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FIXED POINT THEOREMS FOR LOCAL ALMOST CONTRACTIONS

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Abstract. This paper introduces a new class of contraction: the almost local contractions. Then, we prove the existence and uniqueness of a fixed-point for local almost contractions in two cases: with constant and variable coefficients of contraction.

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1. Introduction

The concept of local contraction was presented by Martins da Rocha and Filipe Vailakis in [4], meanwhile the almost contractive mappings was introduced by V. Berinde in [2]. The aim of this paper is to combine this two concepts and to study the fixed points of almost local contractions. First, we present the concept of almost contraction, following V. Berinde in [2] (2004).

Definition 1. Let (X,D) be a metric space and $T : X \to X$ is called almost contraction or (δ, L)- contraction if there exist a constant $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(Tx,Ty) \leq \delta \cdot d(x,y) + L \cdot d(y,Tx), \forall x,y \in X. \tag{1.1}$$

Remark 1. The term of almost contraction is equivalent to weak contraction, and it was first introduced by V. Berinde in [2].

Remark 2. Because of the simmetry of the distance, the almost contraction condition (1.1) includes the following dual one:

$$d(Tx,Ty) \leq \delta \cdot d(x,y) + L \cdot d(x,Ty), \forall x,y \in X \tag{1.2}$$

obtained from (1.1) by replacing d(Tx,Ty) by d(Ty,Tx) and d(x,y) by d(y,x), and after that step, changing x with y, and viceversa.

Obviously, to prove the almost contactiveness of T, it is necesarry to check both (1.1) and (1.2).

A strict contraction satisfies (1.1), with $\delta = a$ and L = 0, therefore is an almost contraction with a unique fixed point.
The concept of local contraction was first introduced by Martins da Rocha and Filipe Vailakis in [4] (2010)

Definition 2. Let F be a set and let $\mathcal{D} = (d_j)_{j\in J}$ a family of semidistances defined on F. We let σ be the weak topology on F defined by the family $\mathcal{D}$.
Let r be a function from J to J. An operator $T : F \to F$ is a *local contraction* with respect ($\mathcal{D}$,r) if, for every j, there exists $\beta_j \in [0,1)$ such that

$$\forall f, g \in F, \quad d_j(Tf, Tg) \le \beta_j d_{r(j)}(f, g)$$

We present an existence theorem (Theorem 1), then an existence and uniqueness theorem (Theorem 2), as they are presented in [2]. Their main merit is that they extend uniqueness and existence theorems for contractions to the larger class of almost contractions. They show us a method for approximating the fixed point, for whitch both a priori and a posteriori error estimates are available.

Theorem 1. *Let (X,D) be a complete metric space and $T : X \to X$ a weak (almost) contraction. Then*

(1) $Fix(T) = \{x \in X: \quad Tx = x\} \neq \phi$;
(2) *For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (3) converges to some $x^* \in Fix(T)$;*
$$x_{n+1} = Tx_n, n = 0, 1, 2, \cdots \tag{1.3}$$
(3) *The following estimates*
$$d(x_n, x^*) \le \frac{\delta^n}{1-\delta} d(x_0, x_1), \quad n = 0, 1, 2\ldots \tag{1.4}$$
$$d(x_n, x^*) \le \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \quad n = 1, 2\ldots \tag{1.5}$$
hold, where δ is the constant appearing in (1.1).

It is possible to force the uniqueness of the fixed point of an almost contraction, see [1] and [2], by imposing an additional contractive condition, quite similar to (1.1), as shown by the next theorem.

Theorem 2. *Let (X,d) be a complete metric space and $T : X \to X$ be an almost contraction for which there exist $\theta \in (0,1)$ and some $L_1 \ge 0$ such that*

$$d(Tx, Ty) \le \theta \cdot d(x, y) + L_1 \cdot d(x, Tx), \forall x, y \in X \tag{1.6}$$

Then

(1) *T has a unique fixed point,i.e., $Fix(T) = \{x^*\}$*
(2) *For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (3) converges to x^*;*

(3) *The a priori and a posteriori error estimates*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), \quad n = 0, 1, 2...$$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \quad n = 1, 2...$$

hold.

