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PERIODIC BOUNDARY VALUE PROBLEM FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The periodic boundary value problem for a functional differential equation is considered. The existence of a solution and upper and lower solutions is proved under assumptions that there exist lower and upper functions and the right side of the functional differential equation satisfies one-sided growth conditions. Results are proved by the Borsuk antipodal theorem and the Leray-Schauder degree.

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1. Introduction

Let $J = [0, T]$ be a compact interval and, for any positive constant K, $\mathcal{L}^K(J) = \{x : x \in L_1(J), |x(t)| \le K \text{ for each } t \in J\}$. We write $x_n \to x$ as $n \to \infty$ in $\mathcal{L}^K(J)$ if $\lim_{n\to\infty} x_n(t) = x(t)$ for a.e. $t \in J$. We denote by $\|x\| = \max\{|x(t)| : t \in J\}$ and $\|x\|_{L_1} = \int_a^b |x(t)|\, dt$ the norms in the Banach spaces $C^0(J)$ and $L_1(J)$, respectively. Set $\mathcal{L}(J) = \bigcup_{K>0} \mathcal{L}^K(J)$.

Consider the periodic boundary value problem (PBVP for short)

$$(g(x'(t)))' = F(x, x(t), x'(t))(t), \tag{1.1}$$

$$x(0) = x(T), \quad x'(0) = x'(T). \tag{1.2}$$

Here $g : \mathbb{R} \to \mathbb{R}$ is an increasing homomorphism with inverse $g^{-1} : \mathbb{R} \to \mathbb{R}$, $g(0) = 0$ and $F : C^0(J) \times \mathbb{R}^2 \to L_1(J)$, $(x, a, b) \longmapsto F(x, a, b)(t)$ is the operator having the following properties:

(i) $F(x, y(t), z(t))(t) \in L_1(J)$ for $x, y \in C^0(J)$ and $z \in \mathcal{L}(J)$;

(ii) for any $K > 0$, $\lim_{n\to\infty}(x_n, y_n, z_n) = (x, y, z)$ in $C^0(J) \times C^0(J) \times \mathcal{L}^K(J) \Rightarrow$

$$\lim_{n\to\infty} F(x_n, y_n(t), z_n(t))(t) = F(x, y(t), z(t))(t) \text{ in } L_1(J);$$

(iii) $a, b \in \mathbb{R}$, $x_1, x_2 \in C^0(J)$, $x_1(t) \le x_2(t)$ for $t \in J \Rightarrow$
$F(x_1, a, b)(t) \ge F(x_2, a, b)(t)$ for a.e. $t \in J$;

(iv) for any $r > 0$ there exists $h_r \in L_1(J)$ such that $(x, a, b) \in C^0(J) \times \mathbb{R}^2$, $\|x\| + |a| + |b| \le r \Rightarrow$
$|F(x, a, b)(t)| \le h_r(t)$ for a.e. $t \in J$.

The special cases of the operator F having the properties (i)-(iv) are, for example, the operators

$$F_1(x, a, b)(t) = \int_{\varphi_1(t)}^{\varphi_2(t)} f_1(s, \xi(t)x(s), a, b)\, ds + f_2(t, a, b)$$

and

$$F_2(x, a, b)(t) = f_1(t, (Sx)(t), a, b),$$

where f_1 (resp. f_2) satisfies the local Carathéodory conditions on $J \times \mathbb{R}^3$ (resp. $J \times \mathbb{R}^2$), $f_1(t, \cdot, a, b)$ is nonincreasing on $\mathbb{R}$ for a.e. $t \in J$ and each $(a, b) \in \mathbb{R}^2$, $\varphi_1, \varphi_2, \xi \in C^0(J)$, $0 \le \varphi_1(t) \le \varphi_2(t) \le T$, $\xi(t) \ge 0$ for $t \in J$ and the operator $S : C^0(J) \to C^0(J)$ is bounded continuous and $x, y \in C^0(J)$, $x(t) \le y(t)$ for $t \in J$ $\Rightarrow (Sx)(t) \le (Sy)(t)$ for $t \in J$. For the operators F_1 and F_2, equation (1) has the form

$$(g(x'(t)))' = \int_{\varphi_1(t)}^{\varphi_2(t)} f_1(s, \xi(t)x(s), x(t), x'(t))\, ds + f_2(t, x(t), x'(t))$$

and

$$(g(x'(t)))' = f_1(t, (Sx)(t), x(t), x'(t)).$$

Under *a solution of PBVP* (1.1), (1.2) we mean a function $x \in C^1(J)$ such that $g(x'(t))$ is absolutely continuous on J ($AC(J)$ for short), equality (1.1) is satisfied a.e. on J and x satisfies the periodic boundary conditions (1.2).

The purpose of this paper is to give sufficient conditions formulated by upper and lower functions of PBVP (1.1), (1.2) and one-sided growth restrictions on the operator F ensuring the existence of a solution and upper and lower solutions for PBVP (1.1), (1.2) in a subset of $C^1(J)$.

Let α, β be lower and upper functions of PBVP (1.1), (1.2) (see Section 2), $\alpha(t) \le \beta(t)$ for $t \in J$, and set $\mathcal{M}_{\alpha\beta} = \{x : x \in C^1(J), \alpha(t) \le x(t) \le \beta(t) \text{ for } t \in J\}$. The existence results for PBVP (1.1), (1.2) in the set $\mathcal{M}_{\alpha\beta}$ are proved by the Borsuk antipodal theorem and the Leray-Schauder degree (see e.g. [2], [9]). Denote by $\mathcal{D}_{\alpha\beta}$ the set of all solutions of PBVP (1.1), (1.2) in the set $\mathcal{M}_{\alpha\beta}$. The proof of the existence of upper and lower solutions for PBVP (1.1), (1.2) in the set $\mathcal{D}_{\alpha\beta}$ is based on the following two facts: a) for any $u, v \in \mathcal{D}_{\alpha\beta}$ there exist $x, y \in \mathcal{D}_{\alpha\beta}$

such that $x(t) \le \min\{u(t), v(t)\}$, $\max\{u(t), v(t)\} \le y(t)$ for $t \in J$, b) the functional $\varphi : \mathcal{D}_{\alpha\beta} \to \mathbb{R}$, $\varphi(x) = \int_0^T x(t)\,dt$ achieves its extremal values at upper and lower solutions of PBVP (1.1), (1.2).

We observe that there are many papers devoted to PBVP

$$x''(t) = f(t, x(t), x'(t)) \tag{1.3}$$

with the scalar function f where the existence results have been proved by the method of upper and lower functions combined with a priori estimates for solutions (see, e.g., [1], [3-8], [10-15] and references cited therein). To obtain a priori estimates for solutions of the above PBVP authors usually assumed that the function f satisfies either the two-sided Bernstein-Nagumo growth condition ([1], [3, 4], [8], [10], [11], [13-15]) or f satisfies some sign conditions ([12]) or f satisfies one-sided growth conditions ([5-7]). Our results generalized those of [7].

2. Notation, lemmas

Let $G, Q \in C^0([0,\infty))$ be defined on $[0,\infty)$ by the formulas

$$G(u) = \max\{-g(-u), g(u)\}, \quad Q(u) = \max\{-g^{-1}(-u), g^{-1}(u)\}. \tag{2.1}$$

Then

$$|g(u)| \le G(|u|), \quad |g^{-1}(u)| \le Q(|u|), \qquad u \in \mathbb{R}. \tag{2.2}$$

Clearly, $G(u) = g(u)$, $Q(u) = g^{-1}(u)$ on $[0,\infty)$ provided g is an odd function (for example $g(u) = |u|^{p-2}u$ with $p > 1$).

