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NEW INTEGRABILITY CONDITIONS OF DERIVATION EQUATIONS IN A SUBSPACE OF ASYMMETRIC AFFINE CONNECTION SPACE

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Abstract. In the work [10] we obtained derivational equations of a submanifold of a space L_N with asymmetric affine connection. Based on asymmetry of the connection we define four kinds of covariant derivative and obtain four kinds of derivational equations.

In [20] are examined integrability conditions of derivational equations, using the 1^{st} and the 2^{nd} kind of derivative, and in the present work we do it on the base of the 3^{rd} and 4^{th} kind.

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1. Introduction

In 1922 Cartan was put forward a modification of General Relativity Theory (GRT), by relaxing the assumption that the affine connection has vanishing the antisymmetric part (torsion tensor), and relating the torsion to the density of intrinsic angular momentum. Also, the torsion is implicit in the 1928 Einstein theory of gravitation with teleparallelism.

From 1923 to the end of his life Einstein worked on various variants of Unified Field Theory (Non-symmetric Gravitational Theory-NGT) [3]. This theory had the aim to unite gravitation theory and the theory of electromagnetism. Introducing different variants of his NGT, Einstein used a complex basic tensor, with a symmetric real part and a skew-symmetric imaginary part. Starting from 1950, Einstein used the real non-symmetric basic tensor g, sometimes called *generalized Riemannian metric/manifold*.

Notice that in NGT the symmetric part $g_{\underline{ij}}$ of the basic tensor g_{ij} $(g_{ij} = g_{\underline{ij}} + g_{\underset{\vee}{ij}})$ is related to gravitation, and the skew-symmetric one $g_{\underset{\vee}{ij}}$ to electromagnetism.

While on a Riemannian space the connection coefficients are expressed by virtue of the metric, g_{ij}, in Einstein's work on NGT the connection between these magnitudes is determined by the so-called *Einstein metricity condition,* i.e. the non-symmetric metric tensor g and the connection components L^k_{ij} are connected with the equations

$$\frac{\partial g_{ij}}{\partial x^m} - L^p_{im} g_{pj} - L^p_{mj} g_{ip} = 0. \tag{1.1}$$

The choice of a connection in NGT is not uniquely determined. In particular, in NGT there exist two kinds of the covariant derivative. For example, for tensor a^i_j :

$$a^{\overset{i}{+}}_{\underset{+}{j}|m} = \frac{\partial a^i_j}{\partial x^m} + L^i_{pm} a^p_j - L^p_{jm} a^i_p; \quad a^{\overset{i}{-}}_{\underset{-}{j}|m} = \frac{\partial a^i_j}{\partial x^m} + L^i_{mp} a^p_j - L^p_{mj} a^i_p,$$

where the lowering and the rising of indices one defines by equations

$$g_{pi} g^{pj} = g_{ip} g^{jp} = \delta^j_i, \tag{1.2}$$

Einstein considered only one curvature tensor:

$$R^i_{klm} = L^i_{kl,m} - L^i_{km,l} - L^i_{sl} L^s_{km} + L^i_{sm} L^s_{kl} \tag{1.3}$$

and proved Bianchi type identity for covariant curvature tensor (see [2])

$$R_{\underset{-+-+}{iklm}|n} + R_{\underset{-+++}{ikmn}|l} + R_{\underset{-+--}{iknl}|m} = 0,$$

where is $R_{iklm} = g_{si} R^s{}_{klm}$.

Afterwards, several mathematicians dealt with non-symmetric affine connection, for example, Eisenhart [4], Prvanović [13], Minčić [9–12], Stanković [16, 17] etc. Sinyukov [14] introduced the concept of almost geodesic mappings between affine connected spaces without torsion. Mikeš [1], [5–8, 13], [15], [18] gave some significant contributions to the study of geodesic and almost geodesic mappings of affine connected, Riemannian and Einstein spaces.

Let L_N be a space with asymmetric affine connection L^i_{jk} (in local coordinates), and torsion tensor T^i_{jk}.

In L_N one can define four kinds of covariant derivatives. For example, for a tensor a^i_j we have

$$a^i_{j\underset{1}{|}m} = a^i_{j,m} + L^i_{pm} a^p_j - L^p_{jm} a^i_p, \quad a^i_{j\underset{2}{|}m} = a^i_{j,m} + L^i_{mp} a^p_j - L^p_{mj} a^i_p,$$
$$a^i_{j\underset{3}{|}m} = a^i_{j,m} + L^i_{pm} a^p_j - L^p_{mj} a^i_p, \quad a^i_{j\underset{4}{|}m} = a^i_{j,m} + L^i_{mp} a^p_j - L^p_{jm} a^i_p.$$

In [6] is proved very important theorem

Theorem 1. *Let* $(L_N, g_{ij} = g_{\underline{ij}} + g_{\underset{\vee}{ij}})$ *be an asymmetric affine connection space and* Γ^i_{jk} *be the Levi-Civita connection of* $g_{\underline{ij}}$. *Let* L^i_{jk} *be a linear connection with torsion* T^i_{jk}. *Then* L^i_{jk} *is uniquely determined by the following formula*

$$L_{i.jk} = \mathit{\Gamma}_{i.jk} + \frac{1}{2}\Big(T_{i.jk} + T_{k.ij} - T_{j.ki}\Big) - \frac{1}{2}\Big(g_{\underline{jk}\underset{1}{|}i} + g_{\underline{ki}\underset{1}{|}j} - g_{\underline{ji}\underset{1}{|}k}\Big), \quad (1.4)$$

where $L_{i.jk} = g_{\underline{pi}} L^p_{jk}$.

A submanifold $X_M \subset L_N$ is defined by equations

$$x^i = x^i(u^1, \dots, u^M) = x^i(u^\alpha), \quad i = \overline{1, N}.$$

Partial derivatives $B^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}$ $(\mathrm{rank}(B^i_\alpha) = M)$ define tangent vectors on X_M.