(4) *The rate of convergence of the Picard iteration is given by*

$$d(x_n, x^*) \leq \theta \cdot d(x_{n-1}, x^*), \quad n = 1, 2... \tag{1.7}$$

The existence and uniqueness of fixed points for the local contractions was studied in [4].

Theorem 3. *Assume that the space F is σ- Hausdorff, which means: for each pair f,g ∈ F, f ≠ g, there exists* $j \in J$ *such that* $d_j(f, g) > 0$.
Consider a function $r : J \to J$ *and let* $T : F \to F$ *be a local contraction with respect to* $(\mathcal{D}, r)$*. Consider a nonempty, σ- bounded, sequentially σ- complete, and T-invariant subset* $A \subset F$.

(1) *(E) If the condition*

$$\forall j \in J, \quad \lim_{n\to\infty} \beta_j \beta_{r(j)} \cdots \beta_{r^{n+1}(j)} diam_{r^{n+1}(j)}(A) = 0 \tag{1.8}$$

is satisfied, then the operator T admits a fixed point f^* *in A.*

(2) *(S) Moreover, if* $h \in F$ *satisfies*

$$\forall j \in J, \quad \lim_{n\to\infty} \beta_j \beta_{r(j)} \cdots \beta_{r^{n+1}(j)} d_{r^{n+1}(j)}(h, A) = 0 \tag{1.9}$$

then the sequence $(T^n h)_{n\in\mathbb{N}}$ *is* σ*-convergent to* f^*.

If A is a nonempty subset of F, then for each h in F, we let
$d_j(h, A) \equiv inf\{d_j(h, g) : g \in A\}$.

Remark 3. There exists other interpretations of the local contractions, different from Martins da Rocha and Filipe Vailakis in [4], for example the definition given by Fang Jin-xuan in [3]:

Definition 3. A distribution function is a function $F : [-\infty, \infty] \to [0, 1]$ which is left continuous on $\mathbb{R}$, non-decreasing and $F(-\infty) = 0, F(\infty) = 1$.
We will denote by Δ the family of all distribution functions on $[-\infty, \infty]$. H is a special element of Δ defined by

$$H(t) = 0, if \quad t \leq 0 \quad and \quad 1, if \quad t > 0$$

If X is a nonempty set, $F: X \times X \to \Delta$ is called a probabilistic distance on X and $F(x,y)$ is usually denoted by F_{xy}.

Definition 4 (Schweizer and Sklar [5])**.** The ordered pair (X,F) is called a probabilistic metric space (shortly PM-space) if X is a nonempty set and F is a probabilistic distance satisfying the following conditions: for all $x,y,z \in X$ and $t,s > 0$,

(1) (FM-0) $F_{xy}(t) = 1 \Leftrightarrow x = y$
(2) (FM-1) $F_{xy}(0) = 0$
(3) (FM-2) $F_{xy} = F_{yx}$
(4) (FM-3) $F_{xz}(t) = 1, F_{zy}(s) = 1 \Rightarrow F_{xy}(t+s) = 1$.
The ordered triple $(X,F,*)$ is called Menger space if (X,F) is a PM-space, $*$ is a t- norm and the following condition is also satisfied: for all $x,y,z, \in X$ and $t,s > 0$
(5) (FM-4) $F_{xy}(t+s) \geq F_{xz}(t) * F_{zy}(s)$.

Definition 5. The PM- space (X,F) is said to be ε- chainable if for given $\varepsilon > 0$ and any $x,y \in X$ there is a finite set of points in X: $x = x_0, x_1, \cdots x_n = y$ such that $F_{x_{i-1},x_i}(\varepsilon) = 1, \quad i = 1,2,\cdots n$

We denote by $\mathfrak{T}$ the topology on X and it is called the (ε,λ) topology of (X,F,Δ) . Thus we can induce the concept of $\mathfrak{T}$- Cauchy sequence and also $\mathfrak{T}$- complete sequence.

