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CANONICAL ALMOST GEODESIC MAPPINGS OF TYPE $\underset{\theta}{\pi}_2(0,F)$, $\theta \in \{1,2\}$ BETWEEN GENERALIZED PARABOLIC KÄHLER MANIFOLDS

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Abstract. We introduce a generalized parabolic Kähler manifold and consider special canonical almost geodesic mappings of type $\underset{\theta}{\pi}_2(0,F)$, $\theta \in \{1,2\}$ between generalized Riemannian manifolds and between introduced generalized parabolic Kähler manifolds, particularly. Some invariant geometric objects with respect to these mappings are examined.

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1. Introduction

The use of a non-symmetric affine connection became especially interesting after the works of A. Einstein [3] on the Unified Field Theory. In 1951, L.P. Eisenhart [4] introduced a generalized Riemannian space as a differentiable manifold equipped with a non-symmetric basic tensor. Eisenhart's generalized Riemannian space is a particular case of a non-symmetric affine connection space. Some significant contributions to the study of geometry of non-symmetric affine connection spaces were made by E. Brinis, F. Graif, M. Prvanović [17] and S.M. Minčić [11–14].

Geodesic lines play an important role in modeling of various physical processes. A mapping between two manifolds with linear connection, which preserves geodesics is called a geodesic mapping. Generalizing the notions of geodesic lines and geodesic mappings, Sinyukov [18] introduced the concept of almost geodesic lines and almost geodesic mappings of affine connected spaces without torsion. He indicated three types of almost geodesic mappings of manifolds without torsion, π_1, π_2 and π_3.

The theory of geodesic and almost geodesic mappings of affine connected and Riemannian spaces is an active field of differential geometry, see for instance [1, 2, 5–10, 21, 23].

Almost geodesic mappings of type $\pi_2(e)$, $e = \pm 1$, from spaces with affine connection onto Riemannian spaces are considered in [10,23], while the paper [5] is dedicated to canonical almost geodesic mappings of type $\pi_2(e=0)$ between Riemannian spaces with an almost affinor structure, and between parabolic Kählerian spaces, particularly. Several papers are devoted to almost geodesic mappings of type $\underset{\theta}{\pi}_2(e = \pm 1)$, $\theta \in \{1,2\}$ and its special cases $\underset{\theta}{\pi}_2(e = \pm 1, F)$, $\theta \in \{1,2\}$ between manifolds with non-symmetric affine connection, see [15, 19, 21]. In the papers [16, 22] some invariant geometric objects with respect to special almost geodesic mappings of type $\underset{1}{\pi}_1$ and $\underset{1}{\pi}_3$, respectively, are examined, by considering equitorsion mappings. In [15] we presented systems of differential equations of Cauchy type for almost geodesic mappings of the second type of manifolds with non-symmetric linear connection, also we found some invariant geometric object of almost geodesic mappings of type $\underset{\theta}{\pi}_2(e = -1, F)$, $\theta \in \{1,2\}$ under some assumptions.

In the present paper, we extend and improve results from [5]. We consider canonical almost geodesic mappings of type $\underset{\theta}{\pi}_2(0, F)$, $\theta \in \{1,2\}$ between generalized Riemannian manifolds. Also, we introduce a generalized parabolic Kähler manifold and consider canonical almost geodesic mappings of type $\underset{\theta}{\pi}_2(0, F)$, $\theta \in \{1,2\}$ between such manifolds. The wider class of metrics enables us to find more invariant geometric objects than in the classical (symmetric) case [5].

2. Special canonical almost geodesic mappings of generalized Riemannian manifolds

In the sense of Eisenhart (see [4]) a generalized Riemannian space is a differentiable manifold M equipped with a metric g, which is generally non-symmetric. Therefore, the metric g can be described as follows

$$g(X,Y) = \underline{g}(X,Y) + \underset{\vee}{g}(X,Y), \text{ for all } X,Y \in T_p(M).$$

Here $\underline{g}$ denotes the symmetric part of the metric g and $\underset{\vee}{g}$ denotes the skew-symmetric part of g, i.e.

$$\underline{g}(X,Y) = \frac{1}{2}(g(X,Y) + g(Y,X)) \text{ and } \underset{\vee}{g}(X,Y) = \frac{1}{2}(g(X,Y) - g(Y,X)),$$

where $X,Y \in T_p(M)$ and $T_p(M)$ is the tangent vector space of M at $p \in M$.

The non-symmetric linear connection ${}^1\nabla$ of the generalized Riemannian manifold with the metric g is explicitly defined by

$$g({}^1\nabla_X Y, Z) = \frac{1}{2}(Xg(Y,Z) + Yg(Z,X) - Zg(Y,X)), \; X,Y,Z \in T_p(M). \quad (2.1)$$

Let us denote by ∇ the Levi-Civita connection corresponding to the symmetric metric $\underline{g}$. This connection is the symmetric part of non-symmetric linear connection ${}^1\nabla$, i.e.

$$\nabla_X Y = \frac{1}{2}({}^1\nabla_X Y + {}^1\nabla_Y X), X, Y \in T_p(M).$$

Also, it is well know that on the manifold M with non-symmetric linear connection ${}^1\nabla$ can be defined another non-symmetric linear connection ${}^2\nabla$ in the following way

$${}^2\nabla_X Y = {}^1\nabla_Y X + [X,Y],\ X, Y \in \mathcal{X}(M),$$

where as usual $\mathcal{X}(M)$ denotes the set of smooth vector fields on M and $[\cdot,\cdot]$ denotes the Lie bracket [17].