Lemma 1 *Let $x \in C^1(J)$, $g(x') \in AC(J)$, x satisfy the periodic boundary conditions (1.2) and set*

$$\mathcal{C}_1 = \{t : t \in J, x'(t) > 0\}, \quad \mathcal{C}_2 = \{t : t \in J, x'(t) < 0\}.$$

Assume that there exist $\sigma_k \in \{-1, 1\}$ $(k = 1, 2)$ and positive-valued functions $h \in L_1(J)$ and $w \in C^0(\mathbb{R})$ such that

$$\int_{-\infty}^{0} \frac{dt}{w(g^{-1}(t))} = \int_0^{\infty} \frac{dt}{w(g^{-1}(t))} = \infty \tag{2.3}$$

and

$$\sigma_k (g(x'(t)))' \le (h(t) + |x'(t)|) w(x'(t)) \quad \text{for a.e. } t \in \mathcal{C}_k \ (k = 1, 2). \tag{2.4}$$

Let P be a positive constant satisfying the inequality

$$\min\Big\{\int_{g(-P)}^{0} \frac{dt}{w(g^{-1}(t))}, \int_0^{g(P)} \frac{dt}{w(g^{-1}(t))}\Big\} > 2(\|h\|_{L_1} + 2\|x\|). \tag{2.5}$$

Then

$$\|x'\| < P. \tag{2.6}$$

Proof. Let

$$\mathcal{B} = \{t : t \in J, x'(t) = 0\}$$

and $\|x'\| = |x'(\xi)|$, $\xi \in J$. By (2), $\mathcal{B} \neq \emptyset$ and, without loss of generality, we can assume that $\xi \in (0, T]$. For $\|x'\| = 0$ the inequality (2.6) is satisfied. Let $\|x'\| > 0$. We first assume that $x'(\xi) > 0$. The next part of the proof is divided into two cases according to whether $\sigma_1 = 1$ or $\sigma_1 = -1$.

a) Let $\sigma_1 = 1$. If $\mathcal{B} \cap [0, \xi) \neq \emptyset$ then there exists $t_1 \in \mathcal{B} \cap [0, \xi)$ such that $x'(t) > 0$ for $t \in (t_1, \xi]$ and (cf. (2.4))

$$(g(x'(t)))' \leq (h(t) + x'(t))w(x'(t)) \tag{2.7}$$

for a.e. $t \in [t_1, \xi]$, and consequently

$$\begin{aligned}\int_0^{g(\|x'\|)} \frac{dt}{w(g^{-1}(t))} &= \int_{g(x'(t_1))}^{g(x'(\xi))} \frac{dt}{w(g^{-1}(t))} = \int_{t_1}^{\xi} \frac{(g(x'(t)))'}{w(x'(t))}\, dt \\ &\leq \int_{t_1}^{\xi} (h(t) + x'(t))\, dt \leq \|h\|_{L_1} + 2\|x\|.\end{aligned} \tag{2.8}$$

Let $\mathcal{B} \cap [0, \xi) = \emptyset$. Then $x'(t) > 0$ on $[0, \xi]$, $x'(T) > 0$ (by (1.2)) and there exists $t_2 \in \mathcal{B}$ such that $x'(t)) > 0$ for $t \in (t_2, T]$. Then (2.7) is satisfied for a.e. $t \in [0, \xi] \cup [t_2, T]$, and so

$$\begin{aligned}\int_0^{g(x'(0))} \frac{dt}{w(g^{-1}(t))} &= \int_{g(x'(t_2))}^{g(x'(T))} \frac{dt}{w(g^{-1}(t))} = \int_{t_2}^{T} \frac{(g(x'(t)))'}{w(x'(t))}\, dt \\ &\leq \int_{t_2}^{T} (h(t) + x'(t))\, dt \leq \|h\|_{L_1} + 2\|x\|, \\ \int_{g(x'(0))}^{g(x'(\xi))} \frac{dt}{w(g^{-1}(t))} &= \int_0^{\xi} \frac{(g(x'(t)))'}{w(x'(t))}\, dt \leq \int_0^{\xi} (h(t) + x'(t))\, dt \leq \|h\|_{L_1} + 2\|x\|.\end{aligned}$$

Thus

$$\begin{aligned}\int_0^{g(\|x'\|)} \frac{dt}{w(g^{-1}(t))} &= \int_0^{g(x'(0))} \frac{dt}{w(g^{-1}(t))} + \int_{g(x'(0))}^{g(x'(\xi))} \frac{dt}{w(g^{-1}(t))} \\ &\leq 2(\|h\|_{L_1} + 2\|x\|).\end{aligned} \tag{2.9}$$

b) Let $\sigma_1 = -1$. Assume $\xi < T$. If $\mathcal{B} \cap (\xi, T] \neq \emptyset$ then there exists $t_3 \in \mathcal{B} \cap (\xi, T]$ such that $x'(t) > 0$ on $[\xi, t_3)$ and (cf. (2.4))

$$(g(x'(t)))' \geq -(h(t) + x'(t))w(x'(t)) \tag{2.10}$$

for a.e. $t \in [\xi, t_3]$, which yields

$$\begin{aligned}\int_0^{g(\|x'\|)} \frac{dt}{w(g^{-1}(t))} &= -\int_{\xi}^{t_3} \frac{(g(x'(t)))'}{w(x'(t))}\, dt \leq \int_{\xi}^{t_3} (h(t) + x'(t))\, dt \\ &\leq \|h\|_{L_1} + 2\|x\|.\end{aligned} \tag{2.11}$$

Let $\mathcal{B} \cap (\xi, T] = \emptyset$. Then $x'(t) > 0$ on $[\xi, T]$, $x'(0) = x'(T) > 0$, and there exists $t_4 \in \mathcal{B}$ such that $x'(t) > 0$ on $[0, t_4)$. Therefore (2.10) is satisfied for a.e. $t \in [0, t_4] \cup [\xi, T]$, and so

$$\begin{aligned} \int_0^{g(x'(0))} \frac{dt}{w(g^{-1}(t))} &= -\int_0^{t_4} \frac{(g(x'(t)))'}{w(x'(t))}\,dt \le \int_0^{t_4} (h(t) + x'(t))\,dt \\ &\le \|h\|_{L_1} + 2\|x\|, \\ \int_{g(x'(0))}^{g(x'(\xi))} \frac{dt}{w(g^{-1}(t))} &= \int_{g(x'(T))}^{g(x'(\xi))} \frac{dt}{w(g^{-1}(t))} = -\int_\xi^T \frac{(g(x'(t)))'}{w(x'(t))}\,dt \\ &\le \int_\xi^T (h(t) + x'(t))\,dt \le \|h\|_{L_1} + 2\|x\|; \end{aligned}$$

hence

$$\begin{aligned} \int_0^{g(\|x'\|)} \frac{dt}{w(g^{-1}(t))} &= \int_0^{g(x'(0))} \frac{dt}{w(g^{-1}(t))}\,dt + \int_{g(x'(0))}^{g(x'(\xi))} \frac{dt}{w(g^{-1}(t))} \\ &\le 2(\|h\|_{L_1} + 2\|x\|). \end{aligned} \tag{2.12}$$

Assume $\xi = T$. Since $x'(0) = x'(T) > 0$, there exists $t_5 \in \mathcal{B}$ such that $x'(t) > 0$ on $[0, t_5)$ and (2.10) is satisfied for a.e. $t \in [0, t_5]$. Consequently

$$\begin{aligned} \int_0^{g(\|x'\|)} \frac{dt}{w(g^{-1}(t))} &= -\int_0^{t_5} \frac{(g(x'(t)))'}{w(x'(t))}\,dt \le \int_0^{t_5} (h(t) + x'(t))\,dt \\ &\le \|h\|_{L_1} + 2\|x\|. \end{aligned} \tag{2.13}$$

Summarizing, we have (cf. (2.8), (2.9) and (2.12)-(2.13))

$$\int_0^{g(\|x\|)} \frac{dt}{w(g^{-1}(t))} \le 2(\|h\|_{L_1} + 2\|x\|) \tag{2.14}$$

provided $x'(\xi) > 0$. Analogously (by (2.4) with $k = 2$) we can prove

$$\int_{g(-\|x\|)}^{0} \frac{dt}{w(g^{-1}(t))} \le 2(\|h\|_{L_1} + 2\|x\|) \tag{2.15}$$

provided $x'(\xi) < 0$. The inequality (2.6) follows immediately from (2.5), (2.14) and (2.15). □