Consider $N - M$ contravariant vectors C^i_A $(A, B, C, \dots, \in \{M+1, \dots, N\})$ defined on X_M and linearly independent, and let the matrix $\binom{B^\alpha_i}{C^A_i}$ be inverse for the matrix (B^i_α, C^i_A) provided that the following conditions are satisfied [19, 20]:

$$a)\ B^i_\alpha B^\beta_i = \delta^\beta_\alpha; \quad b)\ B^i_\alpha C^A_i = 0; \quad c)\ B^\alpha_i C^i_A = 0;$$
$$d)\ C^i_A C^B_i = \delta^B_A; \quad e)\ B^i_\alpha B^\alpha_j + C^i_A C^A_j = \delta^i_j;$$

The magnitudes B^i_α, B^α_i are **projection factors (tangent vectors)**, and the magnitudes C^i_A, C^A_i are **affine pseudonormals** [10, 20].

The **induced connection** on X_M is [10, 19, 20]:

$$\widetilde{L}^\alpha_{\beta\gamma} = B^\alpha_i (B^i_{\beta,\gamma} + L^i_{jk} B^j_\beta B^k_\gamma),$$

where $B^i_{\beta,\gamma} = \partial B^i_\beta / \partial u^\gamma = \partial^2 x^i / \partial u^\beta \partial u^\gamma$. Because L is asymmetric by virtue of j, k, $\widetilde{L}$ is asymmetric in β, γ too. The submanifold X_M endowed with $\widetilde{L}$ becomes L_M and we write $L_M \subset L_N$.

The set of pseudonormals of the submanifold $X_M \subset L_N$ makes **a pseudonormal bundle** of X_M, and we note it X^N_{N-M}. We have defined in [10] **induced connections of pseudonormal bundle** with coefficients

$$\underset{1}{\overline{L}}{}^A_{B\mu} = C^A_i (C^i_{B,\mu} + L^i_{jk} C^j_B B^k_\mu), \quad \underset{2}{\overline{L}}{}^A_{B\mu} = C^A_i (C^i_{B,\mu} + L^i_{kj} C^j_B B^k_\mu).$$

As the coefficients $L, \widetilde{L}, \overline{L}$ are generally asymmetric, we can define four kinds of covariant derivative for a tensor, defined in the points of X_M. For example:

$$t^{i\alpha A}_{j\beta B\underset{1}{|}\mu} = t^{i\alpha A}_{j\beta B,\mu} + L^i_{pm} t^{p\alpha A}_{j\beta B} - L^p_{jm} t^{i\alpha A}_{p\beta B} + \widetilde{L}^\alpha_{\pi\mu} t^{i\pi A}_{j\beta B} - \widetilde{L}^\pi_{\beta\mu} t^{i\alpha A}_{j\pi B} + \underset{1}{\overline{L}}{}^A_{P\mu} t^{i\alpha P}_{j\beta B} - \underset{1}{\overline{L}}{}^P_{\beta\mu} t^{i\alpha A}_{j\beta P},$$
$$t^{i\alpha A}_{j\beta B\underset{2}{|}\mu} = t^{i\alpha A}_{j\beta B,\mu} + L^i_{mp} t^{p\alpha A}_{j\beta B} - L^p_{mj} t^{i\alpha A}_{p\beta B} + \widetilde{L}^\alpha_{\mu\pi} t^{i\pi A}_{j\beta B} - \widetilde{L}^\pi_{\mu\beta} t^{i\alpha A}_{j\pi B} + \underset{2}{\overline{L}}{}^A_{P\mu} t^{i\alpha P}_{j\beta B} - \underset{2}{\overline{L}}{}^P_{\beta\mu} t^{i\alpha A}_{j\beta P}$$
$$t^{i\alpha A}_{j\beta B\underset{3}{|}\mu} = t^{i\alpha A}_{j\beta B,\mu} + L^i_{pm} t^{p\alpha A}_{j\beta B} - L^p_{mj} t^{i\alpha A}_{p\beta B} + \widetilde{L}^\alpha_{\pi\mu} t^{i\pi A}_{j\beta B} - \widetilde{L}^\pi_{\mu\beta} t^{i\alpha A}_{j\pi B} + \underset{1}{\overline{L}}{}^A_{P\mu} t^{i\alpha P}_{j\beta B} - \underset{2}{\overline{L}}{}^P_{\beta\mu} t^{i\alpha A}_{j\beta P}$$
$$t^{i\alpha A}_{j\beta B\underset{4}{|}\mu} = t^{i\alpha A}_{j\beta B,\mu} + L^i_{mp} t^{p\alpha A}_{j\beta B} - L^p_{jm} t^{i\alpha A}_{p\beta B} + \widetilde{L}^\alpha_{\mu\pi} t^{i\pi A}_{j\beta B} - \widetilde{L}^\pi_{\beta\mu} t^{i\alpha A}_{j\pi B} + \underset{2}{\overline{L}}{}^A_{P\mu} t^{i\alpha P}_{j\beta B} - \underset{1}{\overline{L}}{}^P_{\beta\mu} t^{i\alpha A}_{j\beta P}$$

In this manner four connections $\underset{\theta}{\nabla}$ on $X_M \subset L_N$ are defined. We shall note the obtained structures $(X_M \subset L_N, \underset{\theta}{\nabla}, \theta \in \{1,\dots,4\})$.

2. New integrability conditions of derivational equations for tangents

2.0. We have obtained in [20] integrability conditions of derivational equations and corresponding Gauss-Codazzi equations in the structure $(X_M \subset L_N, \underset{\theta}{\nabla}, \theta \in \{1,2\})$. In the present work we are solving this problem for $\theta \in \{3,4\}$. Based on the Theorem 2.2. in [10], the following derivational equations for tangents are in force

$$B^i_{\alpha\underset{\theta}{|}\mu} = \underset{\theta}{\Omega}{}^P_{\alpha\mu} C^i_P, \quad \theta \in \{3,4\} \tag{2.1}$$

$$B^\alpha_{i\underset{\theta}{|}\mu} = \underset{\theta}{\widehat{\Omega}}{}^\alpha_{P\mu} C^P_i, \quad \theta \in \{3,4\}, \tag{2.2}$$

and, (see [20]), for pseudonormals

$$C^i_{A\underset{\theta}{|}\mu} = -\underset{\theta}{\widehat{\Omega}}{}^\pi_{A\mu} B^i_\pi, \quad \theta \in \{3,4\} \tag{2.3}$$

$$C^A_{i\underset{\theta}{|}\mu} = -\underset{\theta}{\Omega}{}^A_{\pi\mu} B^\pi_i, \quad \theta \in \{3,4\}. \tag{2.4}$$

