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This paper is dedicated to the memory of the Corresponding Member of the National Academy of Sciences of Ukraine, Professor Valeriy S. Mel’nik.

Received 14 February, 2008

Abstract. We consider second-order differential-operators inclusions with Volterra type operators. The problem of the existence of solutions of the Cauchy problem for the given inclusions is investigated. Important a priori estimates are obtained. An example illustrating the approach is given.

2000 Mathematics Subject Classification: 34G25, 35L15

Keywords: second-order evolution inclusion, Volterra type operator, pseudomonotone map

1. INTRODUCTION

The progress in the investigation of non-linear boundary problems for partial differential equations became possible thanks to the intense development of the methods of non-linear analysis which had found their application in various parts of mathematics. It has recently become natural to reduce these problems to the study of non-linear operator and differential-operator equations and inclusions in functional spaces. Within such an approach, the results for concrete systems are obtained as rather simple consequences of operator theorems [2, 10].

The evolution differential equations and inclusions are studied rather actively. To prove the properties of the resolving operator (non-emptiness, compactness, connectedness), the method of monotony, method of compactness, and their combinations are often used.

In the present work, we study the solvability of the evolution inclusion with multi-valued non-coercive maps

\[ y'' + A(y') + B(y) \ni f, \]

which is important for applications.

Supported in part by the Fundamental Researches State Fund of Ukraine, Grant No. \( \Phi 25/539-2007 \).
Recent related investigations concern a class of problems with a strongly monotone operator $A$ and multi-valued operator $B$ that can be presented as the sum of a single-valued linear self-conjugated monotone operator and a multi-valued demiclosed bounded operator. These problems are coercive. They were considered, e.g., by Papageorgiou and Yannakakis [13, 14]. More particular cases of evolution inclusions were studied by Ahmed and Kerbal [1], Gasiński and Smołka [3], Kartsatos and Markov [4], Migórski [12], and other authors.

Our goal here is to extend the approach indicated to a wider class of problems, namely, to problems with a multi-valued non-coercive non-monotone operator $A$ and a multi-valued operator $B$ satisfying similar conditions.

The idea of passing to subsequences in the classical definition of a single-valued pseudomonotone operator was suggested by Skrypnik [15]. It was developed for the first order differential-operator equations and inclusions in infinite-dimensional spaces with $+\gamma$-coercive $W_{\lambda_0}$-pseudomonotone maps by Mel’nik, Zgurovskii, and Novikov [11, 18, 19] and Kas’yanov [5–8]. This gave one the possibility to investigate a substantially wider class of problems arising in applications. In particular, this methodology, combined with the non-coercive theory [2, 9, 18], which we apply to the second-order evolution inclusions, allows one to sufficiently extend the class of problems with multi-valued maps for which we can obtain the solvability. Since the operators are multi-valued, such extension faced with considerable difficulties which are not typical for the differential-operator equations. Here, the proof of the solvability is based on the method of singular perturbations [9, 10] and allows us to obtain important a priori estimates for solutions. It makes possible to study properties for the obtained solutions (e.g., dynamics). As an example illustrating the suggested approach, we consider a class of problems with non-linear operators. The obtained results are new for both inclusions and equations.

We note that the solvability of second-order differential-operator equations was investigated by the authors in [16, 17].

2. Problem setting

Let $H$ be a real Hilbert space with the inner product $(\cdot, \cdot)$, and let $(V_1, \| \cdot \|_{V_1})$ and $(V_2, \| \cdot \|_{V_2})$ be some real reflexive separable Banach spaces continuously embedded into $H$ and such that

$$V := V_1 \cap V_2$$

is dense in the spaces $V_1$, $V_2$, and $H$. We assume that one of the embeddings $V_i \subset H$, $i = 1, 2$, is compact. In what follows, the space topologically conjugate to $H$ (with respect to the bilinear form $(\cdot, \cdot)$) is identified with $H$. Then we have

$$V_i \subset H \subset V_i^* \quad (i = 1, 2)$$
with continuous and dense embeddings, where \( (V^*_i, \| \cdot \|_{V^*_i}) \), \( i = 1, 2 \), is the space topologically conjugate to \( V_i \), \( i = 1, 2 \), with respect to the canonical bilinear form
\[
\langle \cdot, \cdot \rangle_{V_i} : V^*_i \times V_i \to \mathbb{R} \quad (i = 1, 2)
\]
that coincides on \( H \times V \) with the inner product \( \langle \cdot, \cdot \rangle \) in \( H \). Let us consider the reflexive function spaces \( Y = L_2(S; H) \) and
\[
X_i := L_{r_i}(S; H) \cap L_{p_i}(S; V_i) \quad (i = 1, 2)
\]
with
\[
\| y \|_{X_i} := \| y \|_{L_{p_i}(S; V_i)} + \| y \|_{L_{r_i}(S; H)} \quad (i = 1, 2),
\]
where \( S := [0, T] \), \( 1 < p_i \leq r_i < +\infty \), \( i = 1, 2 \), and \( \max \{ r_1; r_2 \} \geq 2 \).

Let us consider the reflexive (it follows from [2, Chapter 1]) Banach space \( X := X_1 \cap X_2 \) with the norm \( \| y \|_X := \| y \|_{X_1} + \| y \|_{X_2} \). We note that the space \( X \) is continuously and densely embedded in \( Y \).

We identify \( L_{q_i}(S; V^*_i) + L_{r_i}(S; H) \) with \( X_i^* \). Similarly, \( Y^* = Y \) and \( X^* = X_1^* + X_2^* = L_{q_1}(S; V^*_1) + L_{q_2}(S; V^*_2) + L_{r_1}(S; H) + L_{r_2}(S; H) \).

Let \( A, B : X \Rightarrow X^* \) be strict multi-valued maps. We consider the Cauchy problem for the differential-operator inclusion with non-coercive multi-valued maps of \( W_{\lambda_0}^* \)-pseudomonotone type
\[
\begin{aligned}
\begin{cases}
y'' + Ay' + By \ni f, \\
y(0) = a_0, \quad y'(0) = \overline{0}, \quad y \in C(S; V), \quad y' \in C(S; H),
\end{cases}
\end{aligned}
\tag{2.1}
\]
where \( a_0 \in V \) and \( f \in X^* \) are fixed.