Remark 2 *Lemma 1 is a generation of Lemma 2.5 in [7] which is proved for $g(u) \equiv u$.*

For any number $V \ge 0$ and any function $z : J \to \mathbb{R}$, we define the functions $r_{\rm V} z$ and $z\Big|_{-V}^{V}$ by

$$(r_{\rm V} z)(t) = \begin{cases} z(t) - V & \text{if } z(t) > V \\ 0 & \text{if } |z(t)| \le V \\ z(t) + V & \text{if } z(t) < -V \end{cases} \tag{2.16}$$

and

$$z(t)\Big|_{-V}^{V} = \begin{cases} V & \text{if } z(t) > V \\ z(t) & \text{if } |z(t)| \le V \\ -V & \text{if } z(t) < -V. \end{cases} \tag{2.17}$$

Let k, l, V be positive constants, $\varrho_1 \in L_1(J)$ be a positive-valued function and set

$$\Omega^* = \Big\{(x,a,b): \ (x,a,b) \in C^1(J) \times \mathbb{R}^2, \ \|x\| < V+2, \\ \|x'\| < k, \ |a| < V+2, \ |b| < l\Big\}. \tag{2.18}$$

Define the operator

$$H: [0,1] \times \overline{\Omega}^* \to C^1(J) \times \mathbb{R}^2$$

by

$$H(\lambda, x, a, b) = \Bigg(a + \int_0^t \Big[(1-\lambda)\Big(b + \int_0^s \varrho_1(\nu)(r_{\mathrm{V}} x)(\nu)\, d\nu\Big) \\ + \lambda g^{-1}\Big(b + \int_0^s \varrho_1(\nu)(r_{\mathrm{V}} x)(\nu)\, d\nu\Big)\Big]\, ds, \\ a + x(0) - x(T), \ b + x'(0) - x'(T)\Bigg) \tag{2.19}$$

Lemma 3 $\mathrm{D}(I - H(1,\cdot), \Omega^*, 0) \neq 0$, *where* I *is the identical operator on* $C^1(J) \times \mathbb{R}^2$ *and "D" stands for the Leray-Schauder degree.*

Proof. Since $r_{\mathrm{V}}(-x) = -r_{\mathrm{V}} x$ for $x \in C^1(J)$, we have

$$H(0, -x, -a, -b) = -H(0, x, a, b), \quad (x, a, b) \in \overline{\Omega}^*$$

and so $H(0,\cdot)$ is an odd operator.

Let $H(\lambda_0, x_0, a_0, b_0) = (x_0, a_0, b_0)$ for some $(\lambda_0, x_0, a_0, b_0) \in [0,1] \times \partial\Omega^*$. Then

$$x_0(t) = a_0 + \int_0^t \Big[(1-\lambda_0)\Big(b_0 + \int_0^s \varrho_1(\nu)(r_{\mathrm{V}} x_0)(\nu)\, d\nu\Big) \\ + \lambda_0 g^{-1}\Big(b_0 + \int_0^s \varrho_1(\nu)(r_{\mathrm{V}} x_0)(\nu)\, d\nu\Big)\Big]\, ds, \tag{2.20}$$

$$x_0(0) = x_0(T), \quad x_0'(0) = x_0'(T). \tag{2.21}$$

Assume $\|x_0\| > V$. Then there exists $\xi \in [0, T)$ (see (2.21)) such that $|x_0(\xi)| = \|x_0\|$. Let $x_0(\xi) > V$ (the case $x_0(\xi) < -V$ can be treated analogously). Then $x_0(t) > V$ on

$[\xi, t_1]$ with a $t_1 \in (\xi, T]$. From the definition (2.16) of the function $r_{\rm V} u$, the equalities $x_0'(\xi) = 0$ and (cf. (2.20))

$$\begin{aligned} x_0'(t) &= (1-\lambda_0)\Big(b_0 + \int_0^t \varrho_1(s)(r_{\rm V} x_0)(s)\,ds\Big) \\ &+\lambda_0 g^{-1}\Big(b_0 + \int_0^t \varrho_1(s)(r_{\rm V} x_0)(s)\,ds\Big), \quad t \in J \end{aligned} \tag{2.22}$$

we see that

$$x_0'(t) = x_0'(t) - x_0'(\xi) = (1-\lambda_0)\int_\xi^t \varrho_1(s)(r_{\rm V} x_0)(s)\,ds$$
$$+\lambda_0\Big[g^{-1}\Big(b_0 + \int_0^t \varrho_1(s)(r_{\rm V} x_0)(s)\,ds\Big) - g^{-1}\Big(b_0 + \int_0^\xi \varrho_1(s)(r_{\rm V} x_0)(s)\,ds\Big)\Big] > 0$$

for $t \in (\xi, t_1]$, and consequently x_0 is increasing on $[\xi, t_1]$, a contradiction. Hence $\|x_0\| \le V$ and then (cf. (2.22))

$$x_0'(t) = (1-\lambda_0)b_0 + \lambda_0 g^{-1}(b_0).$$

Since $x_0(0) = x_0(T)$ implies $x_0'(\tau) = 0$ for a $\tau \in (0, T)$, we infer

$$(1-\lambda_0)b_0 + \lambda_0 g^{-1}(b_0) = 0,$$

which is satisfied if and only if $b_0 = 0$. So $x_0(t) = a_0$. We have proved: $\|x_0\| \le V$, $\|x_0'\| = 0$, $|a_0| \le V$ and $b_0 = 0$, a contradiction. Thus

$$H(\lambda, x, a, b) \neq (x, a, b) \quad \text{for } (\lambda, x, a, b) \in [0,1] \times \partial\Omega^*.$$

Let $(\lambda_n, x_n, a_n, b_n) \to (\lambda, x, a, b)$ as $n \to \infty$ in $[0,1]\times C^1(J)\times\mathbb{R}^2$. Then $\lim_{n\to\infty} r_{\rm V} x_n = r_{\rm V} x$ in $C^0(J)$ and we see that

$$\lim_{n\to\infty} H(\lambda_n, x_n, a_n, b_n) = H(\lambda, x, a, b).$$

Hence H is a continuous operator. Let $\{(\lambda_n, x_n, a_n, b_n)\} \subset [0,1] \times \overline{\Omega}^*$ and set $(u_n, A_n, B_n) = H(\lambda_n, x_n, a_n, b_n)$. Then

$$\begin{aligned} u_n(t) = a_n + \int_0^t \Big[&(1-\lambda_n)\Big(b_n + \int_0^s \varrho_1(\nu)(r_{\rm V} x_n)(\nu)\,d\nu\Big) \\ &+\lambda_n g^{-1}\Big(b_n + \int_0^s \varrho_1(\nu)(r_{\rm V} x_n)(\nu)\,d\nu\Big)\Big]\,ds, \end{aligned}$$

$$A_n = a_n + x_n(0) - x_n(T), \quad B_n = b_n + x_n'(0) - x_n'(T)$$

for $t \in J$ and $n \in \mathbb{N}$. It is easily seen that (cf. (2.1), (2.2) and (2.16))

$$|u_n(t)| \le V + 2 + T(l + 2\|\varrho_1\|_{L_1}) + TQ(l + 2\|\varrho_1\|_{L_1}),$$
$$|u_n'(t)| \le l + 2\|\varrho_1\|_{L_1} + Q(l + 2\|\varrho_1\|_{L_1}),$$
$$A_n \le 3(V + 2), \quad |B_n| \le 2k + l$$

for $t \in J$ and $n \in \mathbb{N}$.