In this manner, one obtains

$$B^i_{\alpha\underset{\theta}{|}\mu\underset{\omega}{|}\nu} - B^i_{\alpha\underset{\omega}{|}\nu\underset{\theta}{|}\mu} = (\underset{\theta}{\widehat{\Omega}}{}^\pi_{P\mu}\underset{\omega}{\Omega}{}^P_{\alpha\nu} - \underset{\omega}{\widehat{\Omega}}{}^\pi_{P\nu}\underset{\theta}{\Omega}{}^P_{\alpha\mu})B^i_\pi + (\underset{\theta}{\Omega}{}^P_{\alpha\mu\underset{\omega}{|}\nu} - \underset{\theta}{\Omega}{}^P_{\alpha\nu\underset{\omega}{|}\mu})C^i_P,$$
$$\theta,\omega \in \{3,4\}. \tag{2.5}$$

and analogously

$$B^\alpha_{i\underset{\theta}{|}\mu\underset{\omega}{|}\nu} - B^\alpha_{i\underset{\omega}{|}\nu\underset{\theta}{|}\mu} = (\underset{\theta}{\Omega}{}^P_{\pi\mu}\underset{\omega}{\widehat{\Omega}}{}^\alpha_{P\nu} - \underset{\omega}{\Omega}{}^P_{\pi\nu}\underset{\theta}{\widehat{\Omega}}{}^\alpha_{P\mu})B^\pi_i + (\underset{\theta}{\widehat{\Omega}}{}^\alpha_{P\mu\underset{\omega}{|}\nu} - \underset{\theta}{\widehat{\Omega}}{}^\alpha_{P\nu\underset{\omega}{|}\mu})C^P_i,$$
$$\theta \in \{3,4\}.$$

where [10]

$$\begin{aligned} \underset{3}{\Omega}{}^P_{\alpha\mu} &= C^P_i(B^i_{\alpha,\mu} + L^i_{pm}B^p_\alpha B^m_\mu) = \underset{1}{\Omega}{}^P_{\alpha\mu}, \\ \underset{4}{\Omega}{}^P_{\alpha\mu} &= C^P_i(B^i_{\alpha,\mu} + L^i_{mp}B^p_\alpha B^m_\mu) = \underset{2}{\Omega}{}^P_{\alpha\mu}, \end{aligned} \tag{2.6}$$

$$\underset{3}{\widehat{\Omega}}{}^\alpha_{P\mu} = C^i_P(B^\alpha_{i,\mu} + L^p_{mi}B^\alpha_p B^m_\mu) = \underset{2}{\widehat{\Omega}}{}^\alpha_{P\mu},\ \underset{4}{\widehat{\Omega}}{}^\alpha_{P\mu} = C^i_P(B^\alpha_{i,\mu} + L^p_{im}B^\alpha_p B^m_\mu) = \underset{1}{\widehat{\Omega}}{}^\alpha_{P\mu}.$$

From the Theorem 2.2. in [10] the induced connection $\widetilde{L}$ in the structure $(X_M \subset L_N, \underset{\theta}{\nabla}, \theta \in \{3,4\})$ is symmetric, i.e.

$$\widetilde{T}^\alpha_{\beta\gamma} = 0, \tag{2.7}$$

and based on the Theorem 3.2. in this case is

$$\underset{1}{\overline{L}} = \underset{2}{\overline{L}} = \overline{L}, \tag{2.8}$$

that is there exists an unique connection in the pseudonormal bundle.

2.1. Using the Ricci-type identities (12,13) in [9], by virtue of (2.7), we obtain

$$B^i_{\alpha\underset{\theta}{|}\mu\underset{\theta}{|}\nu} - B^i_{\alpha\underset{\theta}{|}\nu\underset{\theta}{|}\mu} = \underset{\theta-2}{R}{}^i{}_{pmn} B^p_\alpha B^m_\mu B^n_\nu - \widetilde{R}^\pi_{\alpha\mu\nu} B^i_\pi + (-1)^\theta \widetilde{T}^\pi_{\mu\nu} B^i_{\alpha\underset{\theta}{|}\pi}, \quad \theta \in \{3,4\}. \tag{2.9}$$

where

$$\begin{aligned} &a)\ \underset{1}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{jn,m} + L^p_{jm} L^i_{pn} - L^p_{jn} L^i_{pm}, \\ &b)\ \underset{2}{R}{}^i_{jmn} = L^i_{mj,n} - L^i_{nj,m} + L^p_{mj} L^i_{np} - L^p_{nj} L^i_{mp}, \end{aligned} \tag{2.10}$$

are **curvature tensors of the** 1^{st} **respectively the** 2^{nd} **kind** of the L_N and $\widetilde{R}^\alpha_{\beta\mu\nu}$ is, with respect of (2.8), curvature tensor of $L^o_M \subset L_N$, where L^o_M is a subspace with symmetric affine connection.

Further, we examine integrability conditions for derivational equations of B^i_α and B^α_i, i.e. for $B^i_{\alpha\underset{\theta}{|}\mu}$, $B^\alpha_{i\underset{\theta}{|}\mu}$, $\theta \in \{3,4\}$.

