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THE MIXED BVP FOR THE SECOND ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATION AT RESONANCE

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Abstract. Efficient sufficient conditions are established for the solvability of the mixed problem

$$u''(t) = p(t)u(t) + f(t,u(t)) + h(t), \qquad u(a) = 0, \quad u'(b) = 0,$$

where $h, p \in L([a,b]; R)$ and $f \in K([a,b] \times R; R)$, in the case where the homogeneous linear problem $w''(t) = p(t)w(t), \;\; w(a) = 0, \;\; w'(b) = 0$ has nontrivial solutions.

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1. Introduction

Consider on the set $I = [a,b]$ the second order nonlinear ordinary differential equation

$$u''(t) = p(t)u(t) + f(t,u(t)) + h(t) \quad \text{for} \quad t \in I \tag{1.1}$$

with the boundary conditions

$$u(a) = 0, \quad u'(b) = 0, \tag{1.2}$$

where $h, p \in L(I; R)$ and $f \in K(I \times R; R)$. By a solution of problem (1.1), (1.2) we understand a function $u \in \widetilde{C}'(I, R)$, which satisfies equation (1.1) almost everywhere on I and satisfies conditions (1.2).

Along with (1.1), (1.2) we consider the homogeneous problem

$$w''(t) = p(t)w(t) \quad \text{for} \quad t \in I, \tag{1.3}$$

$$w(a) = 0, \quad w'(b) = 0. \tag{1.4}$$

At present, the foundations of the general theory of two-point boundary value problems are already laid and problems of this type are studied by many authors and investigated in detail (see, for instance, [3, 4, 10, 11, 13, 14] and references therein). On the other hand, in all of these works, only the non-resonance case is considered. An analysis of the available literature shows that, in contrast to the Dirichlet problem, the case where the problem (1.3), (1.4) has nontrivial solutions is practically unstudied. It should be noted that, in the majority of works on this subject, the Dirichlet boundary value problem for the second order ordinary differential equation with the corresponding homogeneous problem possessing a nontrivial solution studied in the case where the first coefficient of the homogeneous linear problem is a constant and, more precisely, only in the simplest case where this constant is the first eigenvalue of the homogeneous linear problem (see, for instance, [1, 2, 4–8, 10, 15] and references therein). In [16], we developed a technique which allowed us to establish efficient sufficient conditions (Landesman–Lazer's type conditions) for the solvability of Dirichlet BVP for second order ODE in the case where the first coefficient of the homogeneous linear equation is a Lebesgue integrable function (not necessarily constant) and no information is assumed on the number of zeros of the solution. (In particular, if the first coefficient in homogeneous linear equation is constant, we are able to study the cases where this constant not necessarily coincides with the first eigenvalue of the corresponding homogeneous linear problem). The theorems proved there significantly generalize and improve a number of previous results of other authors (see [1, 2, 4, 6, 15]).

In the present paper we generalize the method developed in article [16] for the Dirichlet boundary value problem, and prove Landesman–Lazer's type efficient sufficient conditions for solvability of problem (1.1), (1.2) in the case when the function $p \in L(I;R)$ is not necessarily constant, under the assumption that the homogeneous problem (1.3), (1.4) has a nontrivial solution which may have arbitrarily many zeros in the interval $]a,b]$.

The results presented here are new and generalize Fredholm's third theorem for nonlinear ODE in the sense that the known Fredholm theorem is obtained in the special case where $f(t,x) \equiv 0$.

Throughout the paper we use the following notations:

N is the set of all natural numbers. R is the set of all real numbers, $R_+ = [0,+\infty[$.

$C(I;R)$ is the Banach space of continuous functions $u : I \to R$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$.

$\widetilde{C}'(I;R)$ is the set of functions $u : I \to R$ which are absolutely continuous together with their first derivatives.

$L(I;R)$ is the Banach space of the Lebesgue integrable functions $p : I \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$.

$K(I \times R;R)$ is the set of the functions $f : I \times R \to R$ satisfying the Carathéodory conditions, i.e., $f(\cdot,x) : I \to R$ is a measurable function for all $x \in R$, $f(t,\cdot) : R \to R$ is a continuous function for almost all $t \in I$, and for every $r > 0$ there exists $q_r \in L(I;R_+)$ such that $|f(t,x)| \le q_r(t)$ for almost all $t \in I$, $|x| \le r$.

Having $w : I \to R$, we put:

$$N_w \overset{\text{def}}{=} \{t \in]a,b] : w(t) = 0\},$$

$$\Omega_w^+ \overset{\text{def}}{=} \{t \in I : w(t) > 0\}, \quad \Omega_w^- \overset{\text{def}}{=} \{t \in I : w(t) < 0\},$$

and $[w(t)]_+ = (|w(t)|+w(t))/2,\ \ [w(t)]_- = (|w(t)|-w(t))/2$ for $t\in I$.

