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ON SOLVABILITY OF SECOND-ORDER EVOLUTION INCLUSIONS WITH VOLTERRA TYPE OPERATORS

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This paper is dedicated to the memory of the Corresponding Member of the National Academy of Sciences of Ukraine, Professor Valeriy S. Mel'nik.

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Abstract. We consider second-order differential-operators inclusions with Volterra type operators. The problem of the existence of solutions of the Cauchy problem for the given inclusions is investigated. Important *a priori* estimates are obtained. An example illustrating the approach is given.

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Keywords: second-order evolution inclusion, Volterra type operator, pseudomonotone map

1. Introduction

The progress in the investigation of non-linear boundary problems for partial differential equations became possible thanks to the intense development of the methods of non-linear analysis which had found their application in various parts of mathematics. It has recently become natural to reduce these problems to the study of non-linear operator and differential-operator equations and inclusions in functional spaces. Within such an approach, the results for concrete systems are obtained as rather simple consequences of operator theorems [2, 10].

The evolution differential equations and inclusions are studied rather actively. To prove the properties of the resolving operator (non-emptiness, compactness, connectedness), the method of monotony, method of compactness, and their combinations are often used.

In the present work, we study the solvability of the evolution inclusion with multi-valued non-coercive maps

$$y'' + A(y') + B(y) \ni f,$$

which is important for applications.

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Recent related investigations concern a class of problems with a strongly monotone operator A and multi-valued operator B that can be presented as the sum of a single-valued linear self-conjugated monotone operator and a multi-valued demiclosed bounded operator. These problems are coercive. They were considered, e. g., by Papageorgiou and Yannakakis [13, 14]. More particular cases of evolution inclusions were studied by Ahmed and Kerbal [1], Gasiński and Smołka [3], Kartsatos and Markov [4], Migórski [12], and other authors.

Our goal here is to extend the approach indicated to a wider class of problems, namely, to problems with a multi-valued non-coercive non-monotone operator A and a multi-valued operator B satisfying similar conditions.

The idea of passing to subsequences in the classical definition of a single-valued pseudomonotone operator was suggested by Skrypnik [15]. It was developed for the first order differential-operator equations and inclusions in infinite-dimensional spaces with $+$-coercive W_{λ_0}-pseudomonotone maps by Mel'nik, Zgurovskii, and Novikov [11, 18, 19] and Kas'yanov [5–8]. This gave one the possibility to investigate a substantially wider class of problems arising in applications. In particular, this methodology, combined with the non-coercive theory [2, 9, 18], which we apply to the second-order evolution inclusions, allows one to sufficiently extend the class of problems with multi-valued maps for which we can obtain the solvability. Since the operators are multi-valued, such extension faced with considerable difficulties which are not typical for the differential-operator equations. Here, the proof of the solvability is based on the method of singular perturbations [9, 10] and allows us to obtain important a priori estimates for solutions. It makes possible to study properties for the obtained solutions (e. g., dynamics). As an example illustrating the suggested approach, we consider a class of problems with non-linear operators. The obtained results are new for both inclusions and equations.

We note that the solvability of second-order differential-operator equations was investigated by the authors in [16, 17].

2. Problem setting

Let H be a real Hilbert space with the inner product $(\cdot,\cdot)$, and let $(V_1, \|\cdot\|_{V_1})$ and $(V_2, \|\cdot\|_{V_2})$ be some real reflexive separable Banach spaces continuously embedded into H and such that

$$V := V_1 \cap V_2$$

is dense in the spaces V_1, V_2, and H. We assume that one of the embeddings $V_i \subset H$, $i = 1, 2$, is compact. In what follows, the space topologically conjugate to H (with respect to the bilinear form $(\cdot,\cdot)$) is identified with H. Then we have

$$V_i \subset H \subset V_i^* \qquad (i = 1, 2)$$

with continuous and dense embeddings, where $(V_i^*, \|\cdot\|_{V_i^*})$, $i = 1,2$, is the space topologically conjugate to V_i, $i = 1,2$, with respect to the canonical bilinear form

$$\langle\cdot,\cdot\rangle_{V_i}: V_i^* \times V_i \to \mathbb{R} \qquad (i = 1,2)$$

that coincides on $H \times V$ with the inner product $(\cdot,\cdot)$ in H. Let us consider the reflexive function spaces $Y = L_2(S;H)$ and

$$X_i := L_{r_i}(S;H) \cap L_{p_i}(S;V_i) \qquad (i = 1,2)$$

with

$$\|y\|_{X_i} := \|y\|_{L_{p_i}(S;V_i)} + \|y\|_{L_{r_i}(S;H)} \qquad (i = 1,2),$$

where $S := [0,T]$, $1 < p_i \le r_i < +\infty$, $i = 1,2$, and $\max\{r_1; r_2\} \ge 2$.

Let us consider the reflexive (it follows from [2, Chapter 1]) Banach space $X := X_1 \cap X_2$ with the norm $\|y\|_X := \|y\|_{X_1} + \|y\|_{X_2}$. We note that the space X is continuously and densely embedded in Y.

We identify $L_{q_i}(S;V_i^*) + L_{r_i'}(S;H)$ with X_i^*. Similarly, $Y^* \equiv Y$ and

$$X^* = X_1^* + X_2^* \equiv L_{q_1}(S;V_1^*) + L_{q_2}(S;V_2^*) + L_{r_1'}(S;H) + L_{r_2'}(S;H),$$

where $r_i^{-1} + r_i'^{-1} = p_i^{-1} + q_i^{-1} = 1$.

Let $A, B: X \rightrightarrows X^*$ be strict multi-valued maps. We consider the Cauchy problem for the differential-operator inclusion with non-coercive multi-valued maps of W_{λ_0}-pseudomonotone type

$$\begin{cases} y'' + Ay' + By \ni f, \\ y(0) = a_0,\ y'(0) = \bar{0},\ y \in C(S;V),\ y' \in C(S;H), \end{cases} \tag{2.1}$$

where $a_0 \in V$ and $f \in X^*$ are fixed.