Substituting $\theta = \omega \in \{3,4\}$ into (2.5) and comparing with (2.9), taking into consideration (2.7), we get **the** 1^{st} **and the** 2^{nd} **kind integrability condition of derivational equation** (2.1) in the structure $(X_M \subset L_N, \underset{\theta}{\nabla}, \theta \in \{3,4\})$:

$$\underset{\theta-2}{R}{}^i{}_{pmn} B^p_\alpha B^m_\mu B^n_\nu - \widetilde{R}^\pi_{\alpha\mu\nu} B^i_\pi = (\underset{\theta}{\widehat{\Omega}}{}^\pi_{P\mu} \underset{\theta}{\Omega}{}^P_{\alpha\nu} - \underset{\theta}{\widehat{\Omega}}{}^\pi_{P\nu} \underset{\theta}{\Omega}{}^P_{\alpha\mu}) B^i_\pi + (\underset{\theta}{\Omega}{}^P_{\alpha\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Omega}{}^P_{\alpha\nu\underset{\theta}{|}\mu}) C^i_P,$$
$$\theta \in \{3,4\}. \tag{2.11}$$

Multiplying this equation equation with B^λ_i and taking into consideration (1.2), we obtain

$$\underset{\theta-2}{R}{}^i{}_{pmn} B^\lambda_i B^p_\alpha B^m_\mu B^n_\nu - \widetilde{R}^\lambda_{\alpha\mu\nu} = \underset{\theta}{\widehat{\Omega}}{}^\lambda_{P\mu} \underset{\theta}{\Omega}{}^P_{\alpha\nu} - \underset{\theta}{\widehat{\Omega}}{}^\lambda_{P\nu} \underset{\theta}{\Omega}{}^P_{\alpha\mu}, \quad \theta \in \{3,4\}. \tag{2.12}$$

i.e the **Gauss equation of the** 1^{st} **and the** 2^{nd} **kind** in the structure $(X_M \subset L_N, \underset{\theta}{\nabla}, \theta \in \{3,4\})$.

If one multiplies (2.11) with C^L_i, it follows that

$$\underset{\theta-2}{R}{}^i{}_{pmn} C^L_i B^p_\alpha B^m_\mu B^n_\nu = \underset{\theta}{\Omega}{}^L_{\alpha\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Omega}{}^L_{\alpha\nu\underset{\theta}{|}\mu}, \quad \theta \in \{3,4\}. \tag{2.13}$$

which are **the** 1^{st} **Codazzi equation of the** 1^{st} **and the** 2^{nd} **kind** for the cited structure.

2.1′ Further, if we use the Ricci type identities ([9], equation (12))

$$\begin{aligned} B^{\alpha}_{i\underset{3}{|}\mu\underset{3}{|}\nu} - B^{\alpha}_{i\underset{4}{|}\nu\underset{4}{|}\mu} &= -\underset{2}{R}{}^{p}_{imn} B^{\alpha}_{p} B^{m}_{\mu} B^{n}_{\nu} + \widetilde{R}^{\alpha}_{\pi\mu\nu} B^{\pi}_{i}, \\ B^{\alpha}_{i\underset{4}{|}\mu\underset{4}{|}\nu} - B^{\alpha}_{i\underset{4}{|}\nu\underset{4}{|}\mu} &= -\underset{1}{R}{}^{p}_{imn} B^{\alpha}_{p} B^{m}_{\mu} B^{n}_{\nu} + \widetilde{R}^{\alpha}_{\pi\mu\nu} B^{\pi}_{i}, \end{aligned} \tag{2.14}$$

and (2.3′) for $\theta = \omega \in \{3,4\}$, we have

$$\begin{aligned} -\underset{2}{R}{}^{p}_{imn} B^{\alpha}_{p} B^{m}_{\mu} B^{n}_{\nu} + \widetilde{R}^{\alpha}_{\pi\mu\nu} B^{\pi}_{i} &= (\underset{3}{\Omega}{}^{P}_{\pi\mu} \underset{3}{\widehat{\Omega}}{}^{\alpha}_{P\nu} - \underset{\theta}{\Omega}{}^{P}_{\pi\nu} \underset{\theta}{\widehat{\Omega}}{}^{\alpha}_{P\mu}) B^{\pi}_{i} + (\underset{3}{\widehat{\Omega}}{}^{\alpha}_{P\mu\underset{3}{|}\nu} - \underset{3}{\widehat{\Omega}}{}^{\alpha}_{P\nu\underset{3}{|}\mu}) C^{P}_{i}, \\ -\underset{1}{R}{}^{p}_{imn} B^{\alpha}_{p} B^{m}_{\mu} B^{n}_{\nu} + \widetilde{R}^{\alpha}_{\pi\mu\nu} B^{\pi}_{i} &= (\underset{4}{\Omega}{}^{P}_{\pi\mu} \underset{4}{\widehat{\Omega}}{}^{\alpha}_{P\nu} - \underset{\theta}{\Omega}{}^{P}_{\pi\nu} \underset{\theta}{\widehat{\Omega}}{}^{\alpha}_{P\mu}) B^{\pi}_{i} + (\underset{4}{\widehat{\Omega}}{}^{\alpha}_{P\mu\underset{4}{|}\nu} - \underset{4}{\widehat{\Omega}}{}^{\alpha}_{P\nu\underset{4}{|}\mu}) C^{P}_{i} \end{aligned} \tag{2.15}$$

which are **the 1^{st} and the 2^{nd} kind integrability conditions of derivational equation** (2.2) in the structure $(X_M \subset L_N, \underset{\theta}{\nabla}, \theta \in \{3,4\})$.

a') By multiplying the previous equation with B^{i}_{λ}, one obtains

$$\begin{aligned} \widetilde{R}^{\alpha}_{\lambda\mu\nu} - \underset{2}{R}{}^{p}_{imn} B^{\alpha}_{p} B^{i}_{\lambda} B^{m}_{\mu} B^{n}_{\nu} &= \underset{3}{\Omega}{}^{P}_{\lambda\mu} \underset{3}{\widehat{\Omega}}{}^{\alpha}_{P\nu} - \underset{3}{\Omega}{}^{P}_{\lambda\nu} \underset{3}{\widehat{\Omega}}{}^{\alpha}_{P\mu}, \\ \widetilde{R}^{\alpha}_{\lambda\mu\nu} - \underset{1}{R}{}^{p}_{imn} B^{\alpha}_{p} B^{i}_{\lambda} B^{m}_{\mu} B^{n}_{\nu} &= \underset{4}{\Omega}{}^{P}_{\lambda\mu} \underset{4}{\widehat{\Omega}}{}^{\alpha}_{P\nu} - \underset{4}{\Omega}{}^{P}_{\lambda\nu} \underset{4}{\widehat{\Omega}}{}^{\alpha}_{P\mu}, \end{aligned} \tag{2.16}$$

which is **another form of** (2.12)**.**

If we exchange at (2.12) $i \leftrightarrow p, \alpha \leftrightarrow \lambda$, for $\theta = 3$ from that equation one gets

$$\underset{1}{R}{}^{p}_{imn} B^{\alpha}_{p} B^{m}_{\mu} B^{n}_{\nu} - \overline{R}^{\alpha}_{\lambda\mu\nu} = \underset{3}{\widehat{\Omega}}{}^{\alpha}_{P\mu} \underset{3}{\Omega}{}^{P}_{\lambda\nu} - \underset{3}{\widehat{\Omega}}{}^{\alpha}_{P\nu} \underset{3}{\Omega}{}^{P}_{\lambda\mu}.$$