M.S. Stanković in [19] introduced two kinds of almost geodesic lines, as follows. Let $c : I \to M$ be a curve on a manifold M with non-symmetric linear connection ${}^1\nabla$, satisfying the regularity condition $c'(t) \neq 0, t \in I$. Denote by $\xi(t) = (c(t), c'(t))$ the tangent vector field along c, and let us put

$$\underset{\theta}{\xi_1} = {}^\theta\nabla_\xi \xi, \quad \underset{\theta}{\xi_2} = {}^\theta\nabla_\xi \underset{\theta}{\xi_1}, \quad \theta \in \{1,2\}.$$

If the vector fields ξ and $\underset{\theta}{\xi_1}$ are independent at any point (hence the (local) curve c is not a geodesic one) we can put ${}^\theta D = \mathrm{span}(\xi, \underset{\theta}{\xi_1})$, $\theta \in \{1,2\}$. The curve c is an almost geodesic line of the kind θ ($\theta \in \{1,2\}$) if and only if $\underset{\theta}{\xi_2} \in {}^\theta D$. In [15] we gave an equivalent definition of almost geodesic lines of manifolds with non-symmetric linear connection, it is Definition 1.

Definition 1 ([15]). Let $c : I \to M$ be a curve on a manifold M with non-symmetric linear connection satisfying the regularity condition $c'(t) \neq 0$ and let $\xi(t) = (c(t), c'(t))$ be the tangent vector field along c. The curve c is called an *almost geodesic of the kind* θ ($\theta \in \{1,2\}$) if there exist vector fields X_1 and X_2 satisfying ${}^\theta\nabla_\xi X_i = a_i^j X_j$ for some differentiable functions $a_i^j : I \to \mathbb{R}$ and differentiable real functions $b^i(t)$ along c such that $\xi = b^1 X_1 + b^2 X_2$ holds.

Definition 2 ([15, 19]). A diffeomorphism $f : M \to \overline{M}$ of n-dimensional manifolds with non-symmetric linear connection is called an *almost geodesic mapping of the kind* θ ($\theta = 1,2$) if any geodesic line of the manifold M turns into an almost geodesic line of the kind θ of the manifold $\overline{M}$.

Let M and $\overline{M}$ be two generalized Riemannian manifolds of dimension $n > 2$ with the metrics g and $\overline{g}$, respectively. We can consider these manifolds in the *common coordinate system with respect to the diffeomorphism* $f : M \to \overline{M}$. In this coordinate system the corresponding points $p \in M$ and $f(p) \in \overline{M}$ have the same coordinates. Therefore, we can suppose $M \equiv \overline{M}$ and for $\theta \in \{1,2\}$ we can put

$${}^\theta P = {}^\theta\overline{\nabla} - {}^\theta\nabla,$$

where ${}^{\theta}P$ is the tensor field of type $(1,2)$, called the *deformation tensor field of linear connections* ${}^{\theta}\overline{\nabla}$ *and* ${}^{\theta}\nabla$ *with respect to the mapping* f.

In what follows we will use $\sum_{CS(\cdot,\cdot,\cdot)}$ to denote the cyclic sum on arguments in brackets, for instance for an arbitrary tensor field A we have

$$\sum_{CS(X,Y,Z)} A(X,Y,Z) = A(X,Y,Z) + A(Y,Z,X) + A(Z,X,Y).$$

A diffeomorphism $f : M \to \overline{M}$ is an *almost geodesic mapping of the kind* θ, $\theta \in \{1,2\}$ if and only if [15]

$$\underset{\theta}{P}(X_1,X_2,X_3) \wedge {}^{\theta}P(X_4,X_5) \wedge X_6 = 0,\ X_i \in \mathcal{X}(M),\ i = 1,\dots,6,$$

where ${}^{\theta}P$ is the deformation tensor field of connections ${}^{\theta}\overline{\nabla}$ and ${}^{\theta}\nabla$, with respect to the diffeomorphism f, and $\underset{1}{P}$, $\underset{2}{P}$, are tensor fields of type $(1,3)$, defined by

$$\underset{1}{P}(X,Y,Z) = \sum_{CS(X,Y,Z)} {}^{1}\nabla_Z {}^{1}P(X,Y) + {}^{1}P({}^{1}P(X,Y),Z),\ X,Y,Z \in \mathcal{X}(M)$$

and

$$\underset{2}{P}(X,Y,Z) = \sum_{CS(X,Y,Z)} {}^{2}\nabla_Z {}^{2}P(X,Y) + {}^{2}P(Z,{}^{2}P(X,Y)),\ X,Y,Z \in \mathcal{X}(M).$$