On \( X^* \times X \) we consider the pairing
\[
\langle f, y \rangle = \int_S (f_{11}(\tau), y(\tau))_H d\tau + \int_S (f_{12}(\tau), y(\tau))_H d\tau + \int_S (f_{21}(\tau), y(\tau))_{V_1} d\tau + \int_S (f_{22}(\tau), y(\tau))_{V_2} d\tau
\]
\[
= \int_S (f(\tau), y(\tau)) d\tau,
\]
where \( f = f_{11} + f_{12} + f_{21} + f_{22} \), \( f_{1i} \in L_{r_i}(S; H) \), and \( f_{2i} \in L_{q_i}(S; V^*_i) \). Note that, for any \( f \in X^* \),
\[
\| f \|_{X^*} = \inf_{\substack{f = f_{11} + f_{12} + f_{21} + f_{22} : f_{1i} \in L_{r_i}(S; H), f_{2i} \in L_{q_i}(S; V^*_i) \atop (i = 1, 2)}} \varphi(f_{11}, f_{12}, f_{21}, f_{22}).
\]
where

\[ \varphi(f_{11}, f_{12}, f_{21}, f_{22}) = \max \left\{ \|f_{11}\|_{L_{r_1}(S; H)}, \|f_{12}\|_{L_{r_2}(S; H)}, \|f_{21}\|_{L_{q_1}(S; V_1^*)}, \|f_{22}\|_{L_{q_2}(S; V_2^*)} \right\}. \]

Moreover, let

\[ W = \{ y \in X \mid y' \in X^* \} \]

and \( \|y\|_W = \|y\|_X + \|y'\|_{X^*} \) for all \( y \in W \), where the derivative \( y' \) of the element \( y \in X \) is considered in the sense of scalar distribution space \( D^*(S; V^*) = \mathcal{L}(D(S); V_w^*) \) with \( V = V_1 \cap V_2 \) and \( V_w^* = (V^*, \sigma(V^*, V)) \) [2]. We note that \( W \) is a reflexive Banach space with a compact embedding \( W \subset Y \) [10].

3. Classes of Maps

Let \( Y \) be a reflexive Banach space, \( Y^* \) be its topologically conjugated space, \( \langle \cdot, \cdot \rangle : Y^* \times Y \to \mathbb{R} \) be the pairing, and \( A : Y \to Y^* \) be a strict multi-valued map. Let us define its upper support function \([A(y), w]_+ := \sup_{d \in A(y)} \langle d, w \rangle y \) and lower support function \([A(y), w]_- := \inf_{d \in A(y)} \langle d, w \rangle y \), where \( y, w \in Y \), and its upper norm \( \|A(y)\|_+ := \sup_{d \in A(y)} \|d\|_{Y^*} \) and lower norm \( \|A(y)\|_- := \inf_{d \in A(y)} \|d\|_{Y^*} \). Consider the associated maps \( \co A : Y \to Y^* \) and \( \overline{\co} A : Y \to Y^* \) defined by the relations \( \co A(y) = \co \langle A(y) \rangle \) and \( \overline{\co} A(y) = \overline{\co} \langle A(y) \rangle \) respectively, where \( \overline{\co} \langle A(y) \rangle \) is the weak closure of \( \co A(y) \) in \( Y^* \) and \( \co A(y) \) is the convex hull of \( A(y) \subset Y^* \).

**Proposition 1** ([18]). Let \( A, B : Y \to Y^* \). Then

1. for all \( y, v_1, v_2 \in Y \) the relations
   \[
   [A(y), v_1 + v_2]_+ \leq [A(y), v_1]_+ + [A(y), v_2]_+,
   \]
   \[
   [A(y), v_1 + v_2]_- \geq [A(y), v_1]_- + [A(y), v_2]_-,
   \]
   \[
   [A(y), v_1 + v_2]_+ \geq [A(y), v_1]_+ + [A(y), v_2]_-,
   \]
   \[
   [A(y), v_1 + v_2]_- \leq [A(y), v_1]_+ + [A(y), v_2]_-.
   \]

are satisfied;
2. the equalities
   \[
   [A(y), v]_+ = -[A(y), -v]_-,
   \]
   \[
   [A(y) + B(y), v]_{+(-)} = [A(y), v]_{+(-)} + [B(y), v]_{+(-)}
   \]

hold for all \( y, v \in Y \);
3. \([A(y), v]_{+(-)} = [\overline{\co} A(y), v]_{+(-)} \) for all \( y, v \in Y \);
4. for all \( y, v \in Y \) the relations
   \[
   [A(y), v]_{+(-)} \leq \|A(y)\|_{+(-)} \|v\|_Y,
   \]
   \[
d_H(A(y), B(y)) \geq \|A(y)\|_{+(-)} - \|B(y)\|_{+(-)}
   \]
   \[
   \|A(y) - B(y)\|_+ \geq \|A(y)\|_+ - \|B(y)\|_-.\]
are fulfilled, where \( d_H (\cdot, \cdot) \) is the Hausdorff metric.

**Proposition 2** ([18]). The inclusion \( d \in \overline{co} A(y) \) is true if and only if
\[
[A(y), v]_+ \geq \langle d, v \rangle_Y \quad \text{for all } v \in Y.
\]

**Proposition 3** ([18]). Let \( D \subset Y \) and \( a(\cdot, \cdot) : D \times Y \to \mathbb{R} := \mathbb{R} \cup \{+\infty\} \). For every \( y \in D \), the functional \( Y \ni w \mapsto a(y, w) \) is positively homogeneous, convex, and lower semi-continuous if and only if there exists a multi-valued map \( A : Y \rightrightarrows Y^* \) such that \( D(A) = D \) and
\[
a(y, w) = [A(y), w]_+ \quad \text{for all } y \in D(A), \ w \in Y.
\]

**Remark 1.** In what follows, \( y_n \rightharpoonup y \) in \( Y \) means that \( y_n \) weakly converges to \( y \) in a reflexive Banach space \( Y \).

**Definition 1.** Let us denote the family of all non-empty closed convex bounded subsets of the space \( Y \) by \( C_v(Y) \).