Summing this equation with 1^{st} case in (2.16), one concludes:

$$(\underset{1}{R}{}^{p}_{imn} - \underset{2}{R}{}^{p}_{imn}) B^{\alpha}_{p} B^{i}_{\lambda} B^{m}_{\mu} B^{n}_{\nu} = 0. \tag{2.17}$$

Putting $\theta = 4$ at (2.12), we get the 1^{st} case from (2.16), and together with 2^{nd} case it follows.

If we multiply (2.15) with C^{i}_{L}, it follows that

$$\begin{aligned} -\underset{2}{R}{}^{p}_{imn} B^{\alpha}_{p} C^{i}_{L} B^{m}_{\mu} B^{n}_{\nu} &= \underset{3}{\widehat{\Omega}}{}^{\alpha}_{L\mu\underset{3}{|}\nu} - \underset{3}{\widehat{\Omega}}{}^{\alpha}_{L\nu\underset{3}{|}\mu}, \\ -\underset{1}{R}{}^{p}_{imn} B^{\alpha}_{p} C^{i}_{L} B^{m}_{\mu} B^{n}_{\nu} &= \underset{4}{\widehat{\Omega}}{}^{\alpha}_{L\mu\underset{4}{|}\nu} - \underset{4}{\widehat{\Omega}}{}^{\alpha}_{L\nu\underset{4}{|}\mu}, \end{aligned} \tag{2.18}$$

and this is **another form fo the 1^{st} Codazzi equation of the 1^{st} and the 2^{nd} kind in the cited structure.**

2.2. Further we use the Ricci-type identity (equation (46) from [9])

$$B^{i}_{\alpha\underset{3}{|}\mu\underset{4}{|}\nu} - B^{i}_{\alpha\underset{4}{|}\nu\underset{3}{|}\mu} = \underset{4}{R}{}^{i}_{p\mu\nu} B^{p}_{\alpha} - \widetilde{R}^{\pi}_{\alpha\mu\nu} B^{i}_{\pi}, \tag{2.19}$$

where

$$\underset{4}{R}^{i}{}_{j\mu\nu} = (L^{i}_{jm,n} - L^{i}_{jn,m} + L^{p}_{jm}L^{i}_{np} - L^{p}_{nj}L^{i}_{pm})B^{m}_{\mu}B^{n}_{\nu} + T^{i}_{jm}(B^{m}_{\mu,\nu} - \widetilde{L}^{\pi}_{\mu\nu}B^{m}_{\pi}), \tag{2.20}$$

is **the** 4^{th} **kind curvature tensor** of L_N with respect to $X_M \subset L_N$. On the other hand, putting into (2.5) $\theta = 3, \omega = 4$ and comparing with (2.19), we get **the** 3^{rd} **kind integrability condition of derivational equation** (2.1) in the structure $(X_M \subset L_N, \underset{\theta}{\nabla}, \theta \in \{3,4\})$:

$$\underset{4}{R}^{i}{}_{p\mu\nu}B^{p}_{\alpha} - \widetilde{R}^{\pi}_{\alpha\mu\nu}B^{i}_{\pi} = (\underset{3}{\widehat{\Omega}}{}^{\pi}_{P\mu}\underset{4}{\Omega}{}^{P}_{\alpha\nu} - \underset{4}{\widehat{\Omega}}{}^{\pi}_{P\nu}\underset{3}{\Omega}{}^{P}_{\alpha\mu})B^{i}_{\pi} + (\underset{3}{\Omega}{}^{P}_{\alpha\mu\underset{4}{|}\nu} - \underset{4}{\Omega}{}^{P}_{\alpha\nu\underset{3}{|}\mu})C^{i}_{P}. \tag{2.21}$$

a) If one multiplies this equation with B^{λ}_{i}, we have

$$\underset{4}{R}^{i}{}_{p\mu\nu}B^{p}_{\alpha}B^{\lambda}_{i} - \widetilde{R}^{\lambda}_{\alpha\mu\nu} = \underset{3}{\widehat{\Omega}}{}^{\pi}_{P\mu}\underset{4}{\Omega}{}^{P}_{\alpha\nu} - \underset{4}{\widehat{\Omega}}{}^{\pi}_{P\nu}\underset{3}{\Omega}{}^{P}_{\alpha\mu} \tag{2.22}$$

and this is **Gauss equation of the** 3^{rd} **kind** in the cited structure.
b) Multiplying (2.21) with C^{L}_{i}, we get

$$\underset{4}{R}^{i}{}_{p\mu\nu}C^{L}_{i}B^{p}_{\alpha} = \underset{3}{\Omega}{}^{P}_{\alpha\mu\underset{4}{|}\nu} - \underset{4}{\Omega}{}^{P}_{\alpha\nu\underset{3}{|}\mu}, \tag{2.23}$$

i.e. **the** 1^{st} **Codazzi equation of the** 3^{rd} **kind** in the same structure.
2.2′. Based on equation (46) in [9], we have

$$B^{\alpha}_{i\underset{3}{|}\mu\underset{4}{|}\nu} - B^{\alpha}_{i\underset{4}{|}\nu\underset{3}{|}\mu} = \underset{3}{R}^{p}{}_{i\mu\nu}B^{\alpha}_{p} + \widetilde{R}^{\alpha}_{\pi\mu\nu}B^{\pi}_{i},$$

where

$$\underset{3}{R}^{i}{}_{j\mu\nu} = (L^{i}_{jm,n} - L^{i}_{nj,m} + L^{p}_{jm}L^{i}_{np} - L^{p}_{nj}L^{i}_{pm})B^{m}_{\mu}B^{n}_{\nu} + T^{i}_{jm}(B^{m}_{\mu,\nu} - \widetilde{L}^{\pi}_{\nu\mu}B^{m}_{\pi}), \tag{2.24}$$

is **the** 3^{rd} **kind curvature tensor** of L_N with respect to $X_M \subset L_N$.

