n)})^{-p}, \\ \frac{V_0^{(n+1)} - V_0^{(n)}}{\Delta t_n} &= \varepsilon N \frac{2V_1^{(n)} - 2V_0^{(n)}}{h^2} + (2 - U_0^{(n)})^{-q}, \\ \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} &= \varepsilon \left(\frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \frac{(N-1)U_{i+1}^{(n)} - U_{i-1}^{(n)}}{ih} \right) \\ &\quad + (1 - V_i^{(n)})^{-p}, \quad 1 \leq i \leq I-1, \\ \frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t_n} &= \varepsilon \left(\frac{V_{i+1}^{(n)} - 2V_i^{(n)} + V_{i-1}^{(n)}}{h^2} + \frac{(N-1)V_{i+1}^{(n)} - V_{i-1}^{(n)}}{ih} \right) \\ &\quad + (2 - U_i^{(n)})^{-q}, \quad 1 \leq i \leq I-1, \\ U_I^{(n)} &= 0, \quad V_I^{(n)} = 0, \\ U_i^{(0)} &= a \sin(\pi ih), \quad 0 \leq i \leq I, \\ V_i^{(0)} &= b \sin(\pi ih), \quad 0 \leq i \leq I, \end{aligned}$$

where $n \geq 0$ and $U_h^{(n)} = (U_0^{(n)}, U_1^{(n)}, \dots, U_I^{(n)})^T$, $V_h^{(n)} = (V_0^{(n)}, V_1^{(n)}, \dots, V_I^{(n)})^T$.

In order to permit the discrete solution to reproduce the property of the continuous one when the time t approaches the quenching time T , we need to adapt the size of the time step so that we take

$$\Delta t_n = \min \left\{ \frac{h^2}{2N\varepsilon}, h^2(1 - \|V_h^{(n)}\|_\infty)^{p+1}, h^2(2 - \|U_h^{(n)}\|_\infty)^{q+1} \right\}.$$

We also approximate the solution (u, v) of (3.1)–(3.6) by the solution $(U_h^{(n)}, V_h^{(n)})$ of the following implicit scheme:

$$\begin{aligned} \frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} &= \varepsilon N \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} + (1 - V_0^{(n)})^{-p}, \\ \frac{V_0^{(n+1)} - V_0^{(n)}}{\Delta t_n} &= \varepsilon N \frac{2V_1^{(n+1)} - 2V_0^{(n+1)}}{h^2} + (2 - U_0^{(n)})^{-q}, \\ \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} &= \varepsilon \left(\frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + \frac{(N-1)}{ih} \frac{U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h} \right) \\ &\quad + (1 - V_i^{(n)})^{-p}, \quad 1 \leq i \leq I-1, \\ \frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t_n} &= \varepsilon \left(\frac{V_{i+1}^{(n+1)} - 2V_i^{(n+1)} + V_{i-1}^{(n+1)}}{h^2} + \frac{(N-1)}{ih} \frac{V_{i+1}^{(n+1)} - V_{i-1}^{(n+1)}}{2h} \right) \\ &\quad + (2 - U_i^{(n)})^{-q}, \quad 1 \leq i \leq I-1, \\ U_I^{(n+1)} &= 0, \quad V_I^{(n+1)} = 0, \\ U_i^{(0)} &= a \sin(\pi i h), \quad 0 \leq i \leq I, \\ V_i^{(0)} &= b \sin(\pi i h), \quad 0 \leq i \leq I. \end{aligned}$$

Here, similarly to the case of the explicit scheme, we choose

$$\Delta t_n = \min \left\{ h^2 (1 - \|V_h^{(n)}\|_\infty)^{p+1}, h^2 (2 - \|U_h^{(n)}\|_\infty)^{q+1} \right\}.$$

We note that

$$\lim_{r \rightarrow 0} \frac{u_r(r, t)}{r} = u_{rr}(0, t).$$

Hence, if $t = 0$, then we have

$$u_t(0, t) = \varepsilon N u_{rr}(0, t) + (1 - v(0, t))^{-p}.$$

This observation has been used in the construction of our schemes when $i = 0$. Let us notice that in the explicit scheme, the restriction on the time step ensures the nonnegativity of the discrete solution. For the implicit scheme, the existence and nonnegativity are also guaranteed by standard methods (see, e. g., [2]).

We need the following definition.

Definition 1. We say that the discrete solution $(U_h^{(n)}, V_h^{(n)})$ of the explicit or implicit scheme *quenches* in a finite time if

$$\left(\lim_{n \rightarrow +\infty} \|U_h^{(n)}\|_\infty, \lim_{n \rightarrow +\infty} \|V_h^{(n)}\|_\infty \right) \not\leq (2, 1),$$

and the series $\sum_{n=0}^{+\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{+\infty} \Delta t_n$ is called the *numerical quenching time* of the solution $(U_h^{(n)}, V_h^{(n)})$.