On $X^* \times X$ we consider the pairing

$$\begin{aligned} \langle f, y\rangle &= \int_S (f_{11}(\tau), y(\tau))_H \, d\tau + \int_S (f_{12}(\tau), y(\tau))_H \, d\tau \\ &\quad + \int_S \langle f_{21}(\tau), y(\tau)\rangle_{V_1} \, d\tau + \int_S \langle f_{22}(\tau), y(\tau)\rangle_{V_2} \, d\tau \\ &= \int_S (f(\tau), y(\tau)) \, d\tau, \end{aligned}$$

where $f = f_{11} + f_{12} + f_{21} + f_{22}$, $f_{1i} \in L_{r_i'}(S;H)$, and $f_{2i} \in L_{q_i}(S;V_i^*)$. Note that, for any $f \in X^*$,

$$\|f\|_{X^*} = \inf_{\substack{f = f_{11}+f_{12}+f_{21}+f_{22}: \\ f_{1i} \in L_{r_i'}(S;H),\ f_{2i} \in L_{q_i}(S;V_i^*)\ (i=1,2)}} \varphi(f_{11}, f_{12}, f_{21}, f_{22}),$$

where

$$\varphi(f_{11}, f_{12}, f_{21}, f_{22}) = \max\left\{\|f_{11}\|_{L_{r_1'}(S;H)}, \|f_{12}\|_{L_{r_2'}(S;H)}, \|f_{21}\|_{L_{q_1}(S;V_1^*)}, \|f_{22}\|_{L_{q_2}(S;V_2^*)}\right\}.$$

Moreover, let

$$W = \{y \in X \mid y' \in X^*\}$$

and $\|y\|_W = \|y\|_X + \|y'\|_{X^*}$ for all $y \in W$, where the derivative y' of the element $y \in X$ is considered in the sense of scalar distribution space $\mathcal{D}^*(S; V^*) = \mathcal{L}(\mathcal{D}(S); V_w^*)$ with $V = V_1 \cap V_2$ and $V_w^* = (V^*, \sigma(V^*, V))$ [2]. We note that W is a reflexive Banach space with a compact embedding $W \subset Y$ [10].

3. Classes of maps

Let Y be a reflexive Banach space, Y^* be its topologically conjugated space, $\langle\cdot,\cdot\rangle_Y : Y^* \times Y \to \mathbb{R}$ be the pairing, and $A: Y \rightrightarrows Y^*$ be a strict multi-valued map. Let us define its upper support function $[A(y), w]_+ := \sup_{d \in A(y)} \langle d, w\rangle_Y$ and lower support function $[A(y), w]_- := \inf_{d\in A(y)} \langle d, w\rangle_Y$, where $y, w \in Y$, and its upper norm $\|A(y)\|_+ := \sup_{d\in A(y)} \|d\|_{Y^*}$ and lower norm $\|A(y)\|_- := \inf_{d\in A(y)} \|d\|_{Y^*}$. Consider the associated maps $\mathrm{co}A: Y \rightrightarrows Y^*$ and $\overline{\mathrm{co}}A: Y \rightrightarrows Y^*$ defined by the relations $(\mathrm{co}A)(y) = \mathrm{co}(A(y))$ and $(\overline{\mathrm{co}}A(y)) = \overline{\mathrm{co}}(A(y))$ respectively, where $(\overline{\mathrm{co}}^* A(y))$ is the weak closure of $\mathrm{co}(A(y))$ in Y^* and $\mathrm{co}(A(y))$ is the convex hull of $A(y) \subset Y^*$.

Proposition 1 ([18]). *Let $A, B : Y \rightrightarrows Y^*$. Then*

(1) *for all $y, v_1, v_2 \in Y$ the relations*

$$\begin{aligned}
[A(y), v_1 + v_2]_+ &\le [A(y), v_1]_+ + [A(y), v_2]_+,\\
[A(y), v_1 + v_2]_- &\ge [A(y), v_1]_- + [A(y), v_2]_-,\\
[A(y), v_1 + v_2]_+ &\ge [A(y), v_1]_+ + [A(y), v_2]_-,\\
[A(y), v_1 + v_2]_- &\le [A(y), v_1]_+ + [A(y), v_2]_-
\end{aligned}$$

are satisfied;

(2) *the equalities*

$$\begin{aligned}
[A(y), v]_+ &= -[A(y), -v]_-,\\
[A(y) + B(y), v]_{+(-)} &= [A(y), v]_{+(-)} + [B(y), v]_{+(-)}
\end{aligned}$$

hold for all $y, v \in Y$;

(3) $[A(y), v]_{+(-)} = [\overline{\mathrm{co}}^* A(y), v]_{+(-)}$ *for all $y, v \in Y$;*

(4) *for all $y, v \in Y$ the relations*

$$\begin{aligned}
[A(y), v]_{+(-)} &\le \|A(y)\|_{+(-)} \|v\|_Y,\\
d_H\big(A(y), B(y)\big) &\ge \big|\|A(y)\|_{+(-)} - \|B(y)\|_{+(-)}\big|,\\
\|A(y) - B(y)\|_+ &\ge \big|\|A(y)\|_+ - \|B(y)\|_-\big|
\end{aligned}$$

are fulfilled, where $d_H(\cdot,\cdot)$ *is the Hausdorff metric.*

Proposition 2 ([18]). *The inclusion* $d \in \overline{\text{co}}^* A(y)$ *is true if and only if*

$$[A(y), v]_+ \geq \langle d, v\rangle_Y \quad \textit{for all } v \in Y.$$

Proposition 3 ([18]). *Let* $D \subset Y$ *and* $a(\cdot,\cdot)\colon D \times Y \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. *For every* $y \in D$, *the functional* $Y \ni w \mapsto a(y, w)$ *is positively homogeneous, convex, and lower semi-continuous if and only if there exists a multi-valued map* $A\colon Y \rightrightarrows Y^*$ *such that* $D(A) = D$ *and*

$$a(y, w) = [A(y), w]_+ \quad \textit{for all } y \in D(A),\ w \in Y.$$

Remark 1. In what follows, $y_n \rightharpoonup y$ in Y means that y_n weakly converges to y in a reflexive Banach space Y.

Definition 1. Let us denote the family of all non-empty closed convex bounded subsets of the space Y by $C_v(Y)$.

Definition 2. An operator $A\colon X \rightrightarrows X^*$ is called a *Volterra type operator* if, for any $t \in S$, from the equality $u(s) = v(s)$ for a. e. $s \in [0, t]$ $(u, v \in X)$ it follows that $(\overline{\text{co}}A(u))(s) = (\overline{\text{co}}A(v))(s)$ for a. e. $s \in [0, t]$, i. e., $[A(u), \xi_t]_+ = [A(v), \xi_t]_+$ for all $\xi_t \in X$ such that $\xi_t(s) = 0$ for a. e. $s \in S \setminus [0, t]$.