Definition 1. Let A be a finite (eventually empty) subset of I. We say that $f \in E(A)$, if $f \in K(I\times R;R)$ and, for any measurable set $G\subseteq I$ and an arbitrary constant $r>0$, we can choose $\varepsilon>0$ such that if

$$\int_G |f(s,x)|ds \neq 0 \ \text{ for } \ x\geq r \ (x\leq -r)$$

then

$$\int_{G\setminus U_\varepsilon} |f(s,x)|ds - \int_{U_\varepsilon} |f(s,x)|ds \geq 0 \quad \text{for} \quad x\geq r \ (x\leq -r),$$

where $U_\varepsilon = I\cap\big(\cup_{k=1}^n]t_k-\varepsilon/2n, t_k+\varepsilon/2n[\big)$ if $A=\{t_1,t_2,\dots,t_n\}$, and $U_\varepsilon=\varnothing$ if $A=\varnothing$.

Remark 1. If $f\in K(I\times R;R)$ then $f\in E(\varnothing)$.

Remark 2. It is clear that if $f(t,x)\overset{\text{def}}{\equiv} f_0(t)g_0(x)$, where $f_0\in L(I;R)$ and $g_0\in C(I;R)$, then $f\in E(A)$ for every finite set $A\subset I$.

The example below shows that there exists a function $f\in K(I\times R;R)$ such that $f\notin E(\{t_1,\dots,t_k\})$ for some points $t_1,\dots,t_k\in I$.

Example 1. Let $f(t,x)=|t|^{-1/2}g(t,x)$ for $t\in[-1,0[\cup]0,1]$, $x\in R$, and $f(0,.)\equiv 0$, where $g(-t,x)=g(t,x)$ for $t\in]-1,1]$, $x\in R$, and

$$g(t,x)=\begin{cases} x & \text{for} \quad x\leq 1/t,\ t>0\\ 1/t & \text{for} \quad x>1/t,\ t>0\end{cases}.$$

Then $f\in K([0,1]\times R;R)$ and it is clear that $f\notin E(\{0\})$ because, for every $\varepsilon>0$, if $x\geq 1/\varepsilon$ then $\int_\varepsilon^1 f(s,x)ds-\int_0^\varepsilon f(s,x)ds = 4(\varepsilon^{-1/2}-x^{1/2})-2<0$.

2. Main results

Theorem 1. *Let $i\in\{0,1\}$, w be a nonzero solution of the problem (1.3), (1.4), $f\in E(N_w)$, there exists a constant $r>0$ such that the function $(-1)^i f$ is non-decreasing in the second argument for $|x|\geq r$,*

$$(-1)^i f(t,x)\operatorname{sgn} x\geq 0 \quad \textit{for} \quad t\in I,\ |x|\geq r, \tag{2.1}$$

$$\int_{\Omega_w^+}|f(s,r)|ds+\int_{\Omega_w^-}|f(s,-r)|ds\neq 0, \tag{2.2}$$

and

$$\lim_{|x|\to+\infty}\frac{1}{|x|}\int_a^b |f(s,x)|ds = 0. \tag{2.3}$$

Then there exists $\delta > 0$ such that the problem (1.1)*,* (1.2) *has at least one solution for every h satisfying the condition*

$$\left|\int_a^b h(s)w(s)ds\right| < \delta\|w\|_C. \tag{2.4}$$

Corollary 1. *Let the assumptions of Theorem 1 be satisfied and*

$$\int_a^b h(s)w(s)ds = 0. \tag{2.5}$$

Then the problem (1.1)*,* (1.2) *has at least one solution.*

Example 2. From Theorem 1 it follows that the problem

$$u''(t) = -\lambda^2 u(t) + \sigma|u(t)|^\alpha \operatorname{sgn} u(t) + h(t) \quad \text{for} \quad 0 \le t \le \pi/2 \tag{2.6}$$

$$u(0) = 0, \quad u'(\pi/2) = 0, \tag{2.7}$$

with $\lambda = 2k - 1\,(k \in N)$, $\sigma \in \{-1, 1\}$, and $\alpha \in]0,1[$ has at least one solution if $h \in L([0, \pi/2], R)$ is such that $\int_0^{\pi/2} h(s)\sin\lambda s ds = 0$.

Theorem 2. *Let $i \in \{0, 1\}$, w be a nonzero solution of the problem* (1.3)*,* (1.4)*, $f(t,x) \overset{\text{def}}{\equiv} f_0(t)g_0(x)$ with $f_0 \in L(I; R_+)$, $g_0 \in C(R; R)$, there exists a constant $r > 0$ such that $(-1)^i g_0$ is non-decreasing for $|x| \ge r$ and*

$$(-1)^i g_0(x) \operatorname{sgn} x \ge 0 \qquad \textit{for} \quad |x| \ge r. \tag{2.8}$$