Theorem 4 (The Fixed Point Theorem of One-valued Local Contraction Mappings, Fang [3])**.** *Let (X,F,Δ) be an ε- chainable and $\mathfrak{T}$- complete Menger space. Let the mapping $T: X \to X$ satisfy the following condition: there exists $\delta \in (0,1)$ so that for each $\alpha \in (0,\delta)$ there is a function $\Phi_\alpha(t)$ satisfying the condition Φ such that*

$$F_{Tx,Ty}(\Phi_\alpha)(t) \geq F_{x,y}(t) \tag{1.10}$$

for $F_{x,y}(\varepsilon) \neq 0$ and $F_{x,y}(t) > 1-\alpha$.
Then T has a unique fixed point x_* in X and $T^n x_0 \longrightarrow x_*$ for any $x_0 \in X$

Now, we shall try to combine these two different type of contractive mappings: the almost and local contractions, to study their fixed points.

2. Almost local contractions

Definition 6. The mapping $d(x,y): X \times X \to \mathbb{R}_+$ is said to be a pseudometric if:

(1) $d(x,y) = d(y,x)$
(2) $d(x,y) \leq d(x,z) + d(z,y)$
(3) $x = y$ implies $d(x,y) = 0$
(instead of $x = y \Leftrightarrow d(x,y) = 0$ in the metric case)

Definition 7. Let X be a set and let $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics defined on X. We let σ be the weak topology on X defined by the family $\mathcal{D}$.
A sequence $(x_n)_{n\in\mathbb{N}^*}$ is said to be $\sigma - Cauchy$ if it is d_j-Cauchy, $\forall j \in J$.
The subset A of X is said to be sequencially σ-complete if every σ-Cauchy sequence in X converges in X for the σ-topology.
The subset A$\subset$X is said to be σ-bounded if $diam_j(A) \equiv sup\{d_j(x,y) : x,y \in A\}$ is finite for every $j \in J$.

Definition 8. Let r be a function from J to J. An operator $T : X \to X$ is called an *almost local contraction* with respect ($\mathcal{D}$,r) if, for every j, there exist the constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$d_j(Tx,Ty) \leq \theta \cdot d_j(x,y) + L \cdot d_{r(j)}(y,Tx), \forall x,y \in X \tag{2.1}$$

Remark 4. The almost contractions represent a particular case of almost local contractions, by taking (X,d) metric space instead of the pseudometrics d_j and $d_{r(j)}$ defined on X. Also, to obtain the almost contractions, we take in (1.1) for r the identity function, so we have r(j)=j.

Definition 9. The space X is σ- Hausdorff if the following condition is valid: for each pair x,y $\in$ X, x $\neq$ y, there exists j $\in$J such that $d_j(x,y) > 0$.
If A is a nonempty subset of X, then for each z in X, we let
$d_j(z,A) \equiv inf\{d_j(z,y) : y \in A\}$.

Theorem 5 is an existence fixed point theorem for almost local contractions.

Theorem 5. *Consider a function $r : J \to J$ and let $T : X \to X$ be an almost local contraction with respect to ($\mathcal{D}$,r). Consider a nonempty, σ- bounded, sequentially σ-complete, and T- invariant subset $A \subset X$. If the condition*

$$\forall j \in J, \quad \lim_{n\to\infty} \theta^{n+1} diam_{r^{n+1}(j)}(A) = 0 \tag{2.2}$$

is satisfied, then the operator T admits a fixed point x^* in A.

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}$$

Take $x := x_{n-1}, y := x_n$ in (2.1) to obtain

$$d_j(Tx_{n-1},Tx_n) \leq \theta \cdot d_{r(j)}(x_{n-1},x_n)$$

which yields

$$d_j(x_n,x_{n+1}) \leq \theta \cdot d_{r(j)}(x_{n-1},x_n), \forall j \in J \tag{2.3}$$

Using (2.3), we obtain by induction with respect to n:

$$d_j(x_n,x_{n+1}) \leq \theta^n \cdot d_{r(j)}(x_0,x_1), \quad n = 0,1,2,\cdots \tag{2.4}$$