Definition 3. A strict multi-valued map $A\colon Y \rightrightarrows Y^*$ is called:

(1) $+(-)$-*coercive* if there exists a lower bounded, on bounded in $\mathbb{R}_+$ sets, real function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}$ such that $\gamma(s) \to +\infty$ as $s \to +\infty$ and

$$[A(y), y]_{+(-)} \geq \gamma(\|y\|_Y)\|y\|_Y \quad \text{for all } y \in Y;$$

(2) *bounded* if for any $L > 0$ there is $l > 0$, such that

$$\|A(y)\|_+ \leq l \quad \text{for all } y \in Y,\ \|y\|_Y \leq L;$$

(3) *locally bounded* if for all $y \in Y$ there exist $m > 0$ and $M > 0$ such that

$$\|A(\xi)\|_+ \leq M \quad \text{for all } \xi \in Y,\ \|y - \xi\|_Y \leq m;$$

(4) *finite-dimension locally bounded* if $A|_F$ is locally bounded on $(F, \|\cdot\|_Y)$ for any finite-dimensional subspace $F \subset Y$.

Definition 4. We say that a multi-valued map $A\colon X \rightrightarrows X^*$ possesses the *property* (Π) if the following implication holds: If for some non-empty bounded subset $B \subset Y$, constant $k > 0$, and selector d of A, the relation

$$\langle d(y), y\rangle_Y \leq k \quad \text{for all } y \in B$$

holds, then there is a $K > 0$ such that

$$\|d(y)\|_{Y^*} \leq K \quad \text{for all } y \in B.$$

Definition 5. We say that a function $C: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *belongs to the class Φ* if $C(r_1; \cdot) : \mathbb{R}_+ \to \mathbb{R}$ is continuous for any $r_1 \geq 0$ and

$$\lim_{\tau \to 0+} \tau^{-1} C(r_1; \tau r_2) = 0$$

for all $r_1, r_2 \geq 0$.

Now let W be some normed space with the norm $\|\cdot\|_W$. We suppose that $W \subset Y$ with a continuous embedding. Let also $\|\cdot\|'_W$ be a (semi-)norm on Y which is compact with respect to $\|\cdot\|_W$ on W and continuous with respect to $\|\cdot\|_Y$ on Y. Moreover, let $C \in \Phi$.

Definition 6. A strict multi-valued map $A: Y \rightrightarrows Y^*$ is called:

(1) *radially lower semi-continuous* (or, shortly, RLSC) if

$$\liminf_{t \to 0+} [A(y + t\xi), \xi]_+ \geq [A(y), \xi]_-$$

for all $y, \xi \in Y$;

(2) *radially upper semi-continuous* (or RUSC) if, for all $y, \xi \in Y$, the real function $t \mapsto [A(y + t\xi), \xi]_+$ is upper semi-continuous from the right at the point $t = 0$;

(3) *operator with semi-bounded variation on W* (or (Y, W)-SBV) if, for all $R \geq 0$ and $y_1, y_2 \in Y$ such that $\|y_1\|_Y \leq R$ and $\|y_2\|_Y \leq R$, the inequality

$$[A(y_1), y_1 - y_2]_- \geq [A(y_2), y_1 - y_2]_+ - C(R; \|y_1 - y_2\|'_W)$$

is satisfied;

(4) *operator with N-semi-bounded variation on W* (or N-SBV *on W*) if, for all $R \geq 0$ and every $y_1, y_2 \in Y$ such that $\|y_1\|_Y \leq R$ and $\|y_2\|_Y \leq R$, the condition

$$[A(y_1), y_1 - y_2]_- \geq [A(y_2), y_1 - y_2]_- - C(R; \|y_1 - y_2\|'_W)$$

holds;

(5) *λ_0-pseudomonotone on W* (or *W_{λ_0}-pseudomonotone*) if, for arbitrary sequences $\{y_n\}_{n \geq 0} \subset W$ and $\{d_n\}_{n \geq 1}$ such that $d_n \in \overline{\mathrm{co}} A(y_n)$ for all $n \geq 1$, $y_n \rightharpoonup y_0$ in W, and $d_n \rightharpoonup d_0$ in Y^*, from the inequality

$$\limsup_{n \to \infty} \langle d_n, y_n - y_0 \rangle_Y \leq 0$$

it follows that there exist subsequences $\{y_{n_k}\}_{k \geq 1} \subset \{y_n\}_{n \geq 1}$ and $\{d_{n_k}\}_{k \geq 1} \subset \{d_n\}_{n \geq 1}$ for which the inequality

$$\liminf_{k \to \infty} \langle d_{n_k}, y_{n_k} - w \rangle_Y \geq [A(y_0), y_0 - w]_-$$

holds for all $w \in Y$.

Remark 2. The idea on passing to a subsequence in the definition of a single-valued pseudomonotone operator was proposed by Skrypnik [15].

Lemma 1 ([18]). *Any strict multi-valued operator* $A: Y \rightrightarrows Y^*$ *with* $(Y;W)$-SBV *is bounded-valued, locally bounded, and satisfies property* (Π). *Furthermore, if A is* RLSC, *then it is also* λ_0*-pseudomonotone on* W.

Let $Y := Y_1 \cap Y_2$, where $(Y_1, \|\cdot\|_{Y_1})$ and $(Y_2, \|\cdot\|_{Y_2})$ are some reflexive Banach spaces.

Definition 7. A pair $(A;B)$ of maps $A: Y_1 \to C_v(Y_1^*)$ and $B: Y_2 \to C_v(Y_2^*)$ is called *s-mutually bounded* if, for any constant $M > 0$, bounded set $D \subset Y$, and selectors d_A of A and d_B of B, there exists a $K > 0$ such that the relations $y \in D$ and

$$\langle d_A(y), y\rangle_{Y_1} + \langle d_B(y), y\rangle_{Y_2} \le M$$

imply that $\|d_A(y)\|_{Y_1^*} \le K$ or $\|d_B(y)\|_{Y_2^*} \le K$.

Remark 3. A bounded strict multi-valued map $A: Y \rightrightarrows Y^*$ satisfies condition (Π). If one operator of the pair $(A;B)$ is bounded, then the pair $(A;B)$ is s-mutually bounded. Moreover, if both operators from $(A;B)$ satisfy condition (Π), then their sum also satisfies condition (Π) and the pair $(A;B)$ is s-mutually bounded.

Let now $W := W_1 \cap W_2$, where $(W_1, \|\cdot\|_{W_1})$ and $(W_2, \|\cdot\|_{W_2})$ are Banach spaces such that $W_i \subset Y_i$, $i = 1, 2$, with a continuous embedding.

Lemma 2 ([7]). *Let* $A : Y_1 \to C_v(Y_1^*)$ *and* $B : Y_2 \to C_v(Y_2^*)$ *be multi-valued maps which are* λ_0*-pseudomonotone on* W_1 *and* W_2*, respectively, and such that the pair* $(A;B)$ *is s-mutually bounded. Then the map* $C := A + B : Y \to C_v(Y^*)$ *is* λ_0*-pseudomonotone on* W.