Let, moreover,

$$|g_0(r)| \int_{\Omega_w^+} f_0(s)ds + |g_0(-r)| \int_{\Omega_w^-} f_0(s)ds \ne 0 \tag{2.9}$$

and

$$\lim_{|x|\to+\infty} |g_0(x)| = +\infty, \qquad \lim_{|x|\to+\infty} \frac{g_0(x)}{x} = 0. \tag{2.10}$$

Then, for every $h \in L(I; R)$, the problem (1.1)*,* (1.2) *has at least one solution.*

Example 3. From Theorem 2 it follows that the equation

$$u''(t) = p_0(t)u(t) + p_1(t)|u(t)|^\alpha \operatorname{sgn} u(t) + h(t) \quad \text{for} \quad t \in I, \tag{2.11}$$

with the conditions (1.2) has at least one solution for arbitrary $\alpha \in]0,1[$ $p_0, h \in L(I; R)$, and such $p_1 \in L(I; R)$ that the condition $\sigma p_1(t) > 0$ for $t \in I$ holds, where $\sigma \in \{-1, 1\}$.

Theorem 3. *Let $i \in \{0,1\}$ and w be a nonzero solution of the problem (1.3), (1.4). Let, moreover, there exist constants $r > 0$, $\varepsilon > 0$, and functions $\alpha, f^+, f^- \in L(I;R_+)$ such that the conditions*

$$\begin{aligned} (-1)^i f(t,x) &\le -f^-(t) \quad \textit{for} \quad x \le -r, \\ f^+(t) &\le (-1)^i f(t,x) \quad \textit{for} \quad x \ge r, \end{aligned} \tag{2.12$_i$}$$

$$\sup\{|f(t,x)| : x \in R\} \le \alpha(t) \tag{2.13}$$

hold on I, and let

$$\begin{aligned} &-\int_a^b (f^+(s)[w(s)]_- + f^-(s)[w(s)]_+)ds + \varepsilon||\alpha||_L \\ &\le (-1)^{i+1}\int_a^b h(s)w(s)ds \\ &\le \int_a^b (f^-(s)[w(s)]_- + f^+(s)[w(s)]_+)ds - \varepsilon||\alpha||_L. \end{aligned} \tag{2.14}$$

Then the problem (1.1), (1.2) has at least one solution.

Remark 3. If $f \not\equiv 0$ then the condition (2.12$_i$) ($i = 1,2$) of Theorem 3 can be replaced by

$$\begin{aligned} &-\int_a^b (f^+(s)[w(s)]_- + f^-(s)[w(s)]_+)ds \\ &< (-1)^{i+1}\int_a^b h(s)w(s)ds \\ &< \int_a^b (f^-(s)[w(s)]_- + f^+(s)[w(s)]_+)ds. \end{aligned} \tag{2.15}$$

because from (2.15) there follows the existence of a constant $\varepsilon > 0$ such that the condition (2.12$_i$) is satisfied.

Remark 4. If $\tilde{f}(t) = \min\{f^+(t), f^-(t)\}$ then the condition (2.14) of Theorem 3 can be replaced by

$$\left|\int_a^b h(s)w(s)ds\right| \le \int_a^b \tilde{f}(s)|w(s)|ds - \varepsilon||\alpha||_L.$$

Example 4. From Theorem 3 it follows that the equation

$$u''(t) = -\lambda^2 u(t) + \frac{|u(t)|^\alpha}{1+|u(t)|^\alpha}\operatorname{sgn} u(t) + h(t) \quad \text{for} \quad 0 \le t \le \pi/2, \tag{2.16}$$

where $\lambda = 2k-1$ ($k \in N$) and $\alpha \in]0,+\infty[$, with the conditions (2.7) has at least one solution if $h \in L([0,\pi/2],R)$ is such that $|h(t)| < 1$ for $0 \le t \le \pi/2$.

3. Auxiliary propositions

Let $u_n \in \widetilde{C}'(I;R)$, $\|u_n\|_C \neq 0$ $(n \in N)$, w be an arbitrary solution of the problem (1.3), (1.4), and $r > 0$. Then, for every $n \in N$, we define:

$$A_{n,1} \stackrel{\text{def}}{=} \{t \in I : |u_n(t)| \leq r\}, \qquad A_{n,2} \stackrel{\text{def}}{=} \{t \in I : |u_n(t)| > r\},$$

$$B_{n,i} \stackrel{\text{def}}{=} \{t \in A_{n,2} : \operatorname{sgn} u_n(t) = (-1)^{i-1} \operatorname{sgn} w(t)\} \ \ (i = 1,2),$$

$$C_{n,1} \stackrel{\text{def}}{=} \{t \in A_{n,2} : |w(t)| \geq 1/n\}, \qquad C_{n,2} \stackrel{\text{def}}{=} \{t \in A_{n,2} : |w(t)| < 1/n\},$$

$$D_n \stackrel{\text{def}}{=} \{t \in I : |w(t)| > r||u_n||_C^{-1} + 1/2n\},$$

$$A^{\pm}_{n,2} \stackrel{\text{def}}{=} \{t \in A_{n,2} : \pm u_n(t) > r\}, \quad B^{\pm}_{n,i} \stackrel{\text{def}}{=} A^{\pm}_{n,2} \cap B_{n,i},$$

$$C^{\pm}_{n,i} \stackrel{\text{def}}{=} A^{\pm}_{n,2} \cap C_{n,i} \ \ (i = 1,2), \ D^{\pm}_n \stackrel{\text{def}}{=} \{t \in I : \pm w(t) > r||u_n||_C^{-1} + 1/2n\},$$