**Definition 2.** An operator \( A : X \rightrightarrows X^* \) is called a *Volterra type operator* if, for any \( t \in S \), from the equality \( u(s) = v(s) \) for a.e. \( s \in [0, t] \) \((u, v \in X)\) it follows that \( \langle \overline{co} A(u) \rangle(s) = \langle \overline{co} A(v) \rangle(s) \) for a.e. \( s \in [0, t] \), i.e., \( [A(u), \xi_t]_+ = [A(v), \xi_t]_+ \) for all \( \xi_t \in X \) such that \( \xi_t(s) = 0 \) for a.e. \( s \in S \setminus [0, t] \).

**Definition 3.** A strict multi-valued map \( A : Y \rightrightarrows Y^* \) is called:

1. \(+(-)\)-coercive if there exists a lower bounded, on bounded in \( Y \) sets, real function \( \gamma : \mathbb{R}_+ \to \mathbb{R} \) such that \( \gamma(s) \to +\infty \) as \( s \to +\infty \) and
   \[
   [A(y), y]_+ \gamma(s) \geq \gamma(\|y\|_Y) \|y\|_Y \quad \text{for all } y \in Y;
   \]
2. bounded if for any \( L > 0 \) there is \( l > 0 \), such that
   \[
   \|A(y)\|_+ \leq l \quad \text{for all } y \in Y, \ \|y\|_Y \leq L;
   \]
3. locally bounded if for all \( y \in Y \) there exist \( m > 0 \) and \( M > 0 \) such that
   \[
   \|A(\xi)\|_+ \leq M \quad \text{for all } \xi \in Y, \ \|y - \xi\|_Y \leq m;
   \]
4. finite-dimension locally bounded if \( A|_F \) is locally bounded on \( (F, \|\cdot\|_Y) \) for any finite-dimensional subspace \( F \subset Y \).

**Definition 4.** We say that a multi-valued map \( A : X \rightrightarrows X^* \) possesses the *property* \((\Pi)\) if the following implication holds: If for some non-empty bounded subset \( B \subset Y \), constant \( k > 0 \), and selector \( d \) of \( A \), the relation
\[
\langle d(y), y \rangle_Y \leq k \quad \text{for all } y \in B
\]
holds, then there is a \( K > 0 \) such that
\[
\|d(y)\|_{Y^*} \leq K \quad \text{for all } y \in B.
\]
Definition 5. We say that a function \( C: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \) belongs to the class \( \Phi \) if \( C(r_1:\cdot): \mathbb{R}_+ \to \mathbb{R} \) is continuous for any \( r_1 \geq 0 \) and
\[
\lim_{\tau \to 0^+} \tau^{-1} C(r_1 ; \tau r_2) = 0
\]
for all \( r_1, r_2 \geq 0 \).

Now let \( W \) be some normed space with the norm \( \| \cdot \|_W \). We suppose that \( W \subset Y \) with a continuous embedding. Let also \( \| \cdot \|'_W \) be a (semi-)norm on \( Y \) which is compact with respect to \( \| \cdot \|_W \) on \( W \) and continuous with respect to \( \| \cdot \|_Y \) on \( Y \). Moreover, let \( C \in \Phi \).

Definition 6. A strict multi-valued map \( A: Y \to Y^* \) is called:

1. **radially lower semi-continuous** (or, shortly, RLSC) if
\[
\liminf_{t \to 0^+} [A(y + t \xi), \xi]_+ \geq [A(y), \xi]_-
\]
for all \( y, \xi \in Y \);

2. **radially upper semi-continuous** (or RUSC) if, for all \( y, \xi \in Y \), the real function \( t \mapsto [A(y + t \xi), \xi]_+ \) is upper semi-continuous from the right at the point \( t = 0 \);

3. **operator with semi-bounded variation on** \( W \) (or \( (Y, W);\text{-SBV} \)) if, for all \( R \geq 0 \) and \( y_1, y_2 \in Y \) such that \( \| y_1 \|_Y \leq R \) and \( \| y_2 \|_Y \leq R \), the inequality
\[
[A(y_1), y_1 - y_2]_- \geq [A(y_2), y_1 - y_2]_+ - C(R; \| y_1 - y_2 \|'_W)
\]
is satisfied;

4. **operator with \( N \)-semi-bounded variation on** \( W \) (or \( N;\text{-SBV} \) on \( W \)) if, for all \( R \geq 0 \) and every \( y_1, y_2 \in Y \) such that \( \| y_1 \|_Y \leq R \) and \( \| y_2 \|_Y \leq R \), the condition
\[
[A(y_1), y_1 - y_2]_- \geq [A(y_2), y_1 - y_2]_+ - C(R; \| y_1 - y_2 \|'_W)
\]
holds;

5. **\( \lambda_0 \)-pseudomonotone on** \( W \) (or \( W_{\lambda_0};\text{-pseudomonotone} \)) if, for arbitrary sequences \( \{y_n\}_{n \geq 0} \subset W \) and \( \{d_n\}_{n \geq 1} \) such that \( d_n \in \overline{co}A(y_n) \) for all \( n \geq 1 \), \( y_n \rightharpoonup y_0 \) in \( W \), and \( d_n \rightharpoonup d_0 \) in \( Y^* \), from the inequality
\[
\limsup_{n \to \infty} \langle d_n, y_n - y_0 \rangle_Y \leq 0
\]
it follows that there exist subsequences \( \{y_{nk}\}_{k \geq 1} \subset \{y_n\}_{n \geq 1} \) and \( \{d_{nk}\}_{k \geq 1} \subset \{d_n\}_{n \geq 1} \) for which the inequality
\[
\liminf_{k \to \infty} \langle d_{nk}, y_{nk} - w \rangle_Y \geq [A(y_0), y_0 - w]_-
\]
holds for all \( w \in Y \).

Remark 2. The idea on passing to a subsequence in the definition of a single-valued pseudomonotone operator was proposed by Skrypnik [15].
Lemma 1 ([18]). Any strict multi-valued operator \( A: Y \rightrightarrows Y^* \) with \((Y;W)\)-SBV is bounded-valued, locally bounded, and satisfies property (II). Furthermore, if \( A \) is RLSC, then it is also \( \lambda_0 \)-pseudomonotone on \( W \).

Let \( Y := Y_1 \cap Y_2 \), where \((Y_1, \|\cdot\|_{Y_1})\) and \((Y_2, \|\cdot\|_{Y_2})\) are some reflexive Banach spaces.