Basic equations of canonical almost geodesic mappings of type $\underset{\theta}{\pi}2(e=0)$*,* $\theta \in \{1,2\}$ between generalized Riemannian manifolds are given by

$${}^{\theta}P(X,Y) = \sum_{CS(X,Y)} \varphi(X)FY + (-1)^{(\theta-1)}K(X,Y), \tag{2.2}$$

$$\sum_{CS(X,Y)} \Big({}^{\theta}\nabla_Y FX - (-1)^{\theta}K(FY,X)\Big) = \sum_{CS(X,Y)} \big(\mu(X)FY - \mu(FX)Y\big), \tag{2.3}$$

where $X,Y \in \mathcal{X}(M)$, φ is a 1-form, K is an anti-symmetric tensor field of type $(1,2)$ defined by

$$K(X,Y) = \frac{1}{2}\big({}^{1}P(X,Y) - {}^{1}P(Y,X)\big) = \frac{1}{2}\big({}^{2}P(Y,X) - {}^{2}P(X,Y)\big),$$

and F is a tensor field of type $(1,1)$ satisfying

$$F^2 = 0.$$

If the affinor structure F satisfies an additional condition

$$\mathrm{Tr}(F) = F^p_p = 0,$$

then we denote by $\underset{\theta}{\pi}2(0,F)$, $\theta \in \{1,2\}$.

A canonical almost geodesic mapping $f : M \to \overline{M}$ of type $\underset{\theta}{\pi}_2(0,F)$, $\theta \in \{1,2\}$ has the *property of reciprocity* if its inverse mapping $f^{-1} : \overline{M} \to M$ is a canonical almost geodesic mapping of type $\underset{\theta}{\pi}_2(0,F)$. Since the deformation tensor fields ${}^1P$ and ${}^1\overline{P}$ of linear connections ${}^1\nabla$ and ${}^1\overline{\nabla}$ with respect to the mappings f and f^{-1}, respectively, satisfy the relation

$$ {}^1\overline{P}(X,Y) = -{}^1P(X,Y), \quad X,Y \in \mathcal{X}(M), $$

without loss of generality we can suppose

$$ \overline{\varphi} = -\varphi, \quad \overline{F} = F, \quad \overline{K} = -K, $$

or in components

$$ \overline{\varphi}_i = -\varphi_i, \quad \overline{F}^h_i = F^h_i, \quad \overline{K}^h_{ij} = -K^h_{ij}. \tag{2.4} $$

Almost geodesic mappings of manifolds with non-symmetric linear connection, which satisfy the property of reciprocity are investigated in [15, 19, 21, 22]. A necessary and sufficient condition for an almost geodesic mapping $f : M \to \overline{M}$ of type $\underset{\theta}{\pi}_2$, $\theta \in \{1,2\}$ to have the property of reciprocity is expressed by

$$ F^2 = \alpha I + \beta F, $$

where I is the identity matrix and α, β are some scalar functions.

2.1. *Invariants*

We use traditional tensor calculus approach "by components". In local coordinates, with respect to a local chart (U,φ), $\varphi = (x^1,\ldots,x^n)$, we have

$$ {}^1\nabla_i \frac{\partial}{\partial x^j} = {}^1\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^h_{ij}\frac{\partial}{\partial x^h}, \qquad {}^2\nabla_i \frac{\partial}{\partial x^j} = {}^2\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^h_{ji}\frac{\partial}{\partial x^h}, $$

and

$$ \nabla_i \frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^h_{\underline{ij}}\frac{\partial}{\partial x^h}, $$

where $\underline{ij}$ signifies a symmetrization with division and the functions Γ^h_{ij} are generalized Christoffel symbols.

A. Einstein [3] used two kinds of covariant differentiation of a tensor a^i_j:

$$ a^i_{j\underset{1}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{jm}a^i_p, \quad a^i_{j\underset{2}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{mj}a^i_p, $$

where $a^i_{j,m}$ denotes the partial derivative of a tensor a^i_j with respect to x^m.

S.M. Minčić [11] has used two more kinds of covariant differentiation of tensors:

$$ a^i_{j\underset{3}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{mj}a^i_p, \quad a^i_{j\underset{4}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{jm}a^i_p. $$

Also, we can consider covariant differentiation with respect to the Levi-Civita connection ∇, that is

$$\nabla_m a^i_j \equiv a^i_{j;m} = a^i_{j,m} + \Gamma^i_{\underline{pm}} a^p_j - \Gamma^p_{\underline{jm}} a^i_p,$$

where $\Gamma^i_{\underline{pm}}$ is the symmetric part of Γ^i_{pm}.

Let us denote by $\underset{\theta}{|}$ and $\underset{\theta}{||}$, $\theta = 1, \ldots, 4$, the covariant derivatives with respect to the generalized Christoffel symbols Γ^h_{ij} and $\overline{\Gamma}^h_{ij}$, respectively.

In local coordinates the basic equation (2.2) reads

$$\overline{\Gamma}^h_{ij} - \Gamma^h_{ij} = \varphi_{(i} F^h_{j)} + K^h_{ij}, \tag{2.5}$$

where φ_i is the covariant vector corresponding to the linear form φ, while F^h_i and K^h_{ij} are components of tensor fields F and K, respectively.