From these definitions it is clear that, for any $n \in N$, we have

$$A_{n,1} \cap A_{n,2} = \varnothing, A^{+}_{n,2} \cap A^{-}_{n,2} = \varnothing, \ B_{n,1} \cap B_{n,2} = \varnothing, \ C_{n,1} \cap C_{n,2} = \varnothing,$$
$$D^{+}_n \cap D^{-}_n = \varnothing, \ B^{+}_{n,2} \cap B^{-}_{n,2} = \varnothing, \ C^{+}_{n,i} \cap C^{-}_{n,i} = \varnothing \ (i = 1,2), \tag{3.1}$$

and

$$\begin{gathered} A_{n,1} \cup A_{n,2} = I, \ A^{+}_{n,2} \cup A^{-}_{n,2} = A_{n,2}, \ B_{n,1} \cup B_{n,2} = A_{n,2} \setminus N_w, \\ C_{n,1} \cup C_{n,2} = A_{n,2}, \ B^{+}_{n,2} \cup B^{-}_{n,2} = B_{n,2}, \ C^{\pm}_{n,1} \cup C^{\pm}_{n,2} = A^{\pm}_{n,2}, \\ C^{+}_{n,i} \cup C^{-}_{n,i} = C_{n,i} \ (i = 1,2), \ D^{+}_n \cup D^{-}_n = D_n. \end{gathered} \tag{3.2}$$

The proofs of the following two lemmas are given in [16].

Lemma 1. *Let $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$, $r > 0$, w be an arbitrary nonzero solution of the problem* (1.3), (1.4), *and*

$$||u_n||_C \geq 2rn \quad \text{for} \quad n \in N, \tag{3.3}$$

$$||v_n - w||_C \leq 1/2n \quad \text{for} \quad n \in N, \tag{3.4}$$

where $v_n(t) = u_n(t)||u_n||_C^{-1}$. Then there exists $n_0 \in N$ such that

$$D^{+}_{n_0} \subset A^{+}_{n,2}, \quad D^{-}_{n_0} \subset A^{-}_{n,2} \quad \text{for} \quad n \geq n_0, \tag{3.5}$$

$$C^{+}_{n_0,1} \subset D^{+}_n \quad C^{-}_{n_0,1} \subset D^{-}_n \quad \text{for} \quad n \geq n_0. \tag{3.6}$$

Moreover

$$\lim_{n \to +\infty} \operatorname{mes} A_{n,1} = 0, \qquad \lim_{n \to +\infty} \operatorname{mes} A_{n,2} = \operatorname{mes} I, \tag{3.7}$$

$$C_{n,1} \subset B_{n,1}, \qquad B_{n,2} \subset C_{n,2}, \tag{3.8}$$

$$B^{+}_{n,2} \subset C^{+}_{n,2}, \qquad B^{-}_{n,2} \subset C^{-}_{n,2}, \tag{3.9}$$

$$C^{+}_{n,1} \subset B^{+}_{n,1}, \qquad C^{-}_{n,1} \subset B^{-}_{n,1}, \tag{3.10}$$

$$\lim_{n\to+\infty} \operatorname{mes} C_{n,1} = \lim_{n\to+\infty} \operatorname{mes} B_{n,1} = \operatorname{mes} I, \\ \lim_{n\to+\infty} \operatorname{mes} C_{n,2} = \lim_{n\to+\infty} \operatorname{mes} B_{n,2} = 0, \tag{3.11}$$

$$r < |u_n(t)| \le ||u_n||_C/2n \qquad \textit{for} \quad t \in B_{n,2}, \tag{3.12}$$

$$|u_n(t)| \ge ||u_n||_C/2n > r \qquad \textit{for} \quad t \in C_{n,1}, \tag{3.13}$$

$$C^{\pm}_{n,2} = \{t \in A_{n,2} : 0 \le \pm w(t) < 1/n\}, \tag{3.14}$$

$$C^{\pm}_{n,1} \subset \Omega^{\pm}_w, \quad \lim_{n\to+\infty} \operatorname{mes} C^{\pm}_{n,1} = \operatorname{mes} \Omega^{\pm}_w. \tag{3.15}$$

Lemma 2. *Let $i \in \{1,2\}$, $r > 0$, $k \in N$, w_0 be a nonzero solution of the problem (1.3), (1.4), $N_{w_0} = \{t_1,\dots,t_k\}$, the function $f_1 \in E(N_{w_0})$ be non-decreasing in the second argument for $|x| \ge r$, and*

$$f_1(t,x)\operatorname{sgn} x \ge 0 \qquad \textit{for} \quad t \in I, \ |x| \ge r. \tag{3.16}$$