Definition 7. A pair \((A;B)\) of maps \( A: Y_1 \rightarrow C_v(Y_1^*) \) and \( B: Y_2 \rightarrow C_v(Y_2^*) \) is called \( s\)-mutually bounded if, for any constant \( M > 0 \), bounded set \( D \subset Y \), and selectors \( d_A \) of \( A \) and \( d_B \) of \( B \), there exists a \( K > 0 \) such that the relations \( y \in D \) and

\[
\langle d_A(y), y \rangle_{Y_1} + \langle d_B(y), y \rangle_{Y_2} \leq M
\]

imply that \( \|d_A(y)\|_{Y_1^*} \leq K \) or \( \|d_B(y)\|_{Y_2^*} \leq K \).

Remark 3. A bounded strict multi-valued map \( A: Y \rightrightarrows Y^* \) satisfies condition (II).

If one operator of the pair \((A;B)\) is bounded, then the pair \((A;B)\) is \( s\)-mutually bounded. Moreover, if both operators from \((A;B)\) satisfy condition (II), then their sum also satisfies condition (II) and the pair \((A;B)\) is \( s\)-mutually bounded.

Let now \( W := W_1 \cap W_2 \), where \((W_1, \|\cdot\|_{W_1})\) and \((W_2, \|\cdot\|_{W_2})\) are Banach spaces such that \( W_i \subset Y_i, i = 1,2 \), with a continuous embedding.

Lemma 2 ([7]). Let \( A: Y_1 \rightarrow C_v(Y_1^*) \) and \( B: Y_2 \rightarrow C_v(Y_2^*) \) be multi-valued maps which are \( \lambda_0 \)-pseudomonotone on \( W_1 \) and \( W_2 \), respectively, and such that the pair \((A;B)\) is \( s\)-mutually bounded. Then the map \( C := A + B : Y \rightarrow C_v(Y^*) \) is \( \lambda_0 \)-pseudomonotone on \( W \).

Definition 8. We say that a multi-valued map \( A: X \rightrightarrows X^* \) satisfies condition (H) if, for any \( y \in X \), \( n \geq 1 \), \( \{d_i\}_{i=1}^n \subset A(y) \) and measurable \( E_j \subset S \) \((j = 1, \ldots, n)\) such that \( \bigcap_{j=1}^n E_j = S \) and \( E_i \cap E_j = \emptyset \) for all \( i, j = 1, \ldots, n, i \neq j \), the inclusion \( d \in \overline{\operatorname{co}}A(y) \) holds, where \( d = \sum_{j=1}^n d_j \mathbb{1}_{E_j} \) and

\[
\mathbb{1}_{E_j}(\tau) := \begin{cases} 
1 & \text{for } \tau \in E_j, \\
0 & \text{for } \tau \in S \setminus E_j.
\end{cases}
\]

4. MAIN RESULT

Theorem 1. Let \( \lambda_A \geq 0 \) be fixed, \( p_0 := \min\{p_1, p_2\} \), the space \( V \) be compactly embedded in some Banach space \( V_0 \), and the embedding \( V_0 \subset V^* \) be continuous. Moreover, let the map \( A + \lambda_A I: X \rightarrow C_v(X^*) \) be \(+\)-coercive and RLSC multi-valued map of the Volterra type with \((X;W)\)-SBV \((\|\cdot\|_W = \|\cdot\|_{L_{p_0}(S;V_0)})\) satisfying condition (H). Let \( B: Y \rightarrow C_v(Y^*) \) be a multi-valued operator of the Volterra type which fulfills condition (H), the growth condition

\[
\|B \|_{Y^*} \leq c_1 \|y\|_Y + c_2 \quad \text{for all } y \in Y
\]

(4.1)

*Here, \( I: X \rightarrow X^* \) is the identical motion.*
with some \( c_1, c_2 \geq 0 \), and the continuity condition
\[
d_H(B(z), B(z_0)) \to 0 \quad \text{as } z \to z_0.
\] (4.2)

Then for any \( a_0 \in V \) and \( f \in X^* \) there exist at least one solution \( u \) of problem (2.1) with \( u' \in W \).

Here, \( d_H(\cdot, \cdot) \) is the Hausdorff metric on \( C_v(Y^*) \), i.e.,
\[
d_H(C, D) := \max\{\text{dist}(C; D), \text{dist}(D, C)\}
\]
with \( \text{dist}(C; D) := \sup_{c \in C} \inf_{d \in D} \|c - d\|_Y \) for \( C, D \in C_v(Y^*) \).

Proof. Let us reduce the evolution inclusion (2.1) to a first-order inclusion. Let \( R: X \to X \) (resp., \( R: Y \to Y \)) be the Volterra type operator defined by the relation
\[
(Rv)(t) = a_0 + \int_0^t v(s)ds \quad \text{for all } t \in S \text{ and every } v \in X \text{ (resp., } v \in Y \).
\]
It is clear that \( R \) is a Lipschitz continuous operator from \( X \) into \( X \) (resp., from \( Y \) into \( Y \)). Consider the problem
\[
\left\{
\begin{array}{l}
v' + A(v) + B(Rv) \ni f; \\
v(0) = 0, \quad v \in W.
\end{array}
\right.
\] (4.3)
If \( v \in W \) is a solution of problem (4.3), then \( u = Rv \in X \) is a solution of problem (2.1) such that \( u' \in W \).

Let us set \( A := A + B \circ R: X \to C_v(X^*) \) and \( \lambda = \lambda_A + \lambda_B \), where \( \lambda_B = 1 + c_1 c_3 \) and \( c_3 \) is the Lipschitz constant for the operator \( R: Y \to Y \). For an arbitrary \( y \in X \) and a.e. \( t \in S \), we set
\[
y_\lambda(t) = e^{-\lambda t} y(t), \quad \gamma_\lambda(t) = e^{\lambda t} y(t),
\] (4.4)
and
\[
(A_\lambda y)(t) = e^{-\lambda t} (A_\lambda \gamma_\lambda)(t) + \lambda y(t).
\]
Then \( g \in A_\lambda(y_\lambda) \iff \langle g, w \rangle_X \leq [A(y) + \lambda y, w_\lambda]_+ \) for all \( w \in X \). The set \( A_\lambda(y_\lambda) \) is non-empty because every \( g \) defined by the relation
\[
g(t) = e^{-\lambda t} d(t) + \lambda y_\lambda(t) \quad \text{for a.e. } t \in S \text{ and all } d \in A(y)
\]
belongs to \( A_\lambda(y_\lambda) \).