Simultaneously, putting into (2) $\theta = 3, \omega = 4$, and comparing with (2), we obtain **the** 3^{rd} **integrability condition of derivational equation** (2.2) in the structure $(X_M \subset L_N, \underset{\theta}{\nabla}, \theta \in \{3,4\})$:

$$\underset{3}{R}^{p}{}_{i\nu\mu}B^{\alpha}_{p} + \widetilde{R}^{\alpha}_{\pi\mu\nu}B^{\pi}_{i} = (\underset{3}{\Omega}{}^{P}_{\pi\mu}\underset{4}{\widehat{\Omega}}{}^{\alpha}_{P\nu} - \underset{4}{\Omega}{}^{P}_{\pi\nu}\underset{3}{\widehat{\Omega}}{}^{\alpha}_{P\mu})B^{\pi}_{i} + (\underset{3}{\widehat{\Omega}}{}^{\alpha}_{P\mu\underset{4}{|}\nu} - \underset{4}{\widehat{\Omega}}{}^{\alpha}_{P\nu\underset{3}{|}\mu})C^{P}_{i}. \tag{2.25}$$

a′) From here, multiplying with B^{i}_{λ}

$$\underset{3}{R}^{p}{}_{i\nu\mu}B^{\alpha}_{p}B^{i}_{\lambda} + \widetilde{R}^{\alpha}_{\pi\mu\nu} = \underset{3}{\Omega}{}^{P}_{\pi\mu}\underset{4}{\widehat{\Omega}}{}^{\alpha}_{P\nu} - \underset{4}{\Omega}{}^{P}_{\pi\nu}\underset{3}{\widehat{\Omega}}{}^{\alpha}_{P\mu}, \tag{2.26}$$

which is **another form of the** 3^{rd} **kind Gauss equation** in the cited structure.

b') Similarly as the equation (2.23), we get

$$\underset{4}{R}^{i}{}_{p\nu\mu} C_i^L B_\alpha^p = \underset{3}{\Omega}{}^P_{\alpha\mu\underset{4}{|}\nu} - \underset{4}{\Omega}{}^P_{\alpha\nu\underset{3}{|}\mu}, \tag{2.27}$$

and that is another form of (2.23).

Now, we can state the following theorems

Theorem 2. *The* 1^{st} *and* 2^{nd} *kind integrability conditions derivational equations* (2.1) *and* (2.1′) *for submanifold* $X_M \subset L_N$ *with the structure* $(X_M \subset L_N, \underset{\theta}{\nabla}, \theta \in \{3,4\})$*, for the connections* $\underset{\theta}{\nabla}$ *are given in* (2.11) *and* (2.15) *respectively. The* 3^{rd} *kind of these conditions are given in* (2.21).

Theorem 3. *Gauss equations of the* 1^{st} *and the* 2^{nd} *kind are given in* (2.12)*, and of the* 3^{rd} *one in* (2.22)*. The* 1^{st} *Codazzi equations of the* 1^{st} *and the* 2^{nd} *kind are given in* (2.18)*, and of the* 3^{rd} *kind in* (2.23)*. The equations* (2.16),(2.18),(2.26),(2.27) *are another forms of previous equations.*

3. Integrability conditions of derivational equations for pseudonormals

3.0. Further, we use the similar procedure on derivational equation of pseudonormals.

Using (2.3),(2.1), we get

$$C^i_{A\underset{\theta}{|}\mu\underset{\omega}{|}\nu} - C^i_{A\underset{\omega}{|}\nu\underset{\theta}{|}\mu} = -(\underset{\theta}{\widehat{\Omega}}{}^\pi_{A\mu\underset{\omega}{|}\nu} - \underset{\omega}{\widehat{\Omega}}{}^\pi_{A\nu\underset{\theta}{|}\mu})B^i_\pi - (\underset{\theta}{\widehat{\Omega}}{}^\pi_{A\mu}\underset{\omega}{\Omega}{}^P_{\pi\nu} - \underset{\omega}{\widehat{\Omega}}{}^\pi_{A\nu}\underset{\theta}{\Omega}{}^P_{\pi\mu})C^i_P \tag{3.1}$$

and from (2.4),(2.2):

$$C^A_{i\underset{\theta}{|}\mu\underset{\omega}{|}\nu} - C^A_{i\underset{\omega}{|}\nu\underset{\theta}{|}\mu} = -(\underset{\theta}{\Omega}{}^A_{\pi\mu\underset{\omega}{|}\nu} - \underset{\omega}{\Omega}{}^A_{\pi\nu\underset{\theta}{|}\mu})B^\pi_i - (\underset{\theta}{\Omega}{}^A_{\pi\mu}\underset{\omega}{\widehat{\Omega}}{}^\pi_{P\nu} - \underset{\omega}{\Omega}{}^A_{\pi\nu}\underset{\theta}{\widehat{\Omega}}{}^\pi_{P\mu})C^P_i$$

$$\theta, \omega \in \{3,4\}.$$

We need also the Ricci type identities ([11], Equation (2.19))

$$C^i_{A\underset{\theta}{|}\mu\underset{\theta}{|}\nu} - C^i_{A\underset{\theta}{|}\nu\underset{\theta}{|}\mu} = \underset{\theta-2}{R}{}^i{}_{pmn} C^p_A B^m_\mu B^n_\nu - \overline{R}{}^P_{A\mu\nu} C^i_P, \quad \theta \in \{3,4\}, \tag{3.2}$$

and in the same way

$$\begin{aligned} C^A_{i\underset{3}{|}\mu\underset{3}{|}\nu} - C^A_{i\underset{3}{|}\nu\underset{3}{|}\mu} &= \overline{R}{}^A_{P\mu\nu} C^P_i - \underset{1}{R}{}^p_{imn} C^A_p B^m_\mu B^n_\nu, \\ C^A_{i\underset{4}{|}\mu\underset{4}{|}\nu} - C^A_{i\underset{4}{|}\nu\underset{4}{|}\mu} &= \overline{R}{}^A_{P\mu\nu} C^P_i - \underset{2}{R}{}^p_{imn} C^A_p B^m_\mu B^n_\nu, \end{aligned} \tag{3.3}$$

where $\underset{1}{R}, \underset{2}{R}$ are given at (2.10) and $\overline{R}$ at [11]

$$\widetilde{R}{}^A_{B\mu\nu} = \overline{L}{}^A_{B\mu,\nu} - \overline{L}{}^A_{B\nu,\mu} + \overline{L}{}^P_{B\mu}\overline{L}{}^A_{P\nu} - \overline{L}{}^P_{B\nu}\overline{L}{}^A_{P\mu}. \tag{3.4}$$