Then:

(a) *If $G \subset I$ and*

$$\int_G |f_1(s,(-1)^i r)w_0(s)|ds \ne 0, \tag{3.17}$$

then there exist $\delta_0 > 0$ and $\varepsilon_1 > 0$ such that

$$\mathbb{I}(G,U_\varepsilon,x) \overset{\text{def}}{\equiv} \int_{G\setminus U_\varepsilon} |f_1(s,x)w_0(s)|ds - \int_{U_\varepsilon} |f_1(s,x)w_0(s)|ds \ge \delta_0 \tag{3.18}$$

for $(-1)^i x \ge r$ and $0 < \varepsilon \le \varepsilon_1$, where $U_\varepsilon = I \cap \left(\cup_{j=1}^k [t_j - \varepsilon/2k, t_j + \varepsilon/2k]\right)$.

(b) *If $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$, $r > 0$, w is an arbitrary nonzero solution of the problem (1.3), (1.4), and the condition (3.3) holds, then there exist $\varepsilon_2 \in]0,\varepsilon_1]$ and $n_0 \in N$ such that*

$$\mathbb{I}(D^+_n, U^+_\varepsilon, x) \ge -\frac{\delta_0}{2} \quad \textit{for} \quad x \ge r, \tag{3.19$_1$}$$

$$\mathbb{I}(D^-_n, U^-_\varepsilon, x) \ge -\frac{\delta_0}{2} \quad \textit{for} \quad x \le -r \tag{3.19$_2$}$$

for $n \ge n_0$ and $0 < \varepsilon \le \varepsilon_2$, where $U^{\pm}_\varepsilon = \{t \in U_\varepsilon : \pm w(t) \ge 0\}$.

Lemma 3. *Let all the conditions of Lemma 1 be fulfilled and there exist $r > 0$ such that the condition (3.16) holds, where $f_1 \in K(I \times R;R)$. Then*

$$\liminf_{n\to+\infty} \int_s^t f_1(\xi,u_n(\xi))\operatorname{sgn} u_n(\xi)d\xi \ge 0 \quad \textit{for} \quad a \le s < t \le b. \tag{3.20}$$

Proof. Let

$$\gamma_r^*(t) \overset{\text{def}}{\equiv} \sup\{|f_1(t,x)| : |x| \le r\} \quad \text{for} \quad t \in I. \tag{3.21}$$

Then, according to (3.1), (3.2), and (3.16), we obtain the estimate

$$\int_s^t f_1(\xi, u_n(\xi)) \operatorname{sgn} u_n(\xi) d\xi$$

$$\ge -\int_{[s,t]\cap A_{n,1}} \gamma_r^*(\xi) d\xi + \int_{[s,t]\cap A_{n,2}} |f_1(\xi, u_n(\xi))| d\xi$$

for $a \le s < t \le b$, $n \in N$. This estimate and (3.7) imply (3.20). □

Lemma 4. *Let w_0 be a nonzero solution of the problem (1.3), (1.4), $r > 0$, the function $f_1 \in E(N_{w_0})$ be non-decreasing in the second argument for $|x| \ge r$, condition (3.16) hold, and*

$$\int_{\Omega_{w_0}^+} |f_1(s,r)| ds + \int_{\Omega_{w_0}^-} |f_1(s,-r)| ds \ne 0. \tag{3.22}$$

Then there exist $\delta > 0$ and $n_1 \in N$ such that if

$$\left| \int_a^b h_1(s) w_0(s) ds \right| < \delta ||w_0||_C \tag{3.23}$$

then, for every nonzero solution w of the problem (1.3), (1.4), and functions $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$ such that the conditions (3.3),

$$|v_n^{(i)}(t) - w^{(i)}(t)| \le 1/2n \quad \text{for} \quad t \in I,\ n \in N,\ (i = 0,1) \tag{3.24}$$

where $v_n(t) = u_n(t) ||u_n||_C^{-1}$ for $t \in I$ and

$$u_n(a) = 0, \ \ u_n'(b) = 0 \tag{3.25}$$

are fulfilled, there exists $n_1 \in N$ such that

$$\mathbb{M}_n(w) \overset{\text{def}}{\equiv} \int_a^b (h_1(s) + f_1(s, u_n(s))) w(s) ds \ge 0 \quad \text{for} \quad n \ge n_1. \tag{3.26}$$