Fix $\varepsilon > 0$. Then there exists δ_1, $0 < \delta_1 < \varepsilon$, such that $|g^{-1}(v_1) - g^{-1}(v_2)| < \varepsilon$ for $v_1, v_2 \in [-l - 2\|\varrho\|_{L_1}, l + 2\|\varrho\|_{L_1}]$, $|v_1 - v_2| < \delta_1$. Let $\delta_2 > 0$ be a number such that $\Big|\int_{t_1}^{t_2} \varrho_1(s)\,ds\Big| < \dfrac{\delta_1}{2}$ for $t_1, t_2 \in J$, $|t_1 - t_2| < \delta_2$. Then (for $t_1, t_2 \in J$, $|t_1 - t_2| < \delta_2$)

$$\Big|\int_0^{t_1} \varrho_1(s)(r_{\mathrm{V}} x_n)(s)\,ds - \int_0^{t_2} \varrho_1(s)(r_{\mathrm{V}} x_n)(s)\,ds\Big| \le 2\Big|\int_{t_1}^{t_2} \varrho_1(s)\,ds\Big| < \delta_1,$$

and so

$$|u_n'(t_1) - u_n'(t_2)| \le \Big|\int_{t_1}^{t_2} \varrho_1(s)(r_{\mathrm{V}} x_n)(s)\,ds\Big|$$
$$+\Big|g^{-1}\Big(b_n + \int_0^{t_1} \varrho_1(s)(r_{\mathrm{V}} x_n)(s)\,ds\Big) - g^{-1}\Big(b_n + \int_0^{t_2} \varrho_1(s)(r_{\mathrm{V}} x_n)(s)\,ds\Big)\Big|$$
$$< \delta_1 + \varepsilon < 2\varepsilon,$$

which implies that $\{u_n'(t)\}$ is equicontinuous on J. Now, by the Arzelà-Ascoli theorem and the Bolzano-Weierstrass theorem, there exists a subsequence of $\{(u_n, A_n, B_n)\}$ converging in $C^1(J) \times \mathbb{R}^2$. Hence H is a compact operator. By the Borsuk theorem and the homotopy, $\mathrm{D}(I - H(0,\cdot), \Omega^*, 0) \ne 0$ and $\mathrm{D}(I - H(1,\cdot), \Omega^*, 0) = \mathrm{D}(I - H(0,\cdot), \Omega^*, 0)$. The proof is complete. □

Let $\gamma, \delta \in AC(J)$ and $\gamma(t) \le \delta(t)$ for $t \in J$. For each $u \in AC(J)$, define the truncation function $\Delta_{\gamma\delta} u : J \to \mathbb{R}$, the penalty function $p_{\gamma\delta} u : J \to [-1, 1]$ and the sets $\mathcal{A}_1(\gamma, \delta)$, $\mathcal{A}_2(\gamma, \delta)$, $\mathcal{A}(\gamma, \delta)$ by

$$(\Delta_{\gamma\delta} u)(t) = \begin{cases} \delta(t) & \text{if } u(t) > \delta(t) \\ u(t) & \text{if } \gamma(t) \le u(t) \le \delta(t) \\ \gamma(t) & \text{if } u(t) < \gamma(t), \end{cases} \tag{2.23}$$

$$(p_{\gamma\delta} u)(t) = \begin{cases} 1 & \text{if } u(t) > \delta(t) + 1 \\ u(t) - \delta(t) & \text{if } \delta(t) < u(t) \le \delta(t) + 1 \\ 0 & \text{if } \gamma(t) \le u(t) \le \delta(t) \\ u(t) - \gamma(t) & \text{if } \gamma(t) - 1 \le u(t) < \gamma(t) \\ -1 & \text{if } u(t) < \gamma(t) - 1, \end{cases} \tag{2.24}$$

$$\begin{aligned}\mathcal{A}_1(\gamma,\delta) &= \Big\{(x,y,b):\ (x,y,b)\in C^0(J)\times C^0(J)\times\mathbb{R},\ \gamma(t)\le x(t)\le\delta(t),\\ &\qquad \gamma(t)\le y(t)\le\delta(t) \text{ for } t\in J,\ b>0\Big\},\\ \mathcal{A}_2(\gamma,\delta) &= \Big\{(x,y,b):\ (x,y,b)\in C^0(J)\times C^0(J)\times\mathbb{R},\ \gamma(t)\le x(t)\le\delta(t),\\ &\qquad \gamma(t)\le y(t)\le\delta(t) \text{ for } t\in J,\ b<0\Big\},\\ \mathcal{A}(\gamma,\delta) &= \Big\{(x,y,b):\ (x,y,b)\in C^0(J)\times C^0(J)\times\mathbb{R},\ \gamma(t)\le x(t)\le\delta(t),\\ &\qquad \gamma(t)\le y(t)\le\delta(t) \text{ for } t\in J,\ b\in\mathbb{R}\Big\}.\end{aligned} \tag{2.25}$$

It is easy to check that $(\Delta_{\gamma\delta}u)(t) = \max\{\gamma(t), \min\{\delta(t), u(t)\}\}$ for $t\in J$. Since $\min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$, $\max\{a,b\} = \frac{1}{2}(a+b+|a-b|)$ for $a,b\in\mathbb{R}$, we see that $\Delta_{\gamma\delta} : AC(J)\to AC(J)$ and

$$(\Delta_{\gamma\delta}u)'(t) = \begin{cases} \delta'(t) & \text{if } u(t)>\delta(t)\\ u'(t) & \text{if } \gamma(t)\le u(t)\le\delta(t)\\ \gamma'(t) & \text{if } u(t)<\gamma(t). \end{cases} \tag{2.26}$$

From $(\Delta_{\gamma\delta}u)'\in L_1(J)$ it may be concluded that $(\Delta_{\gamma\delta}u)'\Big|_{-V}^{V}\in L_1(J)$ for any $u\in AC(J)$ and $V\ge 0$.

A function $\alpha\in C^1(J)$ is called *a lower function of PBVP* (1.1), (1.2) if $g(\alpha')$ is absolutely continuous on J,

$$(g(\alpha'(t)))'\ge F(\alpha,\alpha(t),\alpha'(t))(t) \quad \text{for a.e. } t\in J$$

and

$$\alpha(0)=\alpha(T),\quad \alpha'(0)\ge\alpha'(T).$$

An upper function of PBVP (1.1), (1.2) is defined by reversing the above inequalities.

The following assumption will be needed throughout the paper.

(H_1) There exist lower and upper functions α and β of PBVP (1.1), (1.2), respectively, and

$$\alpha(t)\le\beta(t) \quad \text{for } t\in J;$$

(H_2) There exist $\sigma_k\in\{-1,1\}$ $(k=1,2)$ and positive-valued functions $h\in L_1(J)$, $w\in C^0(\mathbb{R})$ such that (2.3) is satisfied, $w(u)$ and $w(-u)$ are nondecreasing on $[0,\infty)$ and

$$\sigma_k F(x,y(t),b)(t)\le (h(t)+|b|)w(b) \tag{2.27}$$

for a.e. $t\in J$ and each $(x,y,b)\in\mathcal{A}_k(\alpha,\beta)$ $(k=1,2)$.

Set

$$V = \max\{\|\alpha\|, \|\beta\|\} \tag{2.28}$$

and let $P \geq V$ be a positive number such that

$$\min\Big\{\int_{g(-P)}^{0} \frac{dt}{w(g^{-1}(t))}, \int_{0}^{g(P)} \frac{dt}{w(g^{-1}(t))}\Big\} > 2(\|h\|_{L_1} + 2V). \tag{2.29}$$

By the property (iv) of F, there exists a positive-valued function $\varrho \in L_1(J)$ such that

$$\Big|F\Big(\Delta_{\alpha\beta}x, (\Delta_{\alpha\beta}x)(t), (\Delta_{\alpha\beta}x)'(t)\Big|_{-P}^{P}\Big)(t)\Big| \leq \varrho(t) \tag{2.30}$$

for a.e. $t \in J$ and each $x \in C^1(J)$.

Consider the one-parameter family of the functional differential equations

$$\begin{aligned}(g(x'(t)))' \quad &= \quad \lambda F\Big(\Delta_{\alpha\beta}x, (\Delta_{\alpha\beta}x)(t), (\Delta_{\alpha\beta}x)'(t)\Big|_{-P}^{P}\Big)(t) \\ &\quad +\lambda(p_{\alpha\beta}x)(t) + \varrho(t)(r_{\mathrm{V}}x)(t)\end{aligned} \tag{2.31}$$

depending on the parameter $\lambda \in [0, 1]$.