By using covariant differentiation of the first kind and the equation (2.5) we obtain

$$F^h_{i\underset{1}{||}j} = F^h_{i\underset{1}{|}j} + \varphi_p F^p_i F^h_j + K^h_{pj} F^p_i - K^p_{ij} F^h_p. \tag{2.6}$$

After contracting the relation (2.6) over the indices j and h and by using (2.4) we get

$$\overline{F}^\alpha_{i\underset{1}{||}\alpha} + \frac{1}{2}\overline{K}^\alpha_{p\alpha}\overline{F}^p_i - \frac{1}{2}\overline{K}^p_{i\alpha}\overline{F}^\alpha_p = F^\alpha_{i\underset{1}{|}\alpha} + \frac{1}{2}K^\alpha_{p\alpha}F^p_i - \frac{1}{2}K^p_{i\alpha}F^\alpha_p, \tag{2.7}$$

i.e. the tensor $\underset{1}{A}_i$ defined by

$$\underset{1}{A}_i = F^\alpha_{i\underset{1}{|}\alpha} + \frac{1}{2}K^\alpha_{p\alpha}F^p_i - \frac{1}{2}K^p_{i\alpha}F^\alpha_p, \tag{2.8}$$

is invariant with respect to the mapping f.

Analogously, by using covariant differentiation of the kind θ ($\theta = 2, 3, 4$) we can prove that the tensors $\underset{\theta}{A}_i$, $\theta = 2, 3, 4$ given by

$$\begin{aligned}
\underset{2}{A}_i &= F^\alpha_{i\underset{2}{|}\alpha} + \frac{1}{2}K^\alpha_{\alpha p}F^p_i - \frac{1}{2}K^p_{\alpha i}F^\alpha_p,\\
\underset{3}{A}_i &= F^\alpha_{i\underset{3}{|}\alpha} + \frac{1}{2}K^\alpha_{p\alpha}F^p_i - \frac{1}{2}K^p_{\alpha i}F^\alpha_p,\\
\underset{4}{A}_i &= F^\alpha_{i\underset{4}{|}\alpha} + \frac{1}{2}K^\alpha_{\alpha p}F^p_i - \frac{1}{2}K^p_{i\alpha}F^\alpha_p,
\end{aligned} \tag{2.9}$$

are invariant with respect to the mapping f.

In the nontrivial case, when $F^h_i \neq 0$, which is of particular importance for us, there exists a $(1,1)$ tensor $\overset{*}{F}{}^h_i \neq 0$ such that $\overset{*}{F}{}^\alpha_\beta F^\beta_\alpha = n$. After contracting (2.6) with $\overset{*}{F}{}^j_h$

we find

$$n\varphi_p F_i^p = (F^\alpha_{i\underset{1}{||}\beta} - F^\alpha_{i\underset{1}{|}\beta})\overset{*}{F}{}^\beta_\alpha - K^\alpha_{p\beta}\overset{*}{F}{}^\beta_\alpha F^p_i + K^p_{i\beta}\overset{*}{F}{}^\beta_\alpha F^\alpha_p. \tag{2.10}$$

From (2.4) we have

$$F^h_i = \overline{F}{}^h_i, \tag{2.11}$$

so we can conclude

$$\overset{*}{F}{}^h_i = \overset{*}{\overline{F}}{}^h_i. \tag{2.12}$$

Also, the condition (2.4) ensures the relation

$$K^h_{ij} = \frac{1}{2}(K^h_{ij} - \overline{K}{}^h_{ij}). \tag{2.13}$$

Now, from (2.6) by using (2.10)–(2.13) we obtain

$$\underset{1}{B}{}^h_{ij} = \underset{1}{\overline{B}}{}^h_{ij},$$

where the tensor $\underset{1}{B}{}^h_{ij}$ is defined by

$$\underset{1}{B}{}^h_{ij} = F^h_{i\underset{1}{|}j} - \frac{1}{n}(F^\alpha_{i\underset{1}{|}\beta} + \frac{1}{2}K^\alpha_{\gamma\beta}F^\gamma_i - \frac{1}{2}K^\gamma_{i\beta}F^\alpha_\gamma)\overset{*}{F}{}^\beta_\alpha F^h_j + \frac{1}{2}K^h_{\gamma j}F^\gamma_i - \frac{1}{2}K^\gamma_{ij}F^h_\gamma, \tag{2.14}$$

and the tensor $\underset{1}{\overline{B}}{}^h_{ij}$ is defined by

$$\underset{1}{\overline{B}}{}^h_{ij} = \overline{F}{}^h_{i\underset{1}{||}j} - \frac{1}{n}(\overline{F}{}^\alpha_{i\underset{1}{||}\beta} + \frac{1}{2}\overline{K}{}^\alpha_{\gamma\beta}\overline{F}{}^\gamma_i - \frac{1}{2}\overline{K}{}^\gamma_{i\beta}\overline{F}{}^\alpha_\gamma)\overset{*}{\overline{F}}{}^\beta_\alpha \overline{F}{}^h_j + \frac{1}{2}\overline{K}{}^h_{\gamma j}\overline{F}{}^\gamma_i - \frac{1}{2}\overline{K}{}^\gamma_{ij}\overline{F}{}^h_\gamma.$$