Definition 8. We say that a multi-valued map $A: X \rightrightarrows X^*$ satisfies *condition* (H) if, for any $y \in X$, $n \ge 1$, $\{d_i\}_{i=1}^n \subset A(y)$ and measurable $E_j \subset S$ $(j = 1, \dots, n)$ such that $\bigcup_{j=1}^n E_j = S$ and $E_i \cap E_j = \varnothing$ for all $i, j = 1, \dots, n$, $i \ne j$, the inclusion $d \in \overline{\text{co}} A(y)$ holds, where $d = \sum_{j=1}^n d_j \chi_{E_j}$ and

$$\chi_{E_j}(\tau) := \begin{cases} 1 & \text{for } \tau \in E_j, \\ 0 & \text{for } \tau \in S \setminus E_j. \end{cases}$$

4. Main result

Theorem 1. *Let* $\lambda_A \ge 0$ *be fixed,* $p_0 := \min\{p_1, p_2\}$*, the space* V *be compactly embedded in some Banach space* V_0*, and the embedding* $V_0 \subset V^*$ *be continuous. Moreover, let the map** $A + \lambda_A I: X \to C_v(X^*)$ *be* $+$*-coercive and* RLSC *multi-valued map of the Volterra type with* $(X;W)$-SBV *(*$\|\cdot\|'_W = \|\cdot\|_{L_{p_0}(S;V_0)}$*) satisfying condition* (H). *Let* $B: Y \to C_v(Y^*)$ *be a multi-valued operator of the Volterra type which fulfils condition* (H)*, the growth condition*

$$\|By\|_+ \le c_1\|y\|_Y + c_2 \quad \textit{for all } y \in Y \tag{4.1}$$

*Here, $I: X \to X^*$ is the identical motion.

with some $c_1, c_2 \geq 0$*, and the continuity condition*

$$d_H(B(z), B(z_0)) \to 0 \quad \text{as } z \to z_0. \tag{4.2}$$

Then for any $a_0 \in V$ *and* $f \in X^*$ *there exist at least one solution* u *of problem* (2.1) *with* $u' \in W$.

Here, $d_H(\cdot,\cdot)$ is the Hausdorff metric on $C_v(Y^*)$, i. e.,

$$d_H(C, D) := \max\{\operatorname{dist}(C; D), \operatorname{dist}(D, C)\}$$

with $\operatorname{dist}(C; D) := \sup_{c \in C} \inf_{d \in D} \|c - d\|_{Y^*}$ for $C, D \in C_v(Y^*)$.

Proof. Let us reduce the evolution inclusion (2.1) to a first-order inclusion. Let $R{:}\, X \to X$ (resp., $R{:}\, Y \to Y$) be the Volterra type operator defined by the relation

$$(Rv)(t) = a_0 + \int_0^t v(s)ds \quad \text{for all } t \in S \text{ and every } v \in X \text{ (resp., } v \in Y).$$

It is clear that R is a Lipschitz continuous operator from X into X (resp., from Y into Y). Consider the problem

$$\begin{cases} v' + A(v) + B(Rv) \ni f, \\ v(0) = \bar{0}, \ v \in W. \end{cases} \tag{4.3}$$

If $v \in W$ is a solution of problem (4.3), then $u = Rv \in X$ is a solution of problem (2.1) such that $u' \in W \subset X$.

Let us set $\mathcal{A} := A + B \circ R{:}\, X \to C_v(X^*)$ and $\lambda = \lambda_A + \lambda_B$, where $\lambda_B = 1 + c_1 c_3$ and c_3 is the Lipschitz constant for the operator $R{:}\, Y \to Y$. For an arbitrary $y \in X$ and a. e. $t \in S$, we set

$$y_\lambda(t) = e^{-\lambda t} y(t), \qquad \widehat{y}_\lambda(t) = e^{\lambda t} y(t), \tag{4.4}$$

and

$$(A_\lambda y)(t) = e^{-\lambda t}(\mathcal{A}\widehat{y}_\lambda)(t) + \lambda y(t).$$

Then $g \in A_\lambda(y_\lambda) \iff \langle g, w\rangle_X \leq [A(y) + \lambda y, w_\lambda]_+$ for all $w \in X$. The set $A_\lambda(y_\lambda)$ is non-empty because every g defined by the relation

$$g(t) = e^{-\lambda t} d(t) + \lambda y_\lambda(t) \quad \text{for a. e. } t \in S \text{ and all } d \in \mathcal{A}(y)$$

belongs to $A_\lambda(y_\lambda)$.

We note that $A_\lambda{:}\, X \to C_v(X^*)$ and $v \in W$ is a solution of problem (4.3) if and only if $v_\lambda \in W$ is such that

$$v_\lambda' + A_\lambda v_\lambda \ni f_\lambda, \quad v_\lambda(0) = \bar{0}, \tag{4.5}$$

where $f_\lambda(t) = e^{-\lambda t} f(t)$. It turns out that $A_\lambda{:}\, X \to C_v(X^*)$ possesses the following properties:

(α_1) A_λ is $+$-coercive on X,

(α_2) A_λ is λ_0-pseudomonotone on W,
(α_3) A_λ is locally bounded on X,
(α_4) A_λ satisfies condition (Π) on X.

Let us prove the assertion above.

PROPERTY (α_1). Let us fix $y \in X$, $\|y\|_X \neq 0$. As $\|y_\lambda\|_X \le \|y\|_X$, then

$$\|y_\lambda\|_X^{-1}[A_\lambda y_\lambda, y_\lambda]_+ \ge \|y\|_X^{-1} \sup_{\zeta(y)\in A(y)} \int_S e^{-2\lambda t}\big(\zeta(y)(t) + \lambda_A y(t), y(t)\big)dt$$
$$+ \|y\|_X^{-1} \inf_{\zeta(y)\in B(Ry)} \int_S e^{-2\lambda t}\big(\zeta(y)(t) + \lambda_B y(t), y(t)\big)dt. \quad (4.6)$$

We first estimate the first term. We remark that

$$[(A + \lambda_A I)y, y]_+ \ge \widehat{\gamma}(\|y\|_X)\|y\|_X \quad \text{for all } y \in X,$$

where $\widehat{\gamma} : \mathbb{R}_+ \to \mathbb{R}$ can be chosen as a non-decreasing function lower bounded on bounded, in $\mathbb{R}_+$, sets such that $\widehat{\gamma}(r) \to +\infty$ as $r \to \infty$.