Lemma 4 *Let assumptions* (H_1) *and* (H_2) *be satisfied. If* $u(t)$ *is a solution of PBVP (2.31)*$_\lambda$*, (1.2) for some* $\lambda \in [0, 1]$ *then*

$$\|u\| \leq V + 1, \quad \|u'\| \leq Q(2\|\varrho\|_{L_1} + T). \tag{2.32}$$

Proof. Assume $\|u\| = u(\xi) > V + 1$, $\xi \in J$. From (2) we conclude that $u'(\xi) = 0$ and we can assume $\xi \in [0, T)$. Then there is a $t_0 > \xi$ such that $u(t) > V + 1$ for $t \in [\xi, t_0]$, and so (cf. (2.16), (2.24), (2.28) and (2.30))

$$(g(u'(t)))' \geq \lambda(1 - \varrho(t)) + \varrho(t)(u(t) - V) > \lambda(1 - \varrho(t)) + \varrho(t) > 0$$

for a.e. $t \in [\xi, t_0]$. Hence $g(u'(t))$ is increasing on $[\xi, t_0]$ and since $g(u'(\xi)) = 0$ we have $u'(t) > 0$ for $[\xi, t_0]$, a contradiction.

Analogously for $\|u\| = -u(\varepsilon) > V + 1$, $\varepsilon \in J$. Thus $\|u\| \leq V + 1$. Then

$$|(g(u'(t)))'| \leq \lambda(1 + \varrho(t)) + \varrho(t) \leq 2\varrho(t) + 1$$

for a.e. $t \in J$. From the equality $u(0) = u(T)$ it follows that $u'(\nu) = 0$ for a $\nu \in (0, T)$, and consequently

$$|g(u'(t))| = |g(u'(t)) - g(u'(\nu))| \leq \Big|\int_{\nu}^{t} (2\varrho(s) + 1)\, ds\Big| \leq 2\|\varrho\|_{L_1} + T$$

for $t \in J$. Hence (cf. (2.2)) $\|u'\| \leq Q(2\|\varrho\|_{L_1} + T)$. □

3. Existence results

Theorem 5 *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then PBVP (1.1), (1.2) has a solution* $u(t)$ *satisfying the inequalities*

$$\alpha(t) \le u(t) \le \beta(t) \quad \text{for } t \in J. \tag{3.1}$$

Proof. Let V be given by (2.28) and let the constant $P \ge V$ satisfy inequality (2.29). Set

$$\Omega = \Big\{(x,a,b) : (x,a,b) \in C^1(J) \times \mathbb{R}^2,\ \|x\| < V+2,\ \|x'\| < Q(2\|\varrho\|_{L_1} + T) + 1,\ |a| < V+2,\ |b| < G[Q(2\|\varrho\|_{L_1} + T) + 1]\Big\}$$

and $K : [0,1] \times \overline{\Omega} \to C^1(J) \times \mathbb{R}^2$,

$$K(\lambda,x,a,b) = \bigg(a + \int_0^t g^{-1}\Big(b + \int_0^s \Big[\lambda F\Big(\Delta_{\alpha\beta}x, (\Delta_{\alpha\beta}x)(\nu), (\Delta_{\alpha\beta}x)'(\nu)\Big|_{-P}^{P}\Big)(\nu) + \lambda(p_{\alpha\beta}x)(\nu) + \varrho(\nu)(r_{\rm V}x)(\nu)\Big]\,d\nu\Big)\,ds,\ a + x(0) - x(T),\ b + x'(0) - x'(T)\bigg)$$

where $\varrho \in L_1(J)$ is a positive-valued function satisfying (2.30) and the functions G, Q are defined by (2.1).

We now prove that

$$\mathrm{D}(I - K(1,\cdot),\Omega,0) \neq 0. \tag{3.2}$$

Since (cf. (2.19) with $\varrho = \varrho_1$) $K(0,\cdot) = H(1,\cdot)$, Lemma 2 (with $k = Q(2\|\varrho\|_{L_1} + T) + 1$ and $l = G[Q(2\|\varrho\|_{L_1} + T) + 1]$ in Ω^* defined by (2.18)) implies

$$\mathrm{D}(I - K(0,\cdot),\Omega,0) \neq 0. \tag{3.3}$$

Hence to prove (3.2) it is sufficient to verify that

(j) K is a compact operator, and

(jj) $K(\lambda,x,a,b) \neq (x,a,b)$ for $(\lambda,x,a,b) \in [0,1] \times \partial\Omega$.

Let $\lim_{n\to\infty}(\lambda_n,x_n,a_n,b_n) = (\lambda,x,a,b)$ in the Banach space $[0,1] \times C^1(J) \times \mathbb{R}^2$. Then $\lim_{n\to\infty}\Delta_{\alpha\beta}x_n = \Delta_{\alpha\beta}x$ in $C^0(J)$, $\{(\Delta_{\alpha\beta}x_n)'\Big|_{-P}^{P}\} \subset \mathcal{L}^P(J)$,

$$\lim_{n\to\infty}(\Delta_{\alpha\beta}x_n)'(t)\Big|_{-P}^{P} = (\Delta_{\alpha\beta}x)'(t)\Big|_{-P}^{P} \quad \text{a.e. on } J$$

$\Big(\Rightarrow (\Delta_{\alpha\beta}x_n)'\Big|_{-P}^{P} \to (\Delta_{\alpha\beta}x)'\Big|_{-P}^{P}$ in $\mathcal{L}^P(J)$ as $n\to\infty\Big)$, $\lim_{n\to\infty} p_{\alpha\beta}x_n = p_{\alpha\beta}x$ and $\lim_{n\to\infty} r_{\rm V}x_n = r_{\rm V}x$ in $C^0(J)$, and consequently (see property (ii) of F)

$$\lim_{n\to\infty} K(\lambda_n, x_n, a_n, b_n) = K(\lambda, x, a, b) in C^1(J)\times\mathbb{R}^2$$

which shows that K is a continuous operator.

Let $\{(\lambda_n, x_n, a_n, b_n)\}\subset[0,1]\times\overline{\Omega}$ and set

$$(u_n, A_n, B_n) = K(\lambda_n, x_n, a_n, b_n),\quad n\in\mathbb{N}.$$

Then

$$u_n(t) = a_n + \int_0^t g^{-1}\Big(b_n + \int_0^s \Big[\lambda_n F\Big(\Delta_{\alpha\beta}x_n, (\Delta_{\alpha\beta}x_n)(\nu), (\Delta_{\alpha\beta}x_n)'(\nu)\Big|_{-P}^{P}\Big)(\nu) + \lambda_n(p_{\alpha\beta}x_n)(\nu) + \varrho(\nu)(r_{\rm V}x_n)(\nu)\Big]\,d\nu\Big)\,ds,$$

$$A_n = a_n + x_n(0) - x_n(T),\quad B_n = b_n + x_n'(0) - x_n'(T),$$

and consequently

$$|u_n(t)| \le V + 2 + TQ\Big(G[Q(2\|\varrho\|_{L_1}+T)+1] + 3\|\varrho\|_{L_1} + 1\Big),$$
$$|u_n'(t)| \le Q\Big(G(Q(2\|\varrho\|_{L_1}+T)+1) + 3\|\varrho\|_{L_1} + 1\Big),$$
$$|g(u_n'(t_1)) - g(u_n'(t_2))| \le 3\Big|\int_{t_1}^{t_2}\varrho(s)\,ds\Big| + |t_1 - t_2|$$

for $t, t_1, t_2\in J$ and $n\in\mathbb{N}$. Then $\{u_n\}$ is bounded in $C^1(J)$, $\{u_n'(t)\}$ is equicontinuous on J since g is continuous and increasing on $\mathbb{R}$ and $\{A_n\}$, $\{B_n\}$ are bounded (in $\mathbb{R}$), and so there exists a subsequence $\{(u_{k_n}, A_{k_n}, B_{k_n})\}$ converging in $C^1(J)\times\mathbb{R}^2$. Thus K is a compact operator.