Analogously, we can prove that the tensors $\underset{\theta}{B}{}^h_{ij}$, $\theta = 2,3,4$, defined by

$$\begin{aligned}
\underset{2}{B}{}^h_{ij} &= F^h_{i\underset{2}{|}j} - \frac{1}{n}(F^\alpha_{i\underset{2}{|}\beta} + \frac{1}{2}K^\alpha_{\beta\gamma}F^\gamma_i - \frac{1}{2}K^\gamma_{\beta i}F^\alpha_\gamma)\overset{*}{F}{}^\beta_\alpha F^h_j + \frac{1}{2}K^h_{j\gamma}F^\gamma_i - \frac{1}{2}K^\gamma_{ji}F^h_\gamma,\\
\underset{3}{B}{}^h_{ij} &= F^h_{i\underset{3}{|}j} - \frac{1}{n}(F^\alpha_{i\underset{3}{|}\beta} + \frac{1}{2}K^\alpha_{\gamma\beta}F^\gamma_i - \frac{1}{2}K^\gamma_{\beta i}F^\alpha_\gamma)\overset{*}{F}{}^\beta_\alpha F^h_j + \frac{1}{2}K^h_{\gamma j}F^\gamma_i - \frac{1}{2}K^\gamma_{ji}F^h_\gamma,\\
\underset{4}{B}{}^h_{ij} &= F^h_{i\underset{4}{|}j} - \frac{1}{n}(F^\alpha_{i\underset{4}{|}\beta} + \frac{1}{2}K^\alpha_{\beta\gamma}F^\gamma_i - \frac{1}{2}K^\gamma_{i\beta}F^\alpha_\gamma)\overset{*}{F}{}^\beta_\alpha F^h_j + \frac{1}{2}K^h_{j\gamma}F^\gamma_i - \frac{1}{2}K^\gamma_{ij}F^h_\gamma,
\end{aligned} \tag{2.15}$$

are also invariant with respect to the mapping f.

The previous discussion generalize Theorem 1 from [5] to the case of generalized Riemannian manifolds. Namely, the tensors $\underset{\theta}{A}{}^h_{ij}$, $\theta = 1,\dots,4$, given by (2.8) and (2.9) are generalizations of the tensor $A_i = F^\alpha_{i;\alpha}$, while the tensors $\underset{\theta}{B}{}^h_{ij}$, $\theta = 1,\dots,4$, given by (2.14) and (2.15) are generalizations of the tensor B^h_{ij} given by

$$B^h_{ij} = F^h_{i;j} - \frac{1}{n} F^{\alpha}_{i;\beta} \overset{*}{F}{}^{\beta}_{\alpha} F^h_j, \tag{2.16}$$

where $(;)$ denotes covariant differentiation with respect to the Levi-Civita connection.

When $(;)$ denotes covariant differentiation with respect to the symmetric part ∇ of non-symmetric linear connection ${}^1\nabla$, it is obvious that the tensors $A_i = F^{\alpha}_{i;\alpha}$ and B^h_{ij} are invariant with respect to the mapping f of generalized Riemannian manifolds.

3. Special canonical almost geodesic mappings of generalized parabolic Kähler manifolds

We use Eisenhart's idea of generalized Riemannian spaces to generalize the notion of a parabolic Kähler manifold. Namely, we consider a parabolic Kähler manifold with a non-symmetric metric. M.S. Stanković et al. [20] have already considered similar generalization for classical (elliptic) Kähler manifolds. They assumed that the affinor F is covariantly constant with respect to both of connections ${}^1\nabla$ and ${}^2\nabla$. We use weaker condition, by assuming that the affinor F is covariantly constant with respect to the symmetric part of non-symmetric linear connection ${}^1\nabla$.

Definition 3. A generalized Riemannian manifold M with a metric g is called a *generalized parabolic Kähler manifold* if there exists a $(1,1)$ tensor field F on M such that

$$F^2 = 0, \quad \nabla F = 0, \quad g(X,Y) = \omega g(X, FY), \quad \omega = \pm 1, \text{ for all } X, Y \in T_p(M),$$

where ∇ denotes the Levi-Civita connection corresponding to the symmetric part $\underline{g}$ of metric g.

In what follows we consider only generalized parabolic Kähler manifolds for which $\omega = 1$ in Definition 3. Let M and $\overline{M}$ be two generalized parabolic Kähler manifolds of dimension $n > 2$, with the metrics g and $\overline{g}$, respectively and the affinor structure F. As in the case of usual parabolic Kähler manifolds, the conditions

$$F^2 = 0 \text{ and } \operatorname{Tr}(F) = F^p_p = 0$$

are satisfied.

The non-symmetric linear connection ${}^1\nabla$, defined by (2.1), can be represented as follows

$${}^1\nabla_X Y = \nabla_X Y + \frac{1}{2} {}^1T(X,Y), \tag{3.1}$$

where ∇ denotes the symmetric part of non-symmetric connection ${}^1\nabla$ and ${}^1T$ is the torsion tensor field of connection ${}^1\nabla$.