It follows that

$$C^i_{A\underset{3}{|}\mu\underset{4}{|}\nu} - C^i_{A\underset{4}{|}\nu\underset{3}{|}\mu} = \underset{4}{R}{}^i_{pmn} C^p_A - \overline{R}^P_{A\mu\nu} C^i_P, \tag{3.5}$$

and analogously

$$C^A_{i\underset{3}{|}\mu\underset{4}{|}\nu} - C^A_{i\underset{4}{|}\nu\underset{3}{|}\mu} = \underset{3}{R}{}^p_{imn} C^p_A + \overline{R}^A_{P\nu\mu} C^P_i,$$

The magnitude $\overline{R}^A_{B\mu\nu}$ is **curvature tensor of L_N with respect to the pseudonormal submanifold** X^N_{N-M} in the structure $(X_M \subset L_N, \underset{\theta}{\nabla}, \theta \in \{3,4\})$.

3.1. Taking $\theta = \omega \in \{3,4\}$ in (3.1) and comparing with (3.2), we obtain **the** 1^{st} **and the** 2^{nd} **kind integrability condition of derivational equation** (2.3) (for pseudonormals) in the structure $(X_M \subset L_N, \underset{\theta}{\nabla}, \theta \in \{3,4\})$:

$$\begin{aligned}\underset{\theta-2}{R}{}^{\;i}{}_{pmn} C^p_A B^m_\mu B^n_\nu - \overline{R}^P_{A\mu\nu} C^i_P \\ = -(\underset{\theta}{\widehat{\Omega}}{}^\pi_{A\mu\underset{\theta}{|}\nu} + \underset{\theta}{\widehat{\Omega}}{}^\pi_{A\nu\underset{\theta}{|}\mu}) B^i_\pi - (\underset{\theta}{\widehat{\Omega}}{}^\pi_{A\mu} \underset{\theta}{\Omega}{}^P_{\pi\nu} + \underset{\theta}{\widehat{\Omega}}{}^\pi_{A\nu} \underset{\theta}{\Omega}{}^P_{\pi\mu}) C^i_P, \quad \theta \in \{3,4\},\end{aligned} \tag{3.6}$$

a) Multiplying (3.6) with B^λ_i, one gets:

$$\underset{\theta-2}{R}{}^{\;i}{}_{pmn} B^\lambda_i C^p_A B^m_\mu B^n_\nu = -\underset{\theta}{\widehat{\Omega}}{}^\lambda_{A\mu\underset{\theta}{|}\nu} + \underset{\theta}{\widehat{\Omega}}{}^\lambda_{A\nu\underset{\theta}{|}\mu}, \quad \theta \in \{3,4\}. \tag{3.7}$$

which is **one more form of the** 1^{st} **Codazzi equation** (2.13)**.**

b) If we multiply (3.6) with C^L_i, one obtains

$$\underset{\theta-2}{R}{}^{\;i}{}_{pmn} C^L_i C^p_A B^m_\mu B^n_\nu - \overline{R}^L_{A\mu\nu} = -\underset{\theta}{\widehat{\Omega}}{}^\pi_{A\mu} \underset{\theta}{\Omega}{}^L_{\pi\nu} + \underset{\theta}{\widehat{\Omega}}{}^\pi_{A\nu} \underset{\theta}{\Omega}{}^L_{\pi\mu}, \quad \theta \in \{3,4\}. \tag{3.8}$$

and that is **the** 2^{nd} **Codazzi equation of the** 1^{st} **and the** 2^{nd} **kind** in the cited structure.

3.1′. If one takes $\theta = \omega \in \{3,4\}$ in (3) and compare with (3.3) we obtain **the** 1^{st} **and the** 2^{nd} **kind integrability condition of derivational equation** (2.4) in the structure $(X_M \subset L_N, \underset{\theta}{\nabla}, \theta \in \{3,4\})$:

$$\begin{aligned}\underset{2}{R}{}^p_{imn} C^A_p B^m_\mu B^n_\nu - \overline{R}^A_{P\mu\nu} C^P_i &= (\underset{3}{\Omega}{}^A_{\pi\mu\underset{3}{|}\nu} - \underset{3}{\Omega}{}^A_{\pi\nu\underset{3}{|}\mu}) B^\pi_i + (\underset{3}{\Omega}{}^A_{\pi\mu} \underset{3}{\widehat{\Omega}}{}^\pi_{P\nu} - \underset{3}{\Omega}{}^A_{\pi\nu} \underset{3}{\widehat{\Omega}}{}^\pi_{P\mu}) C^P_i \\ \underset{1}{R}{}^p_{imn} C^A_p B^m_\mu B^n_\nu - \overline{R}^A_{P\mu\nu} C^P_i &= (\underset{4}{\Omega}{}^A_{\pi\mu\underset{4}{|}\nu} - \underset{4}{\Omega}{}^A_{\pi\nu\underset{4}{|}\mu}) B^\pi_i + (\underset{4}{\Omega}{}^A_{\pi\mu} \underset{4}{\widehat{\Omega}}{}^\pi_{P\nu} - \underset{4}{\Omega}{}^A_{\pi\nu} \underset{4}{\widehat{\Omega}}{}^\pi_{P\mu}) C^P_i.\end{aligned} \tag{3.9}$$

which is **the** 1^{st} **and the** 2^{nd} **kind integrability conditions** ($\theta = 1,2$) **of derivational equation** (2.4)**.**