Proof. Without loss of generality we can assume that $||w_0||_C = 1$. Also it is not difficult to verify that all the assumption of Lemma 1 are satisfied. Then, by the definition of the sets $B_{n,1}$, $B_{n,2}$, the conditions (3.1), (3.2), and (3.16), we obtain the estimate

$$\int_a^b f_1(s, u_n(s)) w(s) ds \ge -\int_{A_{n,1}} \gamma_r^*(s) |w(s)| ds + \widehat{\mathbb{M}}_n(w), \tag{3.27}$$

where

$$\widehat{\mathbb{M}}_n(w) \overset{\text{def}}{\equiv} -\int_{B_{n,2}} |f_1(s, u_n(s)) w(s)| ds + \int_{B_{n,1}} |f_1(s, u_n(s)) w(s)| ds.$$

On the other hand, from the unique solvability of the Cauchy problem for the equation (1.3) it is clear that

$$w'(a) \neq 0, \;\; w'(b) \neq 0, \;\; w'(t_i) \neq 0 \qquad \text{for } i = 1, \dots, k \tag{3.28}$$

if $N_{w_0} = \{t_1, \dots, t_k\}$. Now note that, for any nonzero solution w of the problem (1.3), (1.4), there exists $\beta \neq 0$ such that $w(t) = \beta w_0(t)$. Consequently

$$\Omega_w^{\pm} = \Omega_{w_0}^{\pm} \quad \text{if} \quad \beta > 0 \quad \text{and} \quad \Omega_w^{\mp} = \Omega_{w_0}^{\pm} \quad \text{if} \quad \beta < 0. \tag{3.29}$$

Then in view of (3.15) and (3.22), there exists $n_2 \geq n_0$ such that

$$\int_{C^+_{n_2,1}} |f_1(s,r)w_0(s)|ds \neq 0 \;\text{ and/or }\; \int_{C^-_{n_2,1}} |f_1(s,-r)w_0(s)|ds \neq 0. \tag{3.30}$$

From (3.30), in view of (3.6), it follows that

$$\int_{D_n^+} |f_1(s,r)w_0(s)|ds \neq 0 \quad \text{for} \quad n \geq n_2 \tag{3.31$_1$}$$

and/or

$$\int_{D_n^-} |f_1(s,-r)w_0(s)|ds \neq 0 \quad \text{for} \quad n \geq n_2. \tag{3.31$_2$}$$

Consequently, all the assumptions of Lemma 3.2 are satisfied with $G = D_n^+$ and/or $G = D_n^-$. Therefore, there exist $\varepsilon_0 \in]0, \varepsilon_2[$, $n_3 \geq n_2$, and $\delta_0 > 0$ such that

$$\begin{aligned} \mathbb{I}(D_n^+, U_{\varepsilon_0}^+, x) \geq \delta_0 \;\text{ for }\; x \geq r, \;\; n \geq n_3,\\ \mathbb{I}(D_n^-, U_{\varepsilon_0}^-, x) \geq -\delta_0/2 \;\text{ for }\; x \leq -r, \;\; n \geq n_3 \end{aligned} \tag{3.32}$$

if (3.31$_1$) holds, and

$$\begin{aligned} \mathbb{I}(D_n^-, U_{\varepsilon_0}^-, x) \geq \delta_0 \quad \text{for} \quad x \leq -r, \; n \geq n_3,\\ \mathbb{I}(D_n^+, U_{\varepsilon_0}^+, x) \geq -\delta_0/2 \quad \text{for} \quad x \geq r, \;\; n \geq n_3 \end{aligned} \tag{3.33}$$

if (3.31$_2$) holds.

On the other hand, the definition of the set U_ε and (3.14), imply that there exists $n_4 > n_3$, such that

$$C^+_{n,2} \subset U^+_{\varepsilon_0}, \;\; C^-_{n,2} \subset U^-_{\varepsilon_0} \quad \text{for} \quad n \geq n_4. \tag{3.34}$$

By these inclusions, (3.2), and (3.5) we obtain

$$C^+_{n,1} = A^+_{n,2} \setminus C^+_{n,2} \supset D^+_{n_0} \setminus U^+_{\varepsilon_0}, \; C^-_{n,1} = A^-_{n,2} \setminus C^-_{n,2} \supset D^-_{n_0} \setminus U^+_{\varepsilon_0} \tag{3.35}$$

for $n \geq n_4$. First suppose that $N_{w_0} \neq \varnothing$ and there exists $n \geq n_4$ such that

$$B_{n,2} \neq \varnothing. \tag{3.36}$$

Then, by taking into account that f_1 is non-decreasing in the second argument for $|x| \geq r$, (3.3), (3.12), (3.16) and the definitions of the sets $B^+_{n,2}, B^-_{n,2}$, we get

$$
\begin{aligned}
&|f_1(t,u_n(t))| = f_1(t,u_n(t)) \\
&\leq f_1\Big(t,\frac{||u_n||_C}{2n}\Big) = \Big|f_1\Big(t,\frac{||u_n||_C}{2n}\Big)\Big| \quad \text{for } t \in B^+_{n,2}, \\
&|f_1(t,u_n(t))| = -f_1(t,-u_n(t)) \\
&\leq -f_1\Big(t,-\frac{||u_n||_C}{2n}\Big) = \Big|f_1\Big(t,-\frac{||u_n||_C}{2n}\Big)\Big| \quad \text{for } t \in B^-_{n,2}.
\end{aligned}
\tag{3.37}
$$