For an anti-symmetric tensor field K given by

$$K(X,Y) = \frac{1}{2}\big({}^1\overline{T}(X,Y) - {}^1T(X,Y)\big), \tag{3.2}$$

according to (3.1) we have

$$\begin{aligned}{}^1\nabla_Y FX + K(Y,FX) &= \nabla_Y FX + \frac{1}{2}{}^1T(Y,FX) + \frac{1}{2}{}^1\overline{T}(Y,FX) - \frac{1}{2}{}^1T(Y,FX)\\ &= \nabla_Y FX + \frac{1}{2}{}^1\overline{T}(Y,FX).\end{aligned}$$

Analogously, we can prove the relation

$${}^2\nabla_Y FX - K(Y,FX) = \nabla_Y FX + \frac{1}{2}{}^2\overline{T}(Y,FX).$$

Therefore the basic equations (2.2) and (2.3) in the case of canonical almost geodesic mappings of type $\underset{\theta}{\pi}_2(0,F)$, $\theta \in \{1,2\}$ (with a priori defined affinor F) between generalized parabolic Kähler manifolds have the following form

$$\begin{aligned}{}^\theta P(X,Y) &= \sum_{CS(X,Y)} \varphi(X)FY + (-1)^{(\theta-1)}K(X,Y),\\ \frac{1}{2}\sum_{CS(X,Y)} {}^\theta\overline{T}(Y,FX) &= \sum_{CS(X,Y)} \big(\mu(X)FY - \mu(FX)Y\big),\end{aligned}$$

where $X,Y \in \mathcal{X}(M)$, φ is a 1-form and K is the anti-symmetric tensor field of type $(1,2)$ given by (3.2).

It is well known that the affinor structure F is locally integrable if and only if on a manifold exists a symmetric linear connection ∇ such that $\nabla F = 0$. Therefore, the affinor structure F of a generalized parabolic Kähler manifold is locally integrable. This fact enables us to consider another affinor structure $\overset{*}{F}$ such that [5]

$$F^h_\alpha \overset{*}{F}{}^\alpha_i + \overset{*}{F}{}^h_\alpha F^\alpha_i = \delta^h_i \tag{3.3}$$

holds on each local chart U of a generalized parabolic Kähler manifold.

In [5] it was proved that the geometric object

$$\Gamma^h_{ij} - \frac{1}{n+1}F^h_{(i}\Gamma^\alpha_{j)\beta}\overset{*}{F}{}^\beta_\alpha \tag{3.4}$$

is invariant with respect to the canonical almost geodesic mapping of type $\pi_2(e=0)$ between parabolic Kähler manifolds. In what follows we give some generalizations of the geometric object given by (3.4), to the case of a canonical almost geodesic mapping of type $\underset{\theta}{\pi}_2(0,F)$, $\theta \in \{1,2\}$ between generalized parabolic Kähler manifolds.

Theorem 1. *Let $f : M \to \overline{M}$ be a canonical almost geodesic mapping of type $\underset{\theta}{\pi}_2(0,F)$, $\theta \in \{1,2\}$ between generalized parabolic Kähler manifolds M and $\overline{M}$.*

Then the geometric objects $\underset{\theta}{C}{}^h_{ij}$, $\theta = 1,\dots,4$, *given by*

$$\underset{1}{C}{}^h_{ij} = \Gamma^h_{ij} - \Big[\frac{1}{n+1}\Big(\Gamma^p_{iq}\overset{*}{F}{}^q_p + \frac{1}{n}[F^\alpha_{p\underset{1}{|}\beta}\overset{*}{F}{}^\beta_\alpha + \frac{1}{2}K^\alpha_{\gamma\beta}\overset{*}{F}{}^\beta_\alpha F^\gamma_p - \frac{1}{2}K^\gamma_{p\beta}F^\alpha_\gamma\overset{*}{F}{}^\beta_\alpha]\overset{*}{F}{}^p_i + \frac{1}{2}K^p_{iq}\overset{*}{F}{}^q_p\Big)F^h_j\Big]_{(ij)} + \frac{1}{2}K^h_{ij}, \tag{3.5}$$

$$\underset{2}{C}{}^h_{ij} = \Gamma^h_{ij} - \Big[\frac{1}{n+1}\Big(\Gamma^p_{iq}\overset{*}{F}{}^q_p + \frac{1}{n}[F^\alpha_{p\underset{2}{|}\beta}\overset{*}{F}{}^\beta_\alpha + \frac{1}{2}K^\alpha_{\beta\gamma}\overset{*}{F}{}^\beta_\alpha F^\gamma_p - \frac{1}{2}K^\gamma_{\beta p}F^\alpha_\gamma\overset{*}{F}{}^\beta_\alpha]\overset{*}{F}{}^p_i + \frac{1}{2}K^p_{iq}\overset{*}{F}{}^q_p\Big)F^h_j\Big]_{(ij)} + \frac{1}{2}K^h_{ij}, \tag{3.6}$$