Assume $K(\lambda_0, x_0, a_0, b_0) = (x_0, a_0, b_0)$ for some $(\lambda_0, x_0, a_0, b_0)\in[0,1]\times\partial\Omega$. Then $x_0(t)$ is a solution of PBVP $(2.31)_{\lambda_0}, (1.2)$ and $a_0 = x_0(0)$, $b_0 = g(x_0'(0))$. By Lemma 3, $\|x_0\|\le V+1$, $\|x_0'\|\le Q(2\|\varrho\|_{L_1}+T)$, and so

$$|a_0|\le V+1,\quad |b_0|\le G[Q(2\|\varrho\|_{L_1}+T)]$$

which contradicts $(x_0, a_0, b_0)\in\partial\Omega$. We have proved that K satisfies (j) and (jj); hence (3.2) holds. Then there exists a solution $u(t)$ of PBVP $(2.31)_1, (1.2)$, that is

$$(g(u'(t)))' = F\Big(\Delta_{\alpha\beta}u, (\Delta_{\alpha\beta}u)(t), (\Delta_{\alpha\beta}u)'(t)\Big|_{-P}^{P}\Big)(t) + (p_{\alpha\beta}u)(t) + \varrho(t)(r_{\rm V}u)(t) \tag{3.4}$$

for a.e. $t\in J$ and

$$u(0) = u(T),\quad u'(0) = u'(T). \tag{3.5}$$

Set $w(t) = u(t) - \beta(t)$ for $t \in J$. Then

$$w(0) = w(T), \quad w'(0) \geq w'(T). \tag{3.6}$$

Let $\max\{w(t) : t \in J\} = w(\xi) > 0$ for some $\xi \in J$. Then (cf. (3.6)) $w'(\xi) = 0$ and without loss of generality we can assume $\xi \in [0, T)$. Since on any interval of the type $[\xi, t_1] \subset J$ where $w(t) > 0$ we have (see property (iii) of F)

$$\begin{aligned}(g(u'(t)))' - (g(\beta'(t)))' \geq F\Big(\Delta_{\alpha\beta}u, (\Delta_{\alpha\beta}u)(t), (\Delta_{\alpha\beta}u)'(t)\Big|_{-P}^{P}\Big)(t) \\ +(p_{\alpha\beta}u)(t) + \varrho(t)(r_{\mathrm{V}}u)(t) - F(\beta, \beta(t), \beta'(t))(t) \\ \geq F(\beta, \beta(t), \beta'(t))(t) + \min\{w(t), 1\} - F(\beta, \beta(t), \beta'(t))(t) > 0,\end{aligned}$$

and so from the equality $g(u'(t)) - g(\beta'(t)) = \displaystyle\int_\xi^t (g(u'(s)) - g(\beta'(s)))'\,ds$ we deduce $w'(t) > 0$ on a right neighbourhood of $t = \xi$, a contradiction. Hence $w(t) \leq 0$ on J and $u(t) \leq \beta(t)$ for $t \in J$.

Set $z(t) = u(t) - \alpha(t)$, $t \in J$. Then

$$z(0) = z(T), \quad z'(0) \leq z'(T). \tag{3.7}$$

Let $\min\{z(t) : t \in J\} = z(\nu) < 0$ with a $\nu \in J$. Then (cf. (3.7)) $z'(\nu) = 0$ and without loss of generality we can assume that $\nu \in [0, T)$. Hence there exists $t_1 \in (\nu, T]$ such that $z(t) < 0$ on $[\nu, t_1]$. Then

$$\begin{aligned}(g(u'(t)))' - (g(\alpha'(t)))' \leq F(\alpha, \alpha(t), \alpha'(t))(t) + (p_{\alpha\beta}u)(t) + \varrho(t)(r_{\mathrm{V}}u)(t) \\ -F(\alpha, \alpha(t), \alpha'(t))(t) \leq (p_{\alpha\beta}u)(t) = \max\{z(t), -1\} < 0\end{aligned}$$

for $t \in [\nu, t_1]$, and consequently $g(u'(t)) - g(\alpha'(t)) < 0$ on $(\nu, t_1]$ since $g(u'(\nu)) - g(\alpha'(\nu)) = 0$. Then $z'(t) = u'(t) - \alpha'(t) < 0$ for $t \in (\nu, t_1]$, a contradiction. Hence $z(t) \geq 0$ on J.

From (3.1) now it follows

$$(g(u'(t)))' = F\Big(u, u(t), u'(t)\Big|_{-P}^{P}\Big)(t) \quad \text{for a.e. } t \in J.$$

Then (cf. (H_2))

$$\sigma_k(g(u'(t)))' \leq \Big(h(t) + |u'(t)|\Big|_0^P\Big) w\Big(u'(t)\Big|_{-P}^{P}\Big) \leq (h(t) + |u'(t)|) w(u'(t))$$

for a.e. $t \in \mathcal{C}_k$ $(k = 1, 2)$, where $\mathcal{C}_k$ is defined in Lemma 1 (with $x = u$). Applying Lemma 1 and using (2.29) and $\|u\| \leq V$, we see that $\|u'\| \leq P$. Thus $(g(u'(t)))' = F(u, u(t), u'(t))(t)$ and u is a solution of PBVP (1.1), (1.2) satisfying the inequalities (3.1). □

If assumptions (H_1) and (H_2) are satisfied then there exists a solution $u(t)$ of PBVP $(1.1),(1.2)$ in the set

$$\mathcal{D}_{\alpha\beta} = \{x : x \text{ is a solution of PBVP } (1),(2) \text{ and } \alpha(t) \le x(t) \le \beta(t) \text{ for } t \in J\},$$

by Theorem 1.

We say that solutions $u_*(t)$ and $u^*(t)$ are respectively *lower and upper solutions of PBVP* $(1.1),(1.2)$ *in the set* $\mathcal{D}_{\alpha\beta}$ if $u_*, u^* \in \mathcal{D}_{\alpha\beta}$ and

$$u_*(t) \le u(t) \le u^*(t), \quad t \in J$$

for any $u \in \mathcal{D}_{\alpha\beta}$.

Theorem 6 *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then there exist lower and upper solutions of PBVP* (1), (2) *in the set* $\mathcal{D}_{\alpha\beta}$.

Proof. By Theorem 1, $\mathcal{D}_{\alpha\beta} \neq \emptyset$. Let V be given by (2.28) and $P \ge V$ satisfies (2.29). From the assumption (H_2) and applying Lemma 1 we conclude (see the proof of Theorem 1) that $\|u'\| \le P$ for every $u \in \mathcal{D}_{\alpha\beta}$. Let $u_1, u_2 \in \mathcal{D}_{\alpha\beta}$ and set

$$w_-(t) = \min\{u_1(t), u_2(t)\}, \quad w_+(t) = \max\{u_1(t), u_2(t\}, \quad t \in J.$$

Then $\alpha(t) \le w_-(t) \le w_+(t) \le \beta(t)$ on J, $w_-, w_+ \in AC(J)$ and $\|u_1'\| \le P$, $\|u_2'\| \le P$. We now show that there exist $u, v \in \mathcal{D}_{\alpha\beta}$ satisfying the inequalities

$$u(t) \le w_-(t), \quad w_+(t) \le v(t), \quad t \in J. \tag{3.8}$$

Consider the one-parameter family of the functional differential equations

$$(g(x'(t)))' = \lambda(S_+x)(t) + \lambda(p_{w_+\beta}x)(t) + 5\varrho(t)(r_{\rm V}x)(t), \quad \lambda \in [0,1], \tag{3.9}$$

where

$$S_+ : C^1(J) \to L_1(J),$$

$$\begin{aligned}(S_+x)(t) \;\; &= \;\; F\Big(\Delta_{w_+\beta}x, (\Delta_{w_+\beta}x)(t), (\Delta_{w_+\beta}x)'(t)\Big|_{-P}^{P}\Big)(t)\\ &- \;\; \Big|F\Big(\Delta_{w_+\beta}x, (\Delta_{w_+\beta}x)(t), (\Delta_{w_+\beta}x)'(t)\Big|_{-P}^{P}\Big)(t)\\ &\quad -F\Big(\Delta_{w_+\beta}x, (\Delta_{u_2\beta}x)(t), (\Delta_{u_2\beta}x)'(t)\Big|_{-P}^{P}\Big)(t)\Big|\\ &- \;\; \Big|F\Big(\Delta_{w_+\beta}x, (\Delta_{w_+\beta}x)(t), (\Delta_{w_+\beta}x)'(t)\Big|_{-P}^{P}\Big)(t)\\ &\quad -F\Big(\Delta_{w_+\beta}x, (\Delta_{u_1\beta}x)(t), (\Delta_{u_1\beta}x)'(t)\Big|_{-P}^{P}\Big)(t)\Big|\end{aligned}$$

and $\varrho \in L_1(J)$ is a positive-valued function satisfying (2.30) for a.e. $t \in J$ and each $x \in C^1(J)$.