Since A is a Volterra type operator, we see that, for any $u \in X$,

$$\sup_{\zeta(u)\in A(u)} \int_0^t \big(\zeta(u)(\tau) + \lambda_A u(\tau), u(\tau)\big)d\tau \ge \widehat{\gamma}(\|u\|_{X_t})\|u\|_{X_t} \quad \text{for all } t \in S,$$

where $\|u\|_{X_t} = \|u_t\|_X$. Let us set

$$g_{\zeta(y)}(\tau) = \big(\zeta(y)(\tau) + \lambda_A y(\tau), y(\tau)\big), \quad \zeta(y) \in A(y),\ \tau \in S,$$

and $h(t) = \widehat{\gamma}(\|y\|_{X_t})\|y\|_{X_t}$ for $t \in S$. Then $h(t) \ge \min\{\widehat{\gamma}(0), 0\}\|y\|_X$ and

$$\sup_{\zeta(y)\in A(y)} \int_0^t g_{\zeta(y)}(\tau)d\tau \ge h(t)$$

for all $t \in S$. Similarly to the definition of A_λ, for any $u \in X$ and a. e. $t \in S$, we put

$$(A_1u)(t) = \big(e^{-2\lambda t} - e^{-2\lambda T}\big)\big((Au)(t) + \lambda_A u(t)\big),$$
$$(A_2u)(t) = e^{-2\lambda T}\big((Au)(t) + \lambda_A u(t)\big),$$

and

$$(\widehat{A}u)(t) = e^{-2\lambda t}\big((Au)(t) + \lambda_A u(t)\big).$$

Then, due to Proposition 1, we get

$$[\widehat{A}y, y]_+ = [A_1y, y]_+ + [A_2y, y]_+$$
$$\ge e^{-2\lambda T}h(T) + 2\lambda T \sup_{\zeta(y)\in A(y)} \inf_{s\in S} e^{-2\lambda s} \int_0^s \big(\zeta(y)(\tau) + \lambda_A y(\tau), y(\tau)\big)d\tau.$$

Further, using condition (H) for the operator A, we prove that

$$\sup_{\zeta(y)\in A(y)} \inf_{s\in S} e^{-2\lambda s}\int_0^s \big(\zeta(y)(\tau)+\lambda_A y(\tau), y(\tau)\big)d\tau \ge -C_1\|y\|_X,$$

where $C_1=\max\{-\widehat{\gamma}(0),0\}\ge 0$ does not depend on y. Consequently, we obtain

$$\|y\|_X^{-1}\sup_{\zeta(y)\in A(y)}\int_0^T e^{-2\lambda\tau}\big(\zeta(y)(\tau)+\lambda_A y(\tau), y(\tau)\big)d\tau \ge e^{-2\lambda T}\widehat{\gamma}(\|y\|_X)-2\lambda C_1 T. \quad (4.7)$$

Let us estimate the second term. Analogously to the previous case, using the Volterra property of the operator $B\circ R$, we obtain that, for all $t\in S$,

$$\inf_{\zeta(y)\in B(Ry)}\int_0^t \big(\zeta(y)(\tau)+\lambda_B y(\tau), y(\tau)\big)d\tau \ge -(c_2+c_1\|R\bar{0}\|_Y)c_4\|y\|_X > -\infty,$$

where $c_4>0$ is such that $\|\cdot\|_Y\le c_4\|\cdot\|_X$. Then

$$\begin{aligned}\inf_{\zeta(y)\in B(Ry)}\int_0^T e^{-2\lambda\tau}\big(\zeta(y)(\tau)+\lambda_B y(\tau), y(\tau)\big)d\tau \ge\\ -e^{-2\lambda T}(c_2+c_1\|R\bar{0}\|_Y)c_4\|y\|_X\\ +2\lambda\int_0^T e^{-2\lambda s}\inf_{\zeta(y)\in B(Ry)}\int_0^s\big(\zeta(y)(\tau)+\lambda_B y(\tau), y(\tau)\big)d\tau ds\\ \ge -(c_2+c_1\|R\bar{0}\|_Y)c_4\|y\|_X.\end{aligned}$$

Therefore, in view of (4.7), it follows from (4.6) that

$$\|y_\lambda\|_X^{-1}[A_\lambda y_\lambda, y_\lambda]_+ \ge e^{-2\lambda T}\widehat{\gamma}(\|y_\lambda\|_X)-2\lambda C_1T-(c_2+c_1\|R\bar{0}\|_Y)c_4,$$

because $\|y\|_X\ge\|y_\lambda\|_X$ and the function $\widehat{\gamma}$ is non-decreasing. Since y is arbitrary, we have proved that $A_\lambda\colon X\to C_v(X^*)$ is $+$-coercive.

PROPERTY (α_2). For any $y\in X$ and a. e. $t\in S$ we set

$$(A_\lambda^1 y)(t)=e^{-\lambda t}(A\widehat{y}_\lambda)(t)+\lambda_A y(t),\quad (A_\lambda^2 y)(t)=e^{-\lambda t}(B(R\widehat{y}_\lambda))(t)+\lambda_B y(t),$$

where $\widehat{y}_\lambda$ is given by (4.4). Let us note that $A_\lambda^1+A_\lambda^2=A_\lambda$.

At first we show that A_λ^1 is an RLSC operator with $(X;W)$-SBV. Let us prove the semi-boundedness of the variation. By virtue of the assumptions of the theorem, for all $R>0$ and $y,\xi\in X$ such that $\|y\|_X\le R$ and $\|\xi\|_X\le R$, we have

$$[A(y)-A(\xi)+\lambda_A y-\lambda_A\xi, y-\xi]_- + C_A(R;\|y-\xi\|'_W)\ge 0.$$

Let us set $\widehat{C}_A(R;t):=\max_{\tau\in[0,t]}C_A(R;\tau)$ for all $R,t\ge 0$ (note that $\widehat{C}_A\in\varPhi$) and

$$z_t(\tau):=\begin{cases} z(\tau) & \text{for } 0\le\tau\le t,\\ \bar{0} & \text{for } t<\tau\le T\end{cases}$$

for a. e. $t \in S$ and all $z \in X$. Let ζ and η be fixed selectors of A. Since A is a Volterra type operator, for all $y, \xi \in X$ and $t \in S$, we have

$$\int_0^t \big(\zeta(y)(\tau) + \lambda_A y(\tau) - \eta(\xi)(\tau) - \lambda_A \xi(\tau), y(\tau) - \xi(\tau)\big) d\tau + \widehat{C}_A(R; \|y - \xi\|'_{W_t})$$
$$\geq [(A + \lambda_A I)(y_t) - (A + \lambda_A I)(\xi_t), y_t - \xi_t]_- + \widehat{C}_A(R; \|y_t - \xi_t\|'_W) \geq 0$$

because $\|y_t\|_X \leq \|y\|_X$ and $\|y_t - \xi_t\|'_W \leq \|y - \xi\|'_W$. Here, $\|\cdot\|'_{W_t} = \|\cdot\|_{L_{p_0}([0,t];V_0)}$.