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256. We take for the numerical quenching time

$$T^n = \sum_{j=0}^{n-1} \Delta t_j,$$

which is computed at the first time when

$$|T^{n+1} - T^n| \leq 10^{-16}.$$

The order s of the method is computed according to the formula

$$s = \frac{\log((T_{4h} - T_{2h}) / (T_{2h} - T_h))}{\log(2)}.$$

3.1. *Numerical experiments for $p = 1$, $q = 1$, $a = 0$, $b = 0$, $N = 2$*

3.1.1. *First case: $\varepsilon = \frac{1}{10}$*

I	T^n	n	CPU time	s
16	1.011490	8172	—	—
32	1.010694	31009	1	—
64	1.010360	118245	4	1.26
128	1.010207	450669	30	1.13
256	1.010135	1714170	230	1.10

TABLE 1. Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU time	s
16	1.015745	9004	—	—
32	1.014494	34740	3	—
64	1.014196	134302	16	2.08
128	1.014124	518394	118	2.06
256	1.014107	1996487	954	2.09

TABLE 2. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

3.1.2. *Second case: $\varepsilon = \frac{1}{100}$*

I	T^n	n	CPU time	s
16	1.000142	8084	–	–
32	1.000100	30610	1	–
64	1.000088	116756	4	1.81
128	1.000022	454329	31	2.46
256	1.000006	1726227	231	2.05

TABLE 3. Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU time	s
16	1.000288	8087	–	–
32	1.000134	30620	1	–
64	1.000100	116520	14	2.18
128	1.000093	443127	101	2.28
256	1.000091	1681491	861	1.81

TABLE 4. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

3.2. *Numerical experiments for $p = 1, q = 1, a = 0, b = 0, N = 2$* 3.2.1. *First case: $\varepsilon = \frac{1}{100}$*

I	T^n	n	CPU time	s
16	0.393420	11468	–	–
32	0.393698	43641	1	–
64	0.393784	166383	6	1.70
128	0.393809	633406	45	1.78
256	0.393817	2405445	338	1.65

TABLE 5. Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU time	s
16	0.393440	11469	—	—
32	0.393703	43641	2	—
64	0.393785	166384	19	1.69
128	0.393810	633407	153	1.72
256	0.393817	2405445	1093	1.84

TABLE 6. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

3.2.2. *Second case:* $\varepsilon = \frac{1}{500}$

I	T^n	n	CPU time	s
16	0.375615	11613	—	—
32	0.375680	44079	—	—
64	0.375702	167676	6	1.57
128	0.375709	636904	45	1.66
256	0.375711	2413063	354	1.81

TABLE 7. Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU time	s
16	0.375619	11613	—	—
32	0.375682	44079	3	—
64	0.375702	167677	19	1.66
128	0.375709	636904	150	1.52
256	0.375711	2413063	1120	1.81

TABLE 8. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

Remark. If we consider the problem (3.1)–(3.6) in the case where the initial data is null and $p = 1$, $q = 1$, it is not difficult to see that the quenching time of the solution of the differential system defined in (1.11)–(1.14) equals one. We observe from Tables 1–4 that when ε diminishes, then the numerical quenching time tends to one. This result has been proved in Theorem 1.

When the initial data $(\varphi(r), \psi(r))$ are such that $\varphi(r) = \frac{1}{2} \sin(\pi r)$ and $\psi(r) = \frac{1}{2} \sin(\pi r)$, and $p = 1$, $q = 1$, then we see that the quenching time of the solution

of the differential system defined in (1.11)–(1.14) equals 0.375. We observe from Tables 5–8 that when ε diminishes, then the numerical quenching time decays to 0.375. This result has been established in Theorem 2.

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*Authors’ addresses***Theodore K. Boni**

Institut National Polytechnique Houphouët-Boigny de Yamoussoukro, Département de Mathématiques et Informatiques, BP 1093 Yamoussoukro, Côte d’Ivoire

E-mail address: theokboni@yahoo.fr

Halima Nachid

Université d’Abobo-Adjame, UFR-SFA, Département de Mathématiques et Informatiques, 16 BP 372, Abidjan 16, Côte d’Ivoire

E-mail address: nachidhalima@yahoo.fr

Diabate Nabongo

Université d’Abobo-Adjame, UFR-SFA, Département de Mathématiques et Informatiques, 16 BP 372, Abidjan 16, Côte d’Ivoire

E-mail address: nabongo_diabate@yahoo.fr