Let

$$\Omega = \Big\{(x,a,b) :\, (x,a,b) \in C^1(J) \times \mathbb{R}^2, \|x\| < V+2,$$
$$\|x'\| < Q(10\|\varrho\|_{L_1} + T) + 1,\, |a| < V+2,$$
$$|b| < G[Q(10\|\varrho\|_{L_1} + T) + 1]\Big\},$$

and $L_+ : [0,1] \times \overline{\Omega} \to C^1(J) \times \mathbb{R}^2$ be defined by the formula

$$L_+(\lambda, x, a, b) =$$
$$= \Bigg(a + \int_0^t g^{-1}\Big(b + \int_0^s \Big[\lambda(S_+x)(\nu) + \lambda(p_{w_+\beta}x)(\nu) + 5\varrho(\nu)(r_{\mathrm{V}}x)(\nu)\Big]\, d\nu\Big)\, ds,$$
$$a + x(0) - x(T),\, b + x'(0) - x'(T)\Bigg).$$

Here the function Q and G are defined by (2.1). We see that (cf. (2.19)) $L_+(0,\cdot) = H(1,\cdot)$ (with $\varrho_1 = 5\varrho$ in (2.19) and $k = Q(10\|\varrho\|_{L_1} + T) + 1$, $l = G[Q(10\|\varrho\|_{L_1} + T) + 1]$ in Ω^* given by (2.18)), and so

$$\mathrm{D}(I - L_+(0,\cdot), \Omega, 0) \neq 0$$

by Lemma 2. Let $L_+(\lambda_0, x_0, a_0, b_0) = (x_0, a_0, b_0)$ for some $(\lambda_0, x_0, a_0, b_0) \in [0,1] \times \partial\Omega$. Then $a_0 = x_0(0)$, $b_0 = g(x_0'(0))$ and $x_0(t)$ is a solution of PBVP $(3.9)_{\lambda_0}$, (1.2). Since (cf. (2.30)) $|(S_+x)(t)| \leq 5\varrho(t)$ for a.e. $t \in J$ and each $x \in C^1(J)$, we can proceed analogously to the proof of Lemma 3 to verify that

$$\|x_0\| \leq V + 1, \quad \|x_0'\| \leq Q(10\|\varrho\|_{L_1} + T),$$

and so $|a_0| \leq V + 1$, $|b_0| \leq G(Q(10\|\varrho\|_{L_1} + T))$. Hence $(x_0, a_0, b_0) \notin \partial\Omega$, which is impossible.

Analysis similar to that of the proof of Theorem 1 shows that L_+ is a compact operator. Thus

$$\mathrm{D}(I - L_+(1,\cdot), \Omega, 0) = \mathrm{D}(I - L_+(0,\cdot), \Omega, 0) \neq 0$$

and then, of course, there exists a fixed point (v,a,b) of the operator $L_+(1,\cdot)$. Clearly, $v(t)$ is a solution of PBVP $(3.9)_1$, (1.2). Assume $\max\{v(t) - \beta(t) : t \in J\} = v(\xi) - \beta(\xi) > 0$ for some $\xi \in J$. Then $v'(\xi) = \beta'(\xi)$ and we can assume $\xi \in [0,T)$. Let

$t_1 \in (\xi, T]$ be a point such that $v(t) - \beta(t) > 0$ for $t \in [\xi, t_1]$. Then (for a.e. $t \in [\xi, t_1]$)

$$\begin{aligned} (g(v'(t)))' - (g(\beta'(t)))' \geq \\ \geq (S_+v)(t) + (p_{w_+\beta}v)(t) + 5\varrho(t)(r_{\mathrm{V}}v)(t) - F(\beta, \beta(t), \beta'(t))(t) \\ = F\Big(\Delta_{w_+\beta}v, (\Delta_{w_+\beta}v)(t), (\Delta_{w_+\beta}v)'(t)\Big|_{-P}^{P}\Big)(t) + (p_{w_+\beta}v)(t) \\ +5\varrho(t)(r_{\mathrm{V}}v)(t) - F(\beta, \beta(t), \beta'(t))(t) \\ > F(\Delta_{w_+\beta}v, \beta(t), \beta'(t))(t) - F(\beta, \beta(t), \beta'(t))(t) \geq 0 \end{aligned} \tag{3.10}$$

since $F(\Delta_{w_+\beta}v, \beta(t), \beta'(t))(t) \geq F(\beta, \beta(t), \beta'(t))(t)$ for a.e. $t \in J$ by property (iii) of F. From (3.10) and the equality $v'(\xi) = \beta'(\xi)$ we deduce $v'(t) > \beta'(t)$ for $t \in (\xi, t_1]$, a contradiction. Hence

$$v(t) \leq \beta(t), \quad t \in J.$$

Assume $\min\{v(t) - u_k(t) : t \in J\} < 0$ for some $k \in \{1, 2\}$. Then there exist $\tau_k \in [0, T)$ and $t_k \in (\tau_k, T]$ such that $v(\tau_k) - u_k(\tau_k) = \min\{v(t) - u_k(t) : t \in J\}$, $v'(\tau_k) = u_k'(\tau_k)$ and $v(t) < u_k(t)$ for $t \in (\tau_k, t_k]$. Then (for a.e. $t \in [\tau_k, t_k]$)

$$\begin{aligned} (g(v'(t)))' - (g(u_k'(t)))' = \\ = (S_+v)(t) + (p_{w_+\beta}v)(t) + 5\varrho(t)(r_{\mathrm{V}}v)(t) - F(u_k, u_k(t), u_k'(t))(t) \\ < F(\Delta_{w_+\beta}v, w_+(t), w_+'(t))(t) - F(u_k, u_k(t), u_k'(t))(t) \\ -\Big|F(\Delta_{w_+\beta}v, w_+(t), w_+'(t))(t) - F(\Delta_{w_+\beta}v, u_k(t), u_k'(t))(t)\Big| \\ \leq F(\Delta_{w_+\beta}v, w_+(t), w_+'(t))(t) - F(\Delta_{w_+\beta}v, u_k(t), u_k'(t))(t) \\ -\Big|F(\Delta_{w_+\beta}v, w_+(t), w_+'(t))(t) - F(\Delta_{w_+\beta}v, u_k(t), u_k'(t))(t)\Big| \leq 0, \end{aligned}$$

and consequently $v'(t) < u_k'(t)$ on $(\tau_k, t_k]$, a contradiction. Hence

$$v(t) \geq u_k(t), \quad t \in J \ (k = 1, 2).$$

We have verified that $w_+(t) \leq v(t) \leq \beta(t)$ for $t \in J$. Then $\|v\| \leq V$ and

$$(g(v'(t)))' = (S_+v)(t) + (p_{w_+\beta}v)(t) + 5\varrho(t)(r_{\mathrm{V}}v)(t) = F\Big(v, v(t), v'(t)\Big|_{-P}^{P}\Big)(t)$$

for a.e. $t \in J$. As in the proof of Theorem 1 we can show that $\|v'\| \leq P$. Hence v is a solution of PBVP (1), (2).