Let us fix $y, \xi \in X$ and set

$$g(\tau) = \big(\zeta(y)(\tau) + \lambda_A y(\tau) - \eta(\xi)(\tau) - \lambda_A \xi(\tau), y(\tau) - \xi(\tau)\big), \quad \tau \in S,$$

and $h(t) = \widehat{C}_A(R; \|y - \xi\|'_{W_t})$, $t \in S$. We have proved that

$$\int_0^t g(\tau) d\tau \geq -h(t) \quad \text{for all } t \in S.$$

The function $S \ni t \mapsto h(t)$ is non-decreasing and, thus,

$$\int_0^T e^{-2\lambda\tau} g(\tau) d\tau = e^{-2\lambda T} \int_0^T g(\tau) d\tau + 2\lambda \int_0^T e^{-2\lambda\tau} \int_0^\tau g(s) ds d\tau \geq -h(T).$$

Consequently,

$$[A^1_\lambda y_\lambda, y_\lambda - \xi_\lambda]_-$$
$$\geq [A^1_\lambda \xi_\lambda, y_\lambda - \xi_\lambda]_+ - \widehat{C}_A(R; \|y - \xi\|_{L_{p_0}(S;V_0)}) \quad \text{for all } y, \xi \in X. \qquad (4.8)$$

Now we consider the space $L_{p_0,\lambda}(S; V_0)$ that consists of measurable functions $g \colon S \to V_0$ for which the integral $\int_S e^{\lambda t p_0} \|g(t)\|^{p_0}_{V_0} dt$ is finite. Then

$$\|y - \xi\|_{L_{p_0}(S;V_0)} = \left(\int_S e^{\lambda t p_0} \|y_\lambda(t) - \xi_\lambda(t)\|^{p_0}_{V_0} dt \right)^{1/p_0} = \|y_\lambda - \xi_\lambda\|_{L_{p_0,\lambda}(S;V_0)}.$$

Therefore, from (4.8) we obtain

$$[A^1_\lambda y_\lambda, y_\lambda - \xi_\lambda]_- \geq [A^1_\lambda \xi_\lambda, y_\lambda - \xi_\lambda]_+ - \widehat{C}_A(R; \|y_\lambda - \xi_\lambda\|_{L_{p_0,\lambda}(S;V_0)}).$$

The proof of the fact that the mapping $A^1_\lambda : X \to C_v(X^*)$ has $(X;W)$-SBV is concluded by taking into account the compactness of the embedding $W \subset L_{p_0,\lambda}(S;V_0)$ [10, Theorem 1.5.1]. The RLSC for A^1_λ is clear.

Since an arbitrary RLSC multi-valued operator with $(X;W)$-SBV is λ_0-pseudomonotone on W [8], we have proved that A^1_λ is λ_0-pseudomonotone on W.

Let us now consider A^2_λ. We first show that A^2_λ is an operator with N-SBV on W (the radial upper semi-continuity is clear).

We first prove that $(B\circ R): X\to C_v(X^*)$ is the operator with N-SBV on W with $\|\cdot\|'_W=\|\cdot\|_Y$. For all $R\ge 0$ and $t\ge 0$, we set

$$C_B(R,t)=t\sup\big\{d_H(B(Rz_1),B(Rz_2))\mid z_1,z_2\in X:\ \|z_1-z_2\|_Y\le t \text{ and } \|z_i\|_X\le R,\ i=1,2\big\}.$$

Similarly to [17], we show that $C_B\in\Phi$. Then for any $R>0$ and $y,\xi\in X$ such that $\|y\|_X\le R$, $\|\xi\|_X\le R$ it follows that

$$[B(Ry)+\lambda_B y, y-\xi]_- - [B(R\xi)+\lambda_B\xi, y-\xi]_- + C_B(R;\|y-\xi\|_Y)\ge 0.$$

Let us set $\widehat{C}_B(R;\cdot)=\max_{\tau\in[0,t]}C_B(R;\tau)$ for all $R,t\ge 0$ (note that $\widehat{C}_B\in\Phi$).

Since $B\circ R$ is the Volterra type operator, for all $y,\xi\in X$ and $t\in S$, we have

$$\inf_{\zeta(y)\in B(Ry)}\int_0^t\big(\zeta(y)(\tau)+\lambda_B y(\tau), y(\tau)-\xi(\tau)\big)d\tau - \inf_{\eta(\xi)\in B(R\xi)}\int_0^t\big(\eta(\xi)(\tau)+\lambda_B\xi(\tau), y(\tau)-\xi(\tau)\big)d\tau + \widehat{C}_B(R;\|y-\xi\|_{Y_t})\ge 0,$$

where $\|\cdot\|_{Y_t}=\|\cdot\|_{L_2([0,t];H)}$. Let us fix some $y,\xi\in X$ and set

$$g_\beta(\tau)=\big(\beta(\tau),y(\tau)-\xi(\tau)\big),\quad \beta\in X^*,\ \tau\in S,$$

and $h(t)=\widehat{C}_B(R;\|y-\xi\|_{Y_t})$, $t\in S$. We have thus proved that

$$\inf_{\zeta\in B(Ry)+\lambda_B y}\int_0^t g_\zeta(\tau)d\tau-\inf_{\eta\in B(R\xi)+\lambda_B\xi}\int_0^t g_\eta(\tau)d\tau\ge -h(t)\quad\text{for all } t\in S.$$