We note that \( A_\lambda: X \to C_v(X^*) \) and \( v \in W \) is a solution of problem (4.3) if and only if \( v_\lambda \in W \) is such that
\[
v_\lambda' + A_\lambda v_\lambda \ni f_\lambda, \quad v_\lambda(0) = 0,
\] (4.5)
where \( f_\lambda(t) = e^{-\lambda t} f(t) \). It turns out that \( A_\lambda: X \to C_v(X^*) \) possesses the following properties:

\((\alpha_1)\) \( A_\lambda \) is \(+\)-coercive on \( X \),
(α₂) $A_\lambda$ is $\lambda_0$-pseudomonotone on $W$, 
(α₃) $A_\lambda$ is locally bounded on $X$, 
(α₄) $A_\lambda$ satisfies condition (II) on $X$.

Let us prove the assertion above.

**PROPERTY (α₁).** Let us fix $y \in X$, $\|y\|_X \neq 0$. As $\|y\|_X \leq \|y\|_X$, then

$$\|y\|_X^{-1} [A_\lambda y, y]_+ \geq \|y\|_X^{-1} \sup_{\tau \in A(y)} \int_0^T e^{-2\lambda t} (\xi(y)(t) + \lambda_A y(t), y(t)) dt$$

$$+ \|y\|_X^{-1} \inf_{\tau \in B(Ry)} \int_0^T e^{-2\lambda t} (\xi(y)(t) + \lambda_B y(t), y(t)) dt.$$  \hspace{1cm} (4.6)

We first estimate the first term. We remark that

$$[(A + \lambda_A I) y, y]_+ \geq \widehat{\varphi}(\|y\|_X) \|y\|_X \text{ for all } y \in X,$$

where $\widehat{\varphi} : \mathbb{R}_+ \to \mathbb{R}$ can be chosen as a non-decreasing function lower bounded on bounded, in $\mathbb{R}_+$, sets such that $\widehat{\varphi}(r) \to +\infty$ as $r \to \infty$.

Since $A$ is a Volterra type operator, we see that, for any $u \in X$,

$$\sup_{\xi(t) \in A(u)} \int_0^t (\xi(u)(\tau) + \lambda_A u(\tau), u(\tau)) d\tau \geq \widehat{\varphi}(\|u\|_{X_t}) \|u\|_{X_t}, \text{ for all } t \in S,$$

where $\|u\|_{X_t} = \|u_t\|_X$. Let us set

$$g_{\xi(t)}(\tau) = (\xi(t)(\tau) + \lambda_A y(\tau), y(\tau)), \quad \xi(t) \in A(y), \quad \tau \in S,$$

and $h(t) = \widehat{\varphi}(\|y\|_{X_t}) \|y\|_{X_t}$ for $t \in S$. Then $h(t) \geq \min\{\widehat{\varphi}(0), 0\} \|y\|_X$ and

$$\sup_{\xi(t) \in A(y)} \int_0^t g_{\xi(t)}(\tau) d\tau \geq h(t)$$

for all $t \in S$. Similarly to the definition of $A_\lambda$, for any $u \in X$ and a.e. $t \in S$, we put

$$(A_1 u)(t) = (e^{-2\lambda t} - e^{-2\lambda T})(A(u)(t) + \lambda_A u(t)),$$

$$(A_2 u)(t) = e^{-2\lambda T}((A(u)(t) + \lambda_A u(t)).$$

and

$$(A\hat{u})(t) = e^{-2\lambda t}((A(u)(t) + \lambda_A u(t)).$$

Then, due to Proposition 1, we get

$$[\hat{A} y, y]_+ = [A_1 y, y]_+ + [A_2 y, y]_+$$

$$\geq e^{-2\lambda T} h(T) + 2\lambda T \sup_{\tau \in A(y)} \inf_{s \in S} e^{-2\lambda s} \int_0^\tau (\xi(y)(\tau) + \lambda_A y(\tau), y(\tau)) d\tau.$$
Further, using condition \((H)\) for the operator \(A\), we prove that
\[
\sup_{\zeta(y) \in A(y)} \inf_{s \in S} e^{-2\lambda s} \int_{0}^{s} (\zeta(y)(t) + \lambda A y(t), y(t)) \, dt \geq -C_{1} \|y\|_{X},
\]
where \(C_{1} = \max \{|-\tilde{\gamma}(0), 0|\} \geq 0\) does not depend on \(y\). Consequently, we obtain
\[
\|y\|_{X}^{-1} \sup_{\zeta(y) \in A(y)} \int_{0}^{T} e^{-2\lambda t} (\zeta(y)(t) + \lambda A y(t), y(t)) \, dt \geq -C_{1} \|y\|_{X},
\]
(4.7)
Let us estimate the second term. Analogously to the previous case, using the Volterra property of the operator \(B \circ R\), we obtain that, for all \(t \in S\),
\[
\inf_{\zeta(y) \in B(Ry)} \int_{0}^{t} (\zeta(y)(t) + \lambda B y(t), y(t)) \, dt \geq -(c_{2} + c_{1} \|R\|_{Y} c_{4} \|y\|_{X}) > -\infty,
\]
where \(c_{4} > 0\) is such that \(\|\cdot\|_{Y} \leq c_{4} \|\cdot\|_{X}\). Then
\[
\inf_{\zeta(y) \in B(Ry)} \int_{0}^{T} e^{-2\lambda t} (\zeta(y)(t) + \lambda B y(t), y(t)) \, dt \geq -e^{-2\lambda T} (c_{2} + c_{1} \|R\|_{Y} c_{4} \|y\|_{X})
\]
\[
+ 2\lambda \int_{0}^{T} e^{-2\lambda s} \inf_{\zeta(y) \in B(Ry)} \int_{0}^{s} (\zeta(y)(t) + \lambda B y(t), y(t)) \, dt \, ds
\]
\[
\geq -(c_{2} + c_{1} \|R\|_{Y} c_{4} \|y\|_{X}).
\]
Therefore, in view of (4.7), it follows from (4.6) that
\[
\|y\|_{X}^{-1} [A_{\lambda, y, \lambda}]_{+} \geq e^{-2\lambda T} \tilde{\gamma}(\|y\|_{X}) - 2\lambda C_{1} T - (c_{2} + c_{1} \|R\|_{Y} c_{4} \|y\|_{X}).
\]
Because \(\|y\|_{X} \geq \|y\|_{X}\) and the function \(\tilde{\gamma}\) is non-decreasing, since \(y\) is arbitrary, we have proved that \(A_{\lambda, y, \lambda} : X \to C_{0}(X^{*})\) is \(+\) coercive.