According to the triangle rule, by (2.4) we get:

$$d_j(x_n,x_{n+p}) \leq \theta^n(1+\theta+\cdots+\theta^{p-1})d_{r(j)}(x_0,x_1) = \tag{2.5}$$

$$= \frac{\theta^n}{1-\theta}(1-\theta^p)\cdot d_{r(j)}(x_0,x_1), \quad n,p \in \mathbb{N}, p \neq 0 \tag{2.6}$$

Conditions (2.5) show us that the sequence $(x_n)_{n\in\mathbb{N}}$ is d_j- Cauchy for each j. The subset A is assumed to be sequentially σ-complete, there exists f^* in A such that $(T^n x)_{n\in\mathbb{N}}$ is σ- convergent to x^*. Besides, the sequence $(T^n x)_{n\in\mathbb{N}}$ converges for the topology σ to x^*, which implies

$$\forall j \in J, \quad d_j(Tx^*,x^*) = \lim_{n\to\infty} d_j(Tx^*,T^{n+1}x).$$

Recall that the operator T is an almost local contraction with respect to ($\mathcal{D}$,r). From that, we have

$$\forall j \in J, \quad d_j(Tx^*,x^*) \leq \beta_j \lim_{n\to\infty} d_{r(j)}(x^*,T^n x).$$

The convergence for the σ- topology implies convergence for the pseudometric $d_{r(j)}$, we obtain $d_j(Tx^*,x^*) = 0$ for every $j \in J$.
This way, we prove that $Tf^* = f^*$, since σ is Hausdorff.
So, we prove the existence of the fixed point for almost local contractions. □

Remark 5. For T verifies (2.1) with L = 0, we find Theorem Vailakis [4] by taking $\theta_j = \theta$.
Further, for the case $d_j = d, \forall j \in J$, with d = metric on X, we obtain the well known Banach contraction, with his unique fixed point.

Remark 6. In Theorem 5, the coefficient of contraction $\theta \in (0,1)$ is constant, but local contractions have a coefficient of contraction $\theta_j \in [0,1)$ whitch depends on $j \in$ J. Our first goal is to extend the local almost contractions to the most general case of $\theta_j \in (0,1)$.

The next Theorem represent an existence and uniqueness theorem for the almost local contractions with constant coefficient of contraction.

Theorem 6. *If to the conditions of Theorem 5, we add:*
(U) for every fixed $j \in J$ there exists:

$$\lim_{n\to\infty} (\theta+L)^n diam_{r^n(j)}(z,A) = 0, \forall x,y \in X \tag{2.7}$$

then the fixed point x^* of T is unique.

Proof. Suppose, by contradiction, there are two different fixed points x^* and y^* of T. Then for every fixed j∈J we have:

$$0 < d_j(x^*,y^*) = d_j(Tx^*,Ty^*) \leq \theta d_{r(j)}(x^*,y^*) + L d_{r(j)}(y^*,Tx^*) =$$

$$= (\theta+L)\cdot d_{r(j)}(x^*,y^*) \leq \cdots \leq (\theta+L)^n d_{r^n(j)}(x^*,y^*) \leq$$
$$\leq (\theta+L)^n diam_{r^n(j)}(z,A)$$

Now, letting n$\rightarrow \infty$, we obtained a contradiction with condition (2.7), i.e. the fixed point is unique. □

Remark. If the function r is the identity (i.e., $r(j) = j$), then the operator T is said to be a 0- local contraction and, in that case, conditions (2.1) and (2.7) are automatically satisfied. In particular, if a fixed point exists, it is unique on the whole space X.
We extend the Definition 8. to the case of almost local contractions with variable coefficient of contraction.

Definition 10. Let r be a function from J to J. An operator $T : X \to X$ is called *almost local contraction* with respect ($\mathcal{D}$,r) or (θ_j, L_j)- contraction, if there exist a constant $\theta_j \in (0,1)$ and some $L_j \geq 0$ such that

$$d_j(Tx,Ty) \leq \theta_j \cdot d_j(x,y) + L_j \cdot d_{r(j)}(y,Tx), \forall x,y \in X \tag{2.8}$$

Theorem 7. *With the presumptions of Theorem 7, if we modify the condition (2.2) by the following one:*

$$\forall j \in J, \quad \lim_{n\to\infty} \theta_j \theta_{r(j)} \cdots \theta_{r^n(j)} diam_{r^{n+1}(j)}(A) = 0, \tag{2.9}$$

then the operator T admits a fixed point x^* in A.