The function $S\ni t\mapsto h(t)$ is non-decreasing and thus, for an arbitrary $\zeta\in B(Ry)+\lambda_B y$, we get

$$\begin{aligned}
&\int_0^T e^{-2\lambda\tau}\big(\zeta(\tau),y(\tau)-\xi(\tau)\big)d\tau-\inf_{\eta\in B(R\xi)+\lambda_B\xi}\int_0^T e^{-2\lambda\tau}\big(\eta(\tau),y(\tau)-\xi(\tau)\big)d\tau\\
&\quad=\sup_{\eta\in B(R\xi)+\lambda_B\xi}\int_0^T e^{-2\lambda\tau}\big(\zeta(\tau)-\eta(\tau),y(\tau)-\xi(\tau)\big)d\tau\\
&\quad\ge e^{-2\lambda T}\inf_{\eta\in B(R\xi)+\lambda_B\xi}\int_0^T\big(\zeta(\tau)-\eta(\tau),y(\tau)-\xi(\tau)\big)d\tau\\
&\qquad+\sup_{\eta\in B(R\xi)+\lambda_B\xi}\int_0^T\Big(e^{-2\lambda\tau}-e^{-2\lambda T}\Big)\big(\zeta(\tau)-\eta(\tau),y(\tau)-\xi(\tau)\big)d\tau\\
&\quad\ge -e^{-2\lambda T}h(T)+2\lambda T\sup_{\eta\in B(R\xi)+\lambda_B\xi}\inf_{s\in S}e^{-2\lambda s}\int_0^s\big(\zeta(\tau)-\eta(\tau),y(\tau)-\xi(\tau)\big)d\tau.
\end{aligned}$$

Using property (H) of the operator B we can prove that, for an arbitrary $\zeta \in B(Ry)+\lambda_B y$, the relation

$$\sup_{\eta\in B(R\xi)+\lambda_B\xi}\inf_{s\in S} e^{-2\lambda s}\int_0^s \big(\zeta(\tau)-\eta(\tau), y(\tau)-\xi(\tau)\big)\,d\tau \geq -h(T)$$

holds. Consequently,

$$\begin{aligned}\inf_{\zeta\in B(Ry)+\lambda_B y}\int_0^T & e^{-2\lambda\tau}\big(\zeta(\tau), y(\tau)-\xi(\tau)\big)\,d\tau \\ &-\inf_{\eta\in B(R\xi)+\lambda_B\xi}\int_0^T e^{-2\lambda\tau}\big(\eta(\tau), y(\tau)-\xi(\tau)\big)\,d\tau \\ &\qquad\qquad \geq -\big(e^{-2\lambda T}+2\lambda T\big)\widehat{C}_B(R;\|y-\xi\|_Y).\end{aligned}$$

Let us set $\widetilde{C}_B(R;t)=\big(e^{-2\lambda T}+2\lambda T\big)\widehat{C}_B(R;t)$ for $R,t\geq 0$ (note that $\widetilde{C}_B\in\Phi$). Then

$$\begin{aligned}\big[A_\lambda^2 y_\lambda, y_\lambda-\xi_\lambda\big]_- - \big[A_\lambda^2\xi_\lambda, y_\lambda-\xi_\lambda\big]_- &\geq -\widetilde{C}_B(R;\|y-\xi\|_Y) \\ &= -\widetilde{C}_B(R;\|y-\xi\|_{L_2(S;H)}).\end{aligned} \tag{4.9}$$

Now we consider the space $L_{2,\lambda}(S;H)$ consisting of the measurable functions $g\colon S\to H$ for which the integral $\int_S e^{2\lambda t}\|g(t)\|_H^2\,dt$ is finite. Then

$$\|y-\xi\|_{L_2(S;H)} = \left(\int_S e^{2\lambda t}\|y_\lambda(t)-\xi_\lambda(t)\|_H^2\,dt\right)^{1/2} = \|y_\lambda-\xi_\lambda\|_{L_{2,\lambda}(S;H)}.$$

Therefore, from (4.9), we obtain

$$\big[A_\lambda^2 y_\lambda, y_\lambda-\xi_\lambda\big]_- \geq \big[A_\lambda^2\xi_\lambda, y_\lambda-\xi_\lambda\big]_- - \widetilde{C}_B(R;\|y_\lambda-\xi_\lambda\|_{L_{2,\lambda}(S;H)}).$$

To prove that $A_\lambda^2\colon X\to C_v(X^*)$ is N-SBV, it is sufficient to note that embedding $W\subset L_{2,\lambda}(S;H)$ is compact. This is a direct sequence of the compactness of the embedding $W\subset Y$.

Let us now check the λ_0-pseudomonotony of A_λ^2 on W. Let $y_{\lambda,n}\to y_\lambda$ weakly in W (therefore $y_{\lambda,n}\to y_\lambda$ in $Y_\lambda := L_{2,\lambda}(S;H)$), $A_\lambda^2(y_{\lambda,n})\ni d_{\lambda,n}\to d_\lambda\in X^*$ weakly in X^*, and

$$\limsup_{n\to\infty}\langle d_{\lambda,n}, y_{\lambda,n}-y_\lambda\rangle \leq 0.$$

Since A_λ^2 is an operator with N-SBV on W, we conclude that for every $v\in X$

$$\begin{aligned}\liminf_{n\to\infty}\langle d_{\lambda,n}, y_{\lambda,n}-v_\lambda\rangle &\geq \liminf_{n\to\infty}[A_\lambda^2(y_{\lambda,n}), y_{\lambda,n}-v_\lambda]_- \\ &\geq \liminf_{n\to\infty}[A_\lambda^2(v_\lambda), y_{\lambda,n}-v_\lambda]_- - \widetilde{C}_B(R;\|y_\lambda-v_\lambda\|_{Y_\lambda}).\end{aligned} \tag{4.10}$$

At first we estimate the first term at the right-hand side of (4.10). It is easy to show that the function $Y_\lambda\ni h\mapsto [A_\lambda^2(v_\lambda), h]_-$ is continuous for all $v\in X$.

Therefore, from (4.10) we obtain that $\langle d_{\lambda,n}, y_{\lambda,n} - y_\lambda\rangle \to 0$ and

$$\liminf_{n\to\infty}\langle d_{\lambda,n}, y_\lambda - v_\lambda\rangle \geq [A^2_\lambda(v_\lambda), y_\lambda - v_\lambda]_- - \widetilde{C}_B(R; \|y_\lambda - v_\lambda\|_{Y_\lambda})$$

for all $v \in X$. Substituting $tw_\lambda + (1-t)y_\lambda$, where $w \in X$, $t \in [0,1]$, for v_λ in the last inequality, dividing the result by t, and passing to the limit as $t \to 0+$, due to the RUSC for A^2_λ, we obtain

$$\liminf_{n\to\infty}\langle d_{\lambda,n}, y_{\lambda,n} - w_\lambda\rangle \geq [A^2_\lambda(y_\lambda), y_\lambda - w_\lambda]_-$$

for all $w \in X$. Therefore, the λ_0-pseudomonotony A^2_λ on W is proved.

In order to prove the λ_0-pseudomonotony of A_λ on W, we use Lemma 2. Let us note that the pair $(A^1_\lambda, A^2_\lambda)$ is s-mutually bounded because A^2_λ is bounded as a consequence of (4.1) and the boundedness of the identity map.