\textbf{PROPERTY (a2).} For any \(y \in X\) and a.e. \(t \in S\) we set
\[
(A_{\lambda, y}) (t) = e^{-\lambda t} (\tilde{\gamma}\lambda)(t) + \lambda A y(t), \quad (A_{\lambda, y}) (t) = e^{-\lambda t} (B(R\tilde{\gamma}) y)(t) + \lambda B y(t),
\]
where \(\tilde{\gamma}\) is given by (4.4). Let us note that \(A_{\lambda, y} + A_{\lambda, y} = A_{\lambda}\).

At first we show that \(A_{\lambda, y}\) is an RLSC operator with \((X; W)\)-SBV. Let us prove the semi-boundedness of the variation. By virtue of the assumptions of the theorem, for all \(R > 0\) and \(y, \xi \in X\) such that \(\|y\|_{X} \leq R\) and \(\|\xi\|_{X} \leq R\), we have
\[
[A(y) - A(\xi)] + \lambda A y - \lambda A \xi, y - \xi] \geq \lambda C_{A}(\|y - \xi\|_{W}) \geq 0.
\]
Let us set \(\hat{C}\tilde{A}(R):= \max_{t \in [0,T]} C\tilde{A}(R; \tau)\) for all \(R, t \geq 0\) (note that \(\hat{C}\tilde{A} \in \Phi\) and
\[
\hat{z}_{\tau}(t) := \begin{cases} z(t) & \text{for } 0 \leq t \leq T, \\ 0 & \text{for } t < T \leq T. \end{cases}
\]
for a.e. \( t \in S \) and all \( z \in X \). Let \( \zeta \) and \( \eta \) be fixed selectors of \( A \). Since \( A \) is a Volterra type operator, for all \( y, \xi \in X \) and \( t \in S \), we have

\[
\int_0^t \left( \xi(y)(\tau) + \lambda_A y(\tau) - \eta(\xi)(\tau) - \lambda_A \xi \right) d\tau + \tilde{C}_A(R; \|y - \xi\|_{W_t}) \geq \|A + \lambda_A I\| \|y_t - \xi_t\|_{W_t} - \tilde{C}_A(R; \|y_t - \xi_t\|_{W_t}) \geq 0
\]

because \( \|y_t\|_{X} \leq \|y\|_{X} \) and \( \|y_t - \xi_t\|_{W_t} \leq \|y - \xi\|_{W_t} \). Here, \( \|y\|_{W_t} = \|g\|_{L^{p_0}(0,t;V_0)} \).

Let us fix \( y, \xi \in X \) and set \( g(\tau) = (\xi(y)(\tau) + \lambda_A y(\tau) - \eta(\xi)(\tau) - \lambda_A \xi \), \( t \in S \),

and \( h(t) = \tilde{C}_A(R; \|y - \xi\|_{W_t}), t \in S \). We have proved that

\[
\int_0^t g(\tau) d\tau \geq -h(t) \quad \text{for all } t \in S.
\]

The function \( S \ni t \mapsto h(t) \) is non-decreasing and, thus,

\[
\int_0^T e^{-2\lambda t} g(\tau) d\tau = e^{-2\lambda T} \int_0^T g(\tau) d\tau + 2\lambda \int_0^T e^{-2\lambda t} \int_0^\tau g(s) ds d\tau \geq -h(T).
\]

Consequently,

\[
[A^1_\lambda y_\lambda, y_\lambda - \xi_\lambda] - \geq [A^1_\lambda \xi_\lambda, y_\lambda - \xi_\lambda] - \tilde{C}_A(R; \|y - \xi\|_{L^{p_0}(S;V_0)}) \quad \text{for all } y, \xi \in X. \quad (4.8)
\]

Now we consider the space \( L^{p_0}(S;V_0) \) that consists of measurable functions \( g: S \to V_0 \) for which the integral \( \int_S e^{\lambda t} g(t) \|g(t)\|_{V_0}^{p_0} dt \) is finite. Then

\[
\|y - \xi\|_{L^{p_0}(S;V_0)} = \left( \int_S e^{\lambda t} \|y_\lambda(t) - \xi_\lambda(t)\|_{V_0}^{p_0} dt \right)^{1/p_0} = \|y_\lambda - \xi_\lambda\|_{L^{p_0}(S;V_0)}.
\]

Therefore, from (4.8) we obtain

\[
[A^1_\lambda y_\lambda, y_\lambda - \xi_\lambda] - \geq [A^1_\lambda \xi_\lambda, y_\lambda - \xi_\lambda] - \tilde{C}_A(R; \|y_\lambda - \xi_\lambda\|_{L^{p_0}(S;V_0)}).
\]

The proof of the fact that the mapping \( A^1_\lambda : X \to C_b(X^*) \) has \((X;W)\)-SBV is concluded by taking into account the compactness of the embedding \( W \subset L^{p_0}(S;V_0) \) [10, Theorem 1.5.1]. The RLSC for \( A^1_\lambda \) is clear.

Since an arbitrary RLSC multi-valued operator with \((X;W)\)-SBV is \(\lambda_0\)-pseudomonotone on \( W \) [8], we have proved that \( A^1_\lambda \) is \(\lambda_0\)-pseudomonotone on \( W \).

Let us now consider \( A^2_\lambda \). We first show that \( A^2_\lambda \) is an operator with \( N \)-SBV on \( W \) (the radial upper semi-continuity is clear).
We first prove that $(B \circ R) : X \to C_0^v(X^*)$ is the operator with $N$-SBV on $W$ with \( \|\cdot\|_W = \|\cdot\|_Y \). For all $R \geq 0$ and $t \geq 0$, we set
\[
C_B(R, t) = t \sup \{ d_H(B(Rz_1), B(Rz_2)) \mid z_1, z_2 \in X : \|z_1 - z_2\|_Y \leq t \}
\]
and $\|z_i\|_X \leq R, i = 1, 2$.