a') By multiplying of the previous equation with B^i_λ we get

$$\begin{aligned} \underset{2}{R}{}^p_{imn} C^A_p B^\lambda_i B^m_\mu B^n_\nu &= \underset{3}{\Omega}{}^A_{\lambda\mu\underset{3}{|}\nu} - \underset{3}{\Omega}{}^A_{\lambda\nu\underset{3}{|}\mu} \\ \underset{1}{R}{}^p_{imn} C^A_p B^\lambda_i B^m_\mu B^n_\nu &= \underset{4}{\Omega}{}^A_{\lambda\mu\underset{4}{|}\nu} - \underset{4}{\Omega}{}^A_{\lambda\nu\underset{4}{|}\mu}. \end{aligned} \tag{3.10}$$

that is **one more form of the** 1^{st} **Codazzi equation** (2.18)**.**

b') Multiplying (3.9) with C^i_L, we get

$$\begin{aligned} \underset{2}{R}{}^p_{imn} C^A_p C^i_L B^m_\mu B^n_\nu - \overline{R}^A_{L\mu\nu} &= \underset{3}{\Omega}{}^A_{\pi\mu} \underset{3}{\widehat{\Omega}}{}^\pi_{L\nu} - \underset{3}{\Omega}{}^A_{\pi\nu} \underset{3}{\widehat{\Omega}}{}^\pi_{L\mu} \\ \underset{1}{R}{}^p_{imn} C^A_p C^i_L B^m_\mu B^n_\nu - \overline{R}^A_{L\mu\nu} &= \underset{4}{\Omega}{}^A_{\pi\mu} \underset{4}{\widehat{\Omega}}{}^\pi_{L\nu} - \underset{4}{\Omega}{}^A_{\pi\nu} \underset{4}{\widehat{\Omega}}{}^\pi_{L\mu} \end{aligned} \tag{3.11}$$

and this is **another form of the** 2^{nd} **Codazzi equation of the** 1^{st} **and the** 2^{nd} **kind.**

3.2. If one takes $\theta = 3, \omega = 4$ in (3.1) and compares obtained equation with (3.5), we obtain 3^{rd} **integrability condition of derivational equation** (2.3) in the structure $(X_M \subset L_N, \underset{\theta}{\nabla}, \theta \in \{3,4\})$:

$$\underset{4}{R}{}^i_{p\mu\nu} C^p_A - \overline{R}^P_{A\mu\nu} C^i_P = -(\underset{3}{\widehat{\Omega}}{}^\pi_{A\mu\underset{4}{|}\nu} - \underset{4}{\widehat{\Omega}}{}^\pi_{A\nu\underset{3}{|}\mu}) B^i_\pi - (\underset{3}{\widehat{\Omega}}{}^\pi_{A\mu} \underset{4}{\Omega}{}^P_{\pi\nu} - \underset{4}{\widehat{\Omega}}{}^\pi_{A\nu} \underset{3}{\Omega}{}^P_{\pi\mu}) C^i_P. \tag{3.12}$$

a) Multiplying (3.12) with B^λ_i, we get

$$\underset{4}{R}{}^i_{p\mu\nu} B^\lambda_i C^p_A = -\underset{3}{\widehat{\Omega}}{}^\lambda_{A\mu\underset{4}{|}\nu} + \underset{4}{\widehat{\Omega}}{}^\lambda_{A\nu\underset{4}{|}\mu}, \tag{3.13}$$

which is **one more form of** (2.17)**.**

b) Multiplying (3.12) with C^L_i, we have

$$\underset{4}{R}{}^i_{p\mu\nu} C^L_i C^p_A - \overline{R}^L_{A\mu\nu} = \underset{3}{\Omega}{}^L_{\pi\mu} \underset{4}{\Omega}{}^\pi_{A\nu} - \underset{4}{\Omega}{}^L_{\pi\nu} \underset{3}{\Omega}{}^\pi_{A\mu}. \tag{3.14}$$

which is **the** 2^{nd} **Codazzi equation of the** 3^{rd} **kind.**

3.2$'$. Endly, we put $\theta = 3, \omega = 4$ into (3) and compare obtained equation with (3.4$'$). In that manner, one obtains **the** 3^{rd} **kind integrability condition of derivational equation** (2.4) in the structure$(X_M \subset L_N, \underset{\theta}{\nabla}, \theta \in \{3,4\})$:

$$\underset{3}{R}{}^p_{i\nu\mu} C^A_p + \overline{R}^A_{P\mu\nu} C^P_i = -(\underset{3}{\Omega}{}^A_{\pi\mu\underset{4}{|}\nu} - \underset{4}{\Omega}{}^A_{\pi\nu\underset{3}{|}\mu}) B^\pi_i - (\underset{3}{\Omega}{}^A_{\pi\mu} \underset{4}{\widehat{\Omega}}{}^\pi_{P\nu} - \underset{4}{\Omega}{}^A_{\pi\nu} \underset{3}{\widehat{\Omega}}{}^\pi_{P\mu}) C^P_i.$$

a') If one multiplies (3) with B^i_λ, it follows that

$$\underset{3}{R}{}^p_{i\nu\mu} C^A_p B^i_\lambda = -\underset{3}{\Omega}{}^A_{\pi\mu\underset{4}{|}\nu} + \underset{4}{\Omega}{}^A_{\pi\nu\underset{3}{|}\mu}, \tag{3.15}$$

and this is **another form of** (2.27) **or** (3.13)**.**

b') Multiplying (3) with C^{i}_{L}, we have

$$\underset{3}{R}{}^{p}_{i\mu\nu} C^{p}_{A} C^{i}_{L} - \overline{R}{}^{A}_{L\mu\nu} = \underset{3}{\widehat{\Omega}}{}^{\pi}_{L\mu} \underset{4}{\Omega}{}^{A}_{\pi\nu} - \underset{4}{\widehat{\Omega}}{}^{\pi}_{L\nu} \underset{3}{\Omega}{}^{A}_{\pi\mu}. \tag{3.16}$$

which is **another form of the** 2^{nd} **Codazzi equation of the** 3^{rd} **kind** i.e. of (3.14).

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