$$\underset{3}{C}{}^h_{ij} = \Gamma^h_{ij} - \Big[\frac{1}{n+1}\Big(\Gamma^p_{iq}\overset{*}{F}{}^q_p + \frac{1}{n}[F^\alpha_{p\underset{3}{|}\beta}\overset{*}{F}{}^\beta_\alpha + \frac{1}{2}K^\alpha_{\gamma\beta}\overset{*}{F}{}^\beta_\alpha F^\gamma_p - \frac{1}{2}K^\gamma_{\beta p}F^\alpha_\gamma\overset{*}{F}{}^\beta_\alpha]\overset{*}{F}{}^p_i + \frac{1}{2}K^p_{iq}\overset{*}{F}{}^q_p\Big)F^h_j\Big]_{(ij)} + \frac{1}{2}K^h_{ij}, \tag{3.7}$$

$$\underset{4}{C}{}^h_{ij} = \Gamma^h_{ij} - \Big[\frac{1}{n+1}\Big(\Gamma^p_{iq}\overset{*}{F}{}^q_p + \frac{1}{n}[F^\alpha_{p\underset{4}{|}\beta}\overset{*}{F}{}^\beta_\alpha + \frac{1}{2}K^\alpha_{\beta\gamma}\overset{*}{F}{}^\beta_\alpha F^\gamma_p - \frac{1}{2}K^\gamma_{p\beta}F^\alpha_\gamma\overset{*}{F}{}^\beta_\alpha]\overset{*}{F}{}^p_i + \frac{1}{2}K^p_{iq}\overset{*}{F}{}^q_p\Big)F^h_j\Big]_{(ij)} + \frac{1}{2}K^h_{ij}, \tag{3.8}$$

are invariant with respect to the mapping f.

Proof. Contracting the basic equation (2.5) with $\overset{*}{F}{}^j_h$ we obtain

$$\begin{aligned}(\overline{\Gamma}{}^p_{iq} - \Gamma^p_{iq})\overset{*}{F}{}^q_p &= \varphi_i F^p_q\overset{*}{F}{}^q_p + \varphi_q\overset{*}{F}{}^q_p F^p_i + K^p_{iq}\overset{*}{F}{}^q_p\\ &= n\varphi_i + \varphi_q(\overset{*}{F}{}^q_p F^p_i + F^q_p\overset{*}{F}{}^p_i - F^q_p\overset{*}{F}{}^p_i) + K^p_{iq}\overset{*}{F}{}^q_p\\ &\overset{(3.3)}{=} n\varphi_i + \varphi_q\delta^q_i - \varphi_q F^q_p\overset{*}{F}{}^p_i + K^p_{iq}\overset{*}{F}{}^q_p.\end{aligned}$$

Therefore,

$$\begin{aligned}(n+1)\varphi_i &= (\overline{\Gamma}{}^p_{iq} - \Gamma^p_{iq})\overset{*}{F}{}^q_p + \varphi_q F^q_p\overset{*}{F}{}^p_i - K^p_{iq}\overset{*}{F}{}^q_p\\ &\overset{(3.3)}{=} (\overline{\Gamma}{}^p_{iq} - \Gamma^p_{iq})\overset{*}{F}{}^q_p + \frac{1}{n}\Big[(F^\alpha_{p\underset{1}{||}\beta} - F^\alpha_{p\underset{1}{|}\beta})\overset{*}{F}{}^\beta_\alpha - K^\alpha_{\gamma\beta}\overset{*}{F}{}^\beta_\alpha F^\gamma_p\\ &\quad + K^\gamma_{p\beta}F^\alpha_\gamma\overset{*}{F}{}^\beta_\alpha\Big]\overset{*}{F}{}^p_i - K^p_{iq}\overset{*}{F}{}^q_p.\end{aligned} \tag{3.9}$$

Now, after changing (3.9) into the basic equation (2.5), we get

$$\overline{\Gamma}^h_{ij}=\Gamma^h_{ij}+\frac{1}{n+1}\Big[\Big((\overline{\Gamma}^p_{iq}-\Gamma^p_{iq})\overset{*}{F}{}^q_p+\frac{1}{n}\big[(F^\alpha_{p\underset{1}{\|}\beta}-F^\alpha_{p\underset{1}{|}\beta})\overset{*}{F}{}^\beta_\alpha-K^\alpha_{\gamma\beta}\overset{*}{F}{}^\beta_\alpha F^\gamma_p$$
$$+K^\gamma_{p\beta}F^\alpha_\gamma\overset{*}{F}{}^\beta_\alpha\big]\overset{*}{F}{}^p_i-K^p_{iq}\overset{*}{F}{}^q_p\Big)F^h_j\Big]_{(ij)}+K^h_{ij}.$$