PROPERTIES (α_3) AND (α_4). These properties follow from $(X;W)$-SBV of A^1_λ, N-SBV of A^2_λ, and Lemma 1.

In order to prove the solvability for problem (4.3) we use [8, Theorem 3.1]. Let $L : W_{\bar{0}} \subset X \to X^*$ be a densely defined linear operator, $Lz = z'$, $D(L) = W_{\bar{0}} = \{z \in W \mid z(0) = \bar{0}\}$, and $A_\lambda : X \to C_v(X^*)$ be a multi-valued map. Let us consider the problem

$$Lz + A_\lambda z \ni f_\lambda, \quad z \in W_{\bar{0}}. \tag{4.11}$$

Let us note that $D(L) = W_{\bar{0}}$ is a reflexive Banach space with respect to the graph norm of the derivative. Conditions (α_1)–(α_4) guarantee that there exists at least one solution z of problem (4.3) in $W_{\bar{0}}$. The function $\hat{z}_\lambda$ is then a solution of problem (4.3) and $R\hat{z}_\lambda$ is a solution of the original problem. □

Corollary 1. *Assume that $\lambda_A \geq 0$ is fixed, $p_2 \geq 2$, $p_0 = \min\{p_1, p_2\}$, the space V is compactly embedded in a Banach space V_0, and the embedding $V_0 \subset V^*$ is continuous. Moreover, let $A + \lambda_A I : X \to C_v(X^*)$ be a $+$-coercive and* RLSC *multi-valued operator of the Volterra type with $(X;W)$-SBV ($\|\cdot\|'_W = \|\cdot\|_{L_{p_0}(S;V_0)}$) satisfying condition (H), and $B : Y \to C_v(Y^*)$ be a multi-valued operator of the Volterra type satisfying condition (H), the growth condition (4.1), and the continuity condition (4.2)*[†]*, and $C : X \to X^*$ be an operator with the property*

$$(Cu)(t) = C_0(u(t)) \quad \textit{for all } u \in X,\ t \in S,$$

where $C_0 : V_2 \to V_2^*$ is a linear, bounded, self-conjugate, and monotone operator.

Then for arbitrary $a_0 \in V$ and $f \in X^*$ there exists at least one solution of the problem

$$\begin{cases} y'' + Ay' + By + Cy \ni f, \\ y(0) = a_0,\ y'(0) = \bar{0},\ y \in C(S;V),\ y' \in C(S;H). \end{cases} \tag{4.12}$$

[†]We recall that $d_H(\cdot,\cdot)$ is the Hausdorff metric on $C_v(Y^*)$

5. An example

Let $\Omega \subset \mathbb{R}^n$ be a bounded region with the regular boundary $\partial\Omega$, $S = [0,T]$, $Q = \Omega \times S$, $\Gamma_T = \partial\Omega \times S$, $1 < p = p_1 = p_2$, and $\Phi\colon \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the "growth condition"

$$|\Phi(t)| \le c_1|t| + c_2 \quad \text{for all } t \in \mathbb{R}, \tag{5.1}$$

where $c_1, c_2 \in \mathbb{R}$, and the "sign condition"

$$(\Phi(t) - \Phi(s))(t-s) \ge -c_3(s-t)^2 \quad \text{for all } t,s \in \mathbb{R} \tag{5.2}$$

with some $c_3 > 0$. Moreover, let $S \times \mathbb{R} \ni (t,y) \mapsto \theta_i(t,y) \in \mathbb{R}_+$, $i = 1,2$, be single-valued continuous functions such that

$$-c_2(1+|x|) \le \theta_1(t,x) \le \theta_2(t,x) \le c_1(1+|x|) \quad \text{for all } t \in S,\ x \in \mathbb{R}, \tag{5.3}$$

where $c_1, c_2 \ge 0$. For an any $f \in X^* = L_2(S; L_2(\Omega)) + L_q(S; W^{-1,q}(\Omega))$, we consider the problem

$$\begin{aligned} &\frac{\partial^2 y(x,t)}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial^2 y(x,t)}{\partial x_i\, \partial t}\right|^{p-2} \frac{\partial^2 y(x,t)}{\partial x_i\, \partial t}\right) + \left|\frac{\partial y(x,t)}{\partial t}\right|^{p-2} \frac{\partial y(x,t)}{\partial t} \\ &\qquad + \Phi\left(\frac{\partial y(x,t)}{\partial t}\right) - \Delta y(x,t) \\ &\qquad + [\theta_1(t,y(x,t)); \theta_2(t,y(x,t))] \ni f(x,t) \quad \text{a. e. on } Q, \\ &\qquad y(x,0) = 0, \quad \left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = 0 \qquad \text{a.e. on } \Omega, \\ &\qquad\qquad y(x,t) = 0 \quad \text{a. e. on } \partial\Omega. \end{aligned} \tag{5.4}$$

For the operator $A\colon L_p(S; W_0^{1,p}(\Omega)) \to L_q(S; W^{-1,q}(\Omega))$, we take $(Au)(t) = A(u(t))$, $t \in S$ [16, 17], where $A(\varphi) = A_1(\varphi) + A_2(\varphi)$,

$$A_1(\varphi) = -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial \varphi}{\partial x_i}\right|^{p-2} \frac{\partial \varphi}{\partial x_i}\right) + |\varphi|^{p-2}\varphi,$$

and $A_2(\varphi) = \Phi(\varphi)$ for all $\varphi \in C_0^2(\bar{\Omega})$. For the map $B\colon L_2(Q) \to L_2(Q)$, we take

$$B(u) = \{v \in L_2(Q) \mid \theta_1(t,u(x,t)) \le v(x,t) \le \theta_2(t,u(x,t)) \text{ for a. e. } (x,t) \in Q\},$$

and let the operator $C\colon L_2(S; H_0^1(\Omega)) \to L_2(S; H^{-1}(\Omega))$ be defined by the relation $(Cu)(t) = C_0(u(t))$, $t \in S$, where $C_0(v) = -\Delta v$ for $v \in H_0^1(\Omega)$. Moreover, let $H = L_2(\Omega)$, $V_1 = W_0^{1,p}(\Omega)$, $V_2 = H_0^1(\Omega)$, and let $Y = L_2(S;H) = L_2(Q)$,

$$X = L_p(S;V_1) \cap L_2(S;H) \cap L_2(S;V_2), \qquad X^* = L_q(S;V_1^*) + L_2(S;V_2^*).$$

Then, according to Corollary 1, problem (5.4) has a solution $y \in C(S;V)$ such that $y' \in C(S;H)$ and $y'' \in X^*$.

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