Analogously, from (3.3), (3.13), (3.16), and the definitions of the sets $C^+_{n,1}, C^-_{n,1}$, we obtain the estimates

$$
\begin{aligned}
&|f_1(t,u_n(t))| \geq \Big|f_1\Big(t,\frac{||u_n||_C}{2n}\Big)\Big| \quad \text{for} \quad t \in C^+_{n,1}, \\
&|f_1(t,u_n(t))| \geq \Big|f_1\Big(t,-\frac{||u_n||_C}{2n}\Big)\Big| \quad \text{for} \quad t \in C^-_{n,1}.
\end{aligned}
\tag{3.38}
$$

Then from (3.1), (3.2), (3.9), (3.37) and respectively from (3.1), (3.2), (3.8), and (3.38) we have

$$
\begin{aligned}
&\int_{B_{n,2}} |f_1(s,u_n(s))w(s)|ds \\
&\leq \int_{B^+_{n,2}} |f_1(s,\frac{||u_n||_C}{2n})w(s)|ds + \int_{B^-_{n,2}} |f_1(s,-\frac{||u_n||_C}{2n})w(s)|ds \\
&\leq \int_{C^+_{n,2}} |f_1(s,\frac{||u_n||_C}{2n})w(s)|ds + \int_{C^-_{n,2}} |f_1(s,-\frac{||u_n||_C}{2n})w(s)|ds
\end{aligned}
\tag{3.39}
$$

and respectively

$$
\begin{aligned}
&\int_{B_{n,1}} |f_1(s,u_n(s))w(s)|ds \geq \int_{C_{n,1}} |f_1(s,u_n(s))w(s)|ds \\
&\geq \int_{C^+_{n,1}} |f_1(s,\frac{||u_n||_C}{2n})w(s)|ds + \int_{C^-_{n,1}} |f_1(s,-\frac{||u_n||_C}{2n})w(s)|ds.
\end{aligned}
\tag{3.40}
$$

If the condition (3.36) holds, from (3.39) and (3.40) we obtain

$$
\begin{aligned}
\frac{\widehat{M}_n(w)}{|\beta|} \geq &\Big(\int_{C^+_{n,1}} \Big|f_1\Big(s,\frac{||u_n||_C}{2n}\Big)w_0(s)\Big|ds - \int_{C^+_{n,2}} \Big|f_1\Big(s,\frac{||u_n||_C}{2n}\Big)w_0(s)\Big|ds\Big) \\
&+ \Big(\int_{C^-_{n,1}} \Big|f_1\Big(s,-\frac{||u_n||_C}{2n}\Big)w_0(s)\Big|ds - \int_{C^-_{n,2}} \Big|f_1\Big(s,-\frac{||u_n||_C}{2n}\Big)w_0(s)\Big|ds\Big),
\end{aligned}
$$

whence, by (3.34) and (3.35) we get

$$\frac{\widehat{\mathbb{M}}_n(w)}{|\beta|} \geq \mathbb{I}\Big(D_n^+, U_{\varepsilon_0}^+, \frac{||u_n||_C}{2n}\Big) + \mathbb{I}\Big(D_n^-, U_{\varepsilon_0}^-, -\frac{||u_n||_C}{2n}\Big) \tag{3.41}$$

for $n \geq n_4$. From (3.41) by (3.32) and (3.33) we obtain

$$\widehat{\mathbb{M}}_n(w) \geq \frac{\delta_0|\beta|}{2} \qquad \text{for} \quad n \geq n_4. \tag{3.42}$$

On the other hand, in view of (3.10), (3.16), the definition of the sets $A_{n,2}$, $B_{n,1}$, and the fact that f_1 is non-decreasing in the second argument, we obtain the estimate

$$\begin{aligned}\int_{B_{n,1}} |f_1(s,u_n(s))w(s)|ds &\geq \int_{B_{n,1}^+} |f_1(s,r)w(s)|ds + \int_{B_{n,1}^-} |f_1(s,-r)w(s)|ds \\ &\geq \int_{C_{n,1}^+} |f_1(s,r)w(s)|ds + \int_{C_{n,1}^-} |f_1(s,-r)w(s)|ds.\end{aligned} \tag{3.43}$$