To prove the existence of a solution $u(t)$ of PBVP (1), (2) satisfying the inequalities $\alpha(t) \leq u(t) \leq w_-(t)$, $t \in J$, we consider the operator $L_- : [0, 1] \times \overline{\Omega} \to C^1(J) \times \mathbb{R}^2$ given by the formula

$$\begin{aligned} L_-(\lambda, x, a, b) = \\ = \Big(a + \int_0^t g^{-1}(b + \int_0^s [\lambda(S_-x)(\nu) + \lambda(p_{\alpha w_-}x)(\nu) + 5\varrho(\nu)(r_{\mathrm{V}}x)(\nu)]\, d\nu)\, ds, \\ a + x(0) - x(T),\ b + x'(0) - x'(T)\Big). \end{aligned}$$

where $S_- : C^1(J) \to L_1(J)$,

$$\begin{aligned}
(S_-x)(t) &= F\Big(\Delta_{\alpha w_-}x, (\Delta_{\alpha w_-}x)(t), (\Delta_{\alpha w_-}x)'(t)\Big|_{-P}^{P}\Big)(t) \\
&+ \Big|F\Big(\Delta_{\alpha w_-}x, (\Delta_{\alpha w_-}x)(t), (\Delta_{\alpha w_-}x)'(t)\Big|_{-P}^{P}\Big)(t) \\
&\quad -F\Big(\Delta_{\alpha w_-}x, (\Delta_{\alpha u_2}x)(t), (\Delta_{\alpha u_2}x)'(t)\Big|_{-P}^{P}\Big)(t)\Big| \\
&+ \Big|F\Big(\Delta_{\alpha w_-}x, (\Delta_{\alpha w_-}x)(t), (\Delta_{\alpha w_-}x)'(t)\Big|_{-P}^{P}\Big)(t) \\
&\quad -F\Big(\Delta_{\alpha w_-}x, (\Delta_{\alpha u_1}x)(t), (\Delta_{\alpha u_1}x)'(t)\Big|_{-P}^{P}\Big)(t)\Big|.
\end{aligned}$$

The proof is similar to that of the first part of this theorem and therefore it is omitted.

It is easy to check that $\mathcal{D}_{\alpha\beta}$ is a closed set in $C^1(J)$. Since for each $x \in \mathcal{D}_{\alpha\beta}$ and $t_1, t_2 \in J$ we have

$$\|x\| \le V, \quad \|x'\| \le P, \quad |g(x'(t_1)) - g(x'(t_2))| \le \Big|\int_{t_1}^{t_2} \varrho(s)\,ds\Big|,$$

$\mathcal{D}_{\alpha\beta}$ is a compact set in $C^1(J)$. We shall prove only the existence of the maximal solution u^* of PBVP (1.1), (1.2) in the set $\mathcal{D}_{\alpha\beta}$ since the case of the minimal solution u_* of PBVP (1.1), (1.2) in $\mathcal{D}_{\alpha\beta}$ is very similar. By the compactness of $\mathcal{D}_{\alpha\beta}$, there exists $u^* \in \mathcal{D}_{\alpha\beta}$ such that

$$\int_0^T u^*(t)\,dt \ge \int_0^T x(t)\,dt \quad \text{for } x \in \mathcal{D}_{\alpha\beta}. \tag{3.11}$$

Suppose that there exist $x_0 \in \mathcal{D}_{\alpha\beta}$ and $t_0 \in (0,T)$ such that $x_0(t_0) > u^*(t_0)$. By the first part of the proof of our theorem, there exists $x_1 \in \mathcal{D}_{\alpha\beta}$ such that

$$\max\{u^*(t),\, x_0(t)\} \le x_1(t) \le \beta(t), \quad t \in J.$$

Then

$$\int_0^T x_1(t)\,dt > \int_0^T u^*(t)\,dt,$$

contrary to (3.11). Hence u^* is the maximal solution of PBVP (1.1), (1.2) in the set $\mathcal{D}_{\alpha\beta}$. □

Corollary 7 *Let assumption* (H_1) *be satisfied and let there exist positive-valued functions* $h \in L_1(J)$ *and* $w \in C^0(J)$ *such that* (5) *is satisfied and* $w(u)$, $w(-u)$ *are nondecreasing on* $[0,\infty)$. *If at least one of the following inequalities*

$$\begin{aligned}
F(x, y(t), b)(t) &\le (h(t) + |b|)w(b), &(3.12)\\
F(x, y(t), b)(t) &\ge -(h(t) + |b|)w(b), &(3.13)\\
F(x, y(t), b)(t)\,\mathrm{sign}\, b &\le (h(t) + |b|)w(b) &(3.14)
\end{aligned}$$

and

$$F(x, y(t), b)(t) \operatorname{sign} b \geq -(h(t) + |b|)w(b) \tag{3.15}$$

is satisfied for a.e. $t \in J$ and each $(x, y, b) \in \mathcal{A}(\alpha, \beta)$, then PBVP (1.1), (1.2) is solvable and there exist lower and upper solutions in the set $\mathcal{D}_{\alpha\beta}$.

Proof. Corollary 1 follows at once from Theorems 1 and 2 assuming that (H_2) $\sigma_1 = \sigma_2 = 1$ for (3.12), $\sigma_1 = \sigma_2 = -1$ for (3.13), $\sigma_1 = -\sigma_2 = 1$ for (3.14) and $\sigma_1 = -\sigma_2 = -1$ for (3.15). □

Example 8 *Consider the functional differential equation*

$$\begin{aligned}(|x'(t)|^{p-2}x'(t))' = \int_{\frac{t}{2}}^{t} f_1(s, x(T-s), x(t), x'(t))\, ds \\ + f_2(t, x(t), x'(t))(x(t))^{2m-1} + \sigma f_3(t, x(t), x'(t))(x'(t))^n\end{aligned} \tag{3.16}$$

where $p > 1$, f_1 (resp. f_2, f_3) satisfies the local Carathéodory conditions on $J \times \mathbb{R}^3$ (resp. $J \times \mathbb{R}^2$), $f_1(t, \cdot, a, b)$ is nonincreasing on $\mathbb{R}$ for a.e. $t \in J$ and each $a, b \in \mathbb{R}$, $m, n \in \mathbb{N}$ and $\sigma \in \{-1, 1\}$. In addition, there exist positive constants ε and l such that the inequalities

$$|f_1(t, u, x, y)| \leq \frac{2l}{T}(1 + |y|^p), \quad \varepsilon \leq f_2(t, x, y) \leq l(1 + |y|^p), \quad f_3(t, x, y) \geq 0$$

are satisfied for a.e. $t \in J$ and each $u, x \in \left[-\sqrt[2m-1]{\frac{l}{\varepsilon}}, \sqrt[2m-1]{\frac{l}{\varepsilon}}\right]$, $y \in \mathbb{R}$. Set

$$\alpha = -\sqrt[2m-1]{\frac{l}{\varepsilon}} \quad \text{and} \quad \beta = \sqrt[2m-1]{\frac{l}{\varepsilon}}.$$

Then $\alpha(t) \equiv \alpha$ and $\beta(t) \equiv \beta$ are lower and upper functions of PBVP (3.16), (1.2) and

$$\begin{aligned}F(x, y(t), z(t))(t) = \int_{\frac{t}{2}}^{t} f_1(s, x(T-s), y(t), z(t))\, ds \\ + f_2(t, y(t), z(t))(y(t))^{2m-1} + \sigma f_3(t, y(t), z(t))(z(t))^n\end{aligned}$$

satisfies assumption (H_2) with $h(t) = 1$, $\omega(y) = l(1 + \frac{l}{\varepsilon})(1 + |y|^{1-p})$, $g(u) = |u|^{p-2}u$ and

$$\sigma_k = \begin{cases} -\sigma & \text{if } n \text{ is even} \\ (-1)^k \sigma & \text{if } n \text{ is odd.} \end{cases}$$

Theorems 1 and 2 show that PBVP (3.16), (1.2) is solvable and has upper and lower solutions in the set $\mathcal{M}_{\alpha\beta} = \{x : \ x \in C^1(J),\ \alpha \leq x(t) \leq \beta \text{ for } t \in J\}$.

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