Proof. Let x an element in A. From the definition of almost local contraction T, for every pair of integers $q > n > 0$, we have

$$\begin{aligned}
&d_j(T^q x, T^n x)\\
&\leq \theta_j d_{r(j)}(T^{q-1}x, T^{n-1}x) + L_j d_{r(j)}(T^{n-1}x, T^q x) \leq\\
&\leq \theta_j[\theta_{r(j)} d_{r^2(j)}(T^{q-2}x, T^{n-2}x) + L_j d_{r^2(j)}(T^{n-2}x, T^{q-1}x)]+\\
&+ L_j[\theta_{r(j)} d_{r^2(j)}(T^{n-2}x, T^{q-1}x) + L_{r(j)} d_{r^2(j)}(T^{q-1}x, T^{n-1}x)] =\\
&= \theta_j[\theta_{r(j)} d_{r^2(j)}(T^{q-2}x, T^{n-2}x) + L_j(\theta_j + \theta_{r(j)}) d_{r^2(j)}(T^{n-2}x, T^{q-1}x)]+\\
&+ L_j L_{r(j)} d_{r^2(j)}(T^{q-1}x, T^{n-1}x) \leq \cdots \leq\\
&\leq \theta_j \theta_{r(j)} \cdots \theta_{r^{n-1}(j)} d_{r^n(j)}(T^{q-n}x, x)\\
&+ \cdots + L_j L_{r(j)} \cdots L_{r^{n-1}(j)} d_{r^n(j)}(T^{q-n}x, x)
\end{aligned}$$

Since A is T- invariant, $T^{q-n}g$ belongs to A, which yields

$$d_j(T^q x, T^n x) \leq \theta_j \theta_{r(j)} \cdots \theta_{r^{n-1}(j)} diam_{r^n(j)}(A).$$

That means: the sequence $(T^n x)_{n\in\mathbb{N}}$ is d_j-Cauchy for each j. The subset A is assumed to be sequentially σ-complete, there exists x^* in A such that $(T^n x)_{n\in\mathbb{N}}$ is σ-

convergent to x^*. Besides, the sequence $(T^n x)_{n\in\mathbb{N}}$ converges for the topology σ to x^*, which implies

$$\forall j \in J, \quad d_j(Tx^*, x^*) = \lim_{n\to\infty} d_j(Tx^*, T^{n+1}x).$$

Recall that the operator T is an almost local contraction with respect to ($\mathcal{D}$,r). From that, we have

$$\forall j \in J, \quad d_j(Tx^*, x^*) \le \theta_j \lim_{n\to\infty} d_{r(j)}(x^*, T^n x) + L_j d_{r(j)}(T^n x, Tx^*).$$

The convergence for the σ- topology implies convergence for the pseudodistance $d_{r(j)}$, we obtain $d_j(Tx^*, x^*) = 0$ for every $j \in J$. This way, we prove that $Tx^* = x^*$ since σ is Hausdorff. So, we prove the existence of the fixed point. □

REFERENCES

[1] V. Berinde, "On the approximation of fixed points of weak contractive mappings," *Carpathian J. Math.*, vol. 19, no. 1, pp. 7–22, 2003.
[2] V. Berinde, "Approximating fixed points of weak contractions using the picard iteration," *Nonlinear Analysis Forum*, vol. 9, no. 1, pp. 43–53, 2004.
[3] F. Jin-xuan, "Fixed point theorems of local contraction mappings on menger spaces," *Applied Mathematics and Mechanics*, vol. 12, no. 4, April 1991, doi: 10.1007/BF02020399.
[4] F. Martins-da Rocha and V. Yiannis, "Existence and uniqueness of a fixed point for local contractions," *Econometrica*, vol. 78, no. 3, pp. 1127–1141, May 2010, doi: 10.3982/ECTA7920.
[5] B. Schweizer, A. Sklar, and E. Thorp, "The metrization of statistical metric spaces," *Pacific J. Math.*, vol. 10, pp. 673–675, 1960.

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