Similarly to [17], we show that $C_B \in \Phi$. Then for any $R > 0$ and $y, \xi \in X$ such that $\|y\|_X \leq R$, $\|\xi\|_X \leq R$ it follows that
\[
[B(Ry) + \lambda_B y, y - \xi]_+ - [B(R\xi) + \lambda_B \xi, y - \xi]_+ + C_B(R ; y - \xi) \geq 0.
\]

Let us set $\hat{C}_B(R; t) = \max_{\tau \in [0, t]} C_B(R; \tau)$ for all $R, t \geq 0$ (note that $\hat{C}_B \in \Phi$).

Since $B \circ R$ is the Volterra type operator, for all $y, \xi \in X$ and $t \in S$, we have
\[
\inf_{\xi(y) \in B(Ry)} \int_0^t \left( \xi(y)(\tau) + \lambda_B y(\tau), y(\tau) - \xi(\tau) \right) d\tau - \inf_{\eta(\xi) \in B(R\xi)} \int_0^t \left( \eta(\xi)(\tau) + \lambda_B \xi(\tau), y(\tau) - \xi(\tau) \right) d\tau + \hat{C}_B(R ; y - \xi) \geq 0,
\]

where $\|\cdot\|_{Y_{\tau}} = \|\cdot\|_{L_2([0, t]; H)}$. Let us fix some $y, \xi \in X$ and set
\[
g_{\beta} = (\beta(\tau), y(\tau) - \xi(\tau), \beta \in X^*, \tau \in S.
\]

and $h(t) = \hat{C}_B(R; y - \xi)_{|_{Y_{\tau}}}, t \in S$. We have thus proved that
\[
\inf_{\xi \in B(Ry) + \lambda_B y} \int_0^t g_{\xi}(\tau) d\tau - \inf_{\eta \in B(R\xi) + \lambda_B \xi} \int_0^t g_{\eta}(\tau) d\tau \geq -h(t) \quad \text{for all } t \in S.
\]

The function $S \ni t \mapsto h(t)$ is non-decreasing and thus, for an arbitrary $\zeta \in B(Ry) + \lambda_B y$, we get
\[
\int_0^T e^{-2\lambda_T} (\zeta(\tau), y(\tau) - \xi(\tau)) d\tau - \inf_{\eta \in B(R\xi) + \lambda_B \xi} \int_0^T e^{-2\lambda_T} (\eta(\tau), y(\tau) - \xi(\tau)) d\tau
\]
\[
= \sup_{\eta \in B(R\xi) + \lambda_B \xi} \int_0^T e^{-2\lambda_T} (\zeta(\tau) - \eta(\tau), y(\tau) - \xi(\tau)) d\tau
\]
\[
\geq e^{-2\lambda_T} \inf_{\eta \in B(R\xi) + \lambda_B \xi} \int_0^T (\zeta(\tau) - \eta(\tau), y(\tau) - \xi(\tau)) d\tau
\]
\[
+ \sup_{\eta \in B(R\xi) + \lambda_B \xi} \int_0^T (e^{-2\lambda_T} - e^{-2\lambda_T}) (\zeta(\tau) - \eta(\tau), y(\tau) - \xi(\tau)) d\tau
\]
\[
\geq -e^{-2\lambda_T} h(T) + 2\lambda_T \sup_{\eta \in B(R\xi) + \lambda_B \xi} \inf_{s \in S} \int_0^s (\zeta(\tau) - \eta(\tau), y(\tau) - \xi(\tau)) d\tau.
\]
To prove that

\[ \liminf_{n \to \infty} (d_{\lambda,n}, y_{\lambda,n} - v_{\lambda}) \geq \liminf_{n \to \infty} [A_{\lambda}^2(y_{\lambda,n}), y_{\lambda,n} - v_{\lambda}] - C_B(R; y_{\lambda} - v_{\lambda}) \]  

(4.10)

At first we estimate the first term at the right-hand side of (4.10). It is easy to show that the function \( Y_{\lambda} \equiv h \mapsto [A_{\lambda}^2(v_{\lambda}), h] \) is continuous for all \( v \in X \).
Therefore, from (4.10) we obtain that 
\[
\liminf_{n \to \infty} \langle d_{\lambda,n}, y_{\lambda,n} - y_{\lambda} \rangle \to 0
\]
and 
\[
\liminf_{n \to \infty} \langle d_{\lambda,n}, y_{\lambda,n} - v_{\lambda} \rangle \geq \left[ A^2(\lambda)(v_{\lambda}), y_{\lambda} - v_{\lambda} \right]_{-} - \bar{C}_B \| v_{\lambda} - v_{\lambda} \|_{X}
\]
for all \( v \in X \). Substituting \( tw_{\lambda} + (1-t)y_{\lambda} \), where \( w \in X \), \( t \in [0,1] \), for \( v_{\lambda} \) in the last inequality, dividing the result by \( t \), and passing to the limit as \( t \to 0^+ \), due to the RUSC for \( A^2_{\lambda} \), we obtain
\[
\liminf_{n \to \infty} \langle d_{\lambda,n}, y_{\lambda,n} - w_{\lambda} \rangle \geq \left[ A^2_{\lambda}(y_{\lambda}), y_{\lambda} - w_{\lambda} \right]_{-}
\]
for all \( w \in X \). Therefore, the \( \lambda_0 \)-pseudomonotony \( A^2_{\lambda} \) on \( W \) is proved.

In order to prove the \( \lambda_0 \)-pseudomonotony of \( A^2_{\lambda} \) on \( W \), we use Lemma 2. Let us note that the pair \((A^1_{\lambda}, A^2_{\lambda})\) is \( s \)-mutually bounded because \( A^2_{\lambda} \) is bounded as a consequence of (4.1) and the boundedness of the identity map.

**Properties \((a_3)\) and \((a_4)\).** These properties follow from \((X; W)\)-SBV of \( A^1_{\lambda}, N\)-SBV of \( A^2_{\lambda} \), and Lemma 1.