From the previous equation, by using (2.10)–(2.13), we obtain the following relation

$$\overline{\Gamma}^h_{ij}-\Big[\frac{1}{n+1}\Big(\overline{\Gamma}^p_{iq}\overset{*}{\overline{F}}{}^q_p+\frac{1}{n}\big[\overline{F}^\alpha_{p\underset{1}{\|}\beta}\overset{*}{\overline{F}}{}^\beta_\alpha+\frac{1}{2}\overline{K}^\alpha_{\gamma\beta}\overset{*}{\overline{F}}{}^\beta_\alpha\overline{F}^\gamma_p-\frac{1}{2}\overline{K}^\gamma_{p\beta}\overline{F}^\alpha_\gamma\overset{*}{\overline{F}}{}^\beta_\alpha\big]\overset{*}{\overline{F}}{}^p_i$$
$$+\frac{1}{2}\overline{K}^p_{iq}\overset{*}{\overline{F}}{}^q_p\Big)\overline{F}^h_j\Big]_{(ij)}+\frac{1}{2}\overline{K}^h_{ij}$$
$$=\Gamma^h_{ij}-\Big[\frac{1}{n+1}\Big(\Gamma^p_{iq}\overset{*}{F}{}^q_p+\frac{1}{n}\big[F^\alpha_{p\underset{1}{|}\beta}\overset{*}{F}{}^\beta_\alpha+\frac{1}{2}K^\alpha_{\gamma\beta}\overset{*}{F}{}^\beta_\alpha F^\gamma_p-\frac{1}{2}K^\gamma_{p\beta}F^\alpha_\gamma\overset{*}{F}{}^\beta_\alpha\big]\overset{*}{F}{}^p_i$$
$$+\frac{1}{2}K^p_{iq}\overset{*}{F}{}^q_p\Big)F^h_j\Big]_{(ij)}+\frac{1}{2}K^h_{ij},$$

which proves that the geometric object $\underset{1}{C}{}^h_{ij}$ defined by (3.5) is invariant with respect to the mapping f.

In a similar manner one can conclude that the geometric objects $\underset{\theta}{C}{}^h_{ij}$, $\theta=2,3,4$, determined by (3.6)–(3.8) are invariant with respect to the mapping f. □

When we consider a mapping between two affine connected manifolds with torsion, we can consider the so called equitorsion mapping, it is a mapping which preserves the torsion tensor.

Definition 4 ([16, 22])**.** An almost geodesic mapping $f:M\to\overline{M}$ of affine connected manifolds M and $\overline{M}$ with the torsion tensors T^h_{ij} and $\overline{T}^h_{ij}$, respectively, is an *equitorsion almost geodesic mapping* if the following condition holds

$$T^h_{ij}=\overline{T}^h_{ij}.$$

Equation (3.2) in local coordinates reads $K^h_{ij}=\frac{1}{2}(\overline{T}^h_{ij}-T^h_{ij})$. Therefore the geometric objects $\underset{\theta}{C}{}^h_{ij}$, $\theta=1,\dots,4$, given by (3.5)–(3.8), with respect to an equitorsion canonical almost geodesic mapping of type $\underset{\theta}{\pi}2(0,F)$, $\theta\in\{1,2\}$ between generalized parabolic Kähler manifolds take the following forms

$$\underset{\theta}{C}{}^h_{ij}=\Gamma^h_{ij}-\Big[\frac{1}{n+1}\Big(\Gamma^p_{iq}\overset{*}{F}{}^q_p+\frac{1}{n}F^\alpha_{p\underset{\theta}{|}\beta}\overset{*}{F}{}^\beta_\alpha\overset{*}{F}{}^p_i\Big)\Big]_{(ij)},\quad\theta=1,\dots,4. \tag{3.10}$$

Note that the geometric objects given by (3.5)–(3.8) and (3.10) are not tensors, since the generalized Christoffel symbols Γ^h_{ij} are not tensors (see [14], p. 10).

The geometric object

$$C^h_{ij} = \Gamma^h_{\underline{ij}} - \frac{1}{n+1} F^h_{(i} \Gamma^\alpha_{j)\beta} \overset{*}{F}{}^\beta_\alpha, \tag{3.11}$$

where $\Gamma^h_{\underline{ij}}$ is the symmetric part of Γ^h_{ij}, is evidently invariant with respect to the canonical almost geodesic mapping of type $\underset{\theta}{\pi}_2(0, F)$, $\theta \in \{1,2\}$ between generalized parabolic Kähler manifolds. This geometric object is a tensor as well as the geometric object given by (3.4).

Remark 1. The geometric objects, given by (3.5)–(3.8), (3.10) and (3.11) are generalizations of the tensor, given by (3.4).

4. Conclusion

Invariant geometric objects of canonical almost geodesic mappings of type $\underset{\theta}{\pi}_2(0, F)$, $\theta \in \{1,2\}$ are examined. Since the available literature does not contain any results about invariants of almost geodesic mappings of type $\underset{\theta}{\pi}_2(e)$, $\theta \in \{1,2\}$ for $e = 0$, this paper somewise fills the gap in the theory of almost geodesic mappings of manifolds with non-symmetric affine connection.

A generalized parabolic Kähler manifold is introduced and some results concerning invariant geometric objects of canonical almost geodesic mappings of type $\pi_2(e = 0)$, between parabolic Kähler manifolds are extended. This fact opens up possibilities for further extension of results from usual parabolic Kähler manifolds to generalized parabolic Kähler manifolds.

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