In order to prove the solvability for problem (4.3) we use [8, Theorem 3.1]. Let \( L : W_{\lambda} \subset X \to X^* \) be a densely defined linear operator, \( Lz = z', \) \( D(L) = W_{\lambda} = \{ z \in W : z(0) = 0 \} \), and \( A_{\lambda} : X \to C_v(X^*) \) be a multi-valued map. Let us consider the problem
\[
Lz + A_{\lambda}z \ni f_{\lambda}, \quad z \in W_{\lambda}.
\]
Let us note that \( D(L) = W_{\lambda} \) is a reflexive Banach space with respect to the graph norm of the derivative. Conditions \((a_1)-(a_4)\) guarantee that there exists at least one solution \( z \) of problem (4.3) in \( W_{\lambda} \). The function \( \tilde{z}_{\lambda} \) is then a solution of problem (4.3) and \( R\tilde{z}_{\lambda} \) is a solution of the original problem. □

**Corollary 1.** Assume that \( \lambda_0 \geq 0 \) is fixed, \( p_2 \geq 2 \), \( p_0 = \min \{ p_1, p_2 \} \), the space \( V \) is compactly embedded in a Banach space \( V_0 \), and the embedding \( V_0 \subset X^* \) is continuous. Moreover, let \( A + \lambda_0 I : X \to C_v(X^*) \) be a \( \gamma \)-coercive and RLSC multi-valued operator of the Volterra type with \((X; W)\)-SBV \((\| \cdot \|_W = \| \cdot \|_{L_{p_0}(S; V_0)})\) satisfying condition \((H)\), and \( B : Y \to C_v(Y^*) \) be a multi-valued operator of the Volterra type satisfying condition \((H)\), the growth condition (4.1), and the continuity condition (4.2)\(^\dagger\), and \( C : X \to X^* \) be an operator with the property
\[
(Cu)(t) = C_0(u(t)) \quad \text{for all} \; u \in X, \; t \in S,
\]
where \( C_0 : V_2 \to V_2^* \) is a linear, bounded, self-conjugate, and monotone operator.

Then for arbitrary \( a_0 \in V \) and \( f \in X^* \) there exists at least one solution of the problem
\[
\begin{align*}
y'' + Ay' + By + Cy & \ni f, \\
y(0) = a_0, \; y'(0) = \bar{0}, \; y \in C(S; V), \; y' \in C(S; H).
\end{align*}
\]

\(\dagger\)We recall that \( d_H(\cdot, \cdot) \) is the Hausdorff metric on \( C_v(Y^*)\).
5. AN EXAMPLE

Let $\Omega \subset \mathbb{R}^n$ be a bounded region with the regular boundary $\partial \Omega$, $S = [0, T]$, $Q = \Omega \times S$, $I_T = \partial \Omega \times S$, $1 < p = p_1 = p_2$, and $\Phi : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the “growth condition”

$$|\Phi(t)| \leq c_1|t| + c_2 \quad \text{for all } t \in \mathbb{R},$$

(5.1)

where $c_1, c_2 \in \mathbb{R}$, and the “sign condition”

$$(\Phi(t) - \Phi(s))(t - s) \geq -c_3(s-t)^2 \quad \text{for all } t, s \in \mathbb{R}$$

(5.2)

with some $c_3 > 0$. Moreover, let $S \times \mathbb{R} \ni (t, y) \mapsto \theta_i(t, y) \in \mathbb{R}^+, i = 1, 2$, be single-valued continuous functions such that

$$-c_2(1+|x|) \leq \theta_1(t, x) \leq \theta_2(t, x) \leq c_1(1+|x|) \quad \text{for all } t \in \mathbb{S}, \ x \in \mathbb{R},$$

(5.3)

where $c_1, c_2 \geq 0$. For an any $f \in X^* = L_2(S; L_2(\Omega)) + L_q(S; W^{-1,q}(\Omega))$, we consider the problem

$$\begin{align*}
\frac{\partial^2 y(x,t)}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \left| \frac{\partial^2 y(x,t)}{\partial x_i \partial t} \right|^{p-2} \frac{\partial^2 y(x,t)}{\partial x_i \partial t} \right) + \left| \frac{\partial y(x,t)}{\partial t} \right|^{p-2} \frac{\partial y(x,t)}{\partial t} \\
+ \Phi \left( \frac{\partial y(x,t)}{\partial t} \right) - \Delta y(x,t) \\
+ \left[ \theta_1(t, y(x,t)); \theta_2(t, y(x,t)) \right] \ni f(x,t) \quad \text{a.e. on } Q,
\end{align*}$$

(5.4)

$$y(x,0) = 0, \quad \frac{\partial y(x,t)}{\partial t} \bigg|_{t=0} = 0 \quad \text{a.e. on } \Omega,$$

$$y(x,t) = 0 \quad \text{a.e. on } \partial \Omega.$$

For the operator $A : L_p(S; W_0^{1,p}(\Omega)) \to L_q(S; W^{-1,q}(\Omega))$, we take $(Au)(t) = A(u(t)), t \in S$ [16, 17], where $A(\varphi) = A_1(\varphi) + A_2(\varphi),$

$$A_1(\varphi) = -\sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \left| \frac{\partial \varphi}{\partial x_i} \right|^{p-2} \frac{\partial \varphi}{\partial x_i} \right) + |\varphi|^{p-2} \varphi,$$

and $A_2(\varphi) = \Phi(\varphi)$ for all $\varphi \in C^2(\overline{\Omega})$. For the map $B : L_2(Q) \to L_2(Q)$, we take

$$B(u) = \{ v \in L_2(Q) \mid \theta_1(t, u(x,t)) \leq v(x,t) \leq \theta_2(t, u(x,t)) \text{ a.e. } (x,t) \in Q \},$$

and let the operator $C : L_2(S; H_0^1(\Omega)) \to L_2(S; H^{-1}(\Omega))$ be defined by the relation $(Cu)(t) = C_0(u(t)), t \in S$, where $C_0(v) = -\Delta v$ for $v \in H_0^1(\Omega)$. Moreover, let $H = L_2(\Omega)$, $V_1 = W_0^{1,p}(\Omega)$, $V_2 = H_0^1(\Omega)$, and let $Y = L_2(S; H) = L_2(\Omega),$ 

$$X = L_p(S; V_1) \cap L_2(S; H) \cap L_2(S; V_2), \quad X^* = L_q(S; V_1^*) + L_2(S; V_2^*).$$

Then, according to Corollary 1, problem (5.4) has a solution $y \in C(S; V)$ such that $y' \in C(S; H)$ and $y'' \in X^*.$
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