Now suppose that there exists $n \geq n_4$ such that

$$B_{n,2} = \varnothing. \tag{3.44}$$

Then from (3.30) and (3.43), (3.44) there follows the existence of $\delta^* > 0$ such that $\widehat{\mathbb{M}}_n(w) \geq |\beta|\delta^*$. From this inequality and (3.42) it follows that, in both cases when (3.36) or (3.44) are fulfilled, the inequality

$$\widehat{\mathbb{M}}_n(w) \geq |\beta|\delta \quad \text{for} \quad n \geq n_4 \tag{3.45}$$

holds with $\delta = \min\{\delta_0/2, \delta^*\}$. From (3.27) by (3.7) and (3.45), we see that for any $\varepsilon \in]0,\delta[$ there exists $n_1 > n_4$ such that

$$\int_a^b f_1(s,u_n(s))w(s)ds \geq |\beta|(\delta-\varepsilon) \quad \text{for} \quad n \geq n_1,$$

and thus

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq \delta - \varepsilon - \left|\int_a^b h_1(s)w_0(s)ds\right| \quad \text{for} \quad n \geq n_1. \tag{3.46}$$

If $N_{w_0} = \varnothing$ then $|w(t)| > 0$ for $a < t < b$ and in view of (3.3), (3.24), (3.25) and (3.28), the condition (3.44) holds, i.e., the inequality (3.46) also holds.

Consequently since $\varepsilon > 0$ is arbitrary, the inequality (3.26) from (3.46) and (3.23) follows. □

Lemma 5. *Let w_0 be a nonzero solution of the problem* (1.3), (1.4)*, $r > 0$, and the conditions* (3.16), (3.23) *hold with $f_1(t,x) \overset{\text{def}}{\equiv} f_0(t)g_1(x)$, where $f_0 \in L(I;R_+)$ and a non-decreasing function $g_1 \in C(R;R)$ be such that*

$$\lim_{|x|\to+\infty} |g_1(x)| = +\infty. \tag{3.47}$$

Then, for every nonzero solution w of the problem (1.3), (1.4) and functions $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$ fulfilling the conditions (3.3), (3.24), (3.25), the inequality (3.26) holds.

Proof. From the assumptions of our lemma it is clear that the relations (3.27)–(3.35), (3.37)-(3.40) and (3.43) with $f_1(t,x) = f_0(t)g_1(x)$ and $w(t) = \beta w_0(t)$ $(\beta \neq 0)$ are fulfilled.

Assuming $\int_{C^+_{n_2,1}} |f_1(s,r)w_0(s)|ds \neq 0$, the condition (3.31_1) is satisfied i.e., (3.32) holds.

Now notice that from (3.15) and the equality $C^+_{n,1} = \Omega^+_w \setminus (\Omega^+_w \setminus C^+_{n,1})$ it follows that there exist $\varepsilon > 0$ and $n_0 \in N$ such that

$$\int_{C^+_{n,1}} |f_0(s)w_0(s)|ds \geq \int_{\Omega^+_w} |f_0(s)w_0(s)|ds - \varepsilon > 0 \tag{3.48}$$

for $n \geq n_0$.

First consider the case when there exists $n \geq n_4$ such that the condition (3.44) holds. Without loss of generality we can assume that $n_4 > n_0$. Then by (3.29), (3.43), (3.44) and (3.48), we obtain

$$\widehat{\mathrm{M}}_n(w) \geq |\beta||g_1(r)|\left(\int_{\Theta_\beta} |f_0(s)w_0(s)|ds - \varepsilon\right) > 0, \tag{3.49}$$

where $\Theta_\beta = \Omega^+_{w_0}$ if $\beta > 0$ and $\Theta_\beta = \Omega^-_{w_0}$ if $\beta < 0$.

Consider now the case when there exists $n \geq n_4$ such that (3.36) holds. From (3.3) and the definition of the set D^+_n it follows that $D^+_n \subset D^+_{n+1}$, and since g_1 is non-decreasing, from (3.32) we obtain

$$\begin{aligned}\mathbb{I}(D^+_n, U^+_{\varepsilon_0}, x) &= |g_1(x)|\left(\int_{D^+_n \setminus U^+_{\varepsilon_0}} |f_0(t)w_0(s)|ds - \int_{U^+_{\varepsilon_0}} |f_0(t)w_0(s)|ds\right) \\ &\geq |g_1(r)|\mu = \mathbb{I}(D^+_{n_4}, U^+_{\varepsilon_0}, r) \geq \delta_0\end{aligned}$$

for $x \geq r$, with $\mu = \int_{D^+_{n_4} \setminus U^+_{\varepsilon_0}} |f_0(s)w_0(s)|ds - \int_{U^+_{\varepsilon_0}} |f_0(s)w_0(s)|ds > 0$. By the last inequality, (3.3), (3.32), and (3.41) we get

$$\widehat{\mathrm{M}}_n(w) \geq |\beta|(|g_1(r)|\mu - \delta_0/2). \tag{3.50}$$

Applying (3.49), (3.50) in (3.27) and taking (3.7) into account, we conclude that there exist $\varepsilon_1 > 0$ and $n_1 \geq n_4$ such that

$$|\beta|\left(|g_1(r)|\mu_1 - \frac{\delta_0}{2} - \varepsilon_1\right) \leq \int_a^b f_1(s,u_n(s))w(s)ds \quad \text{for} \quad n \geq n_1$$

with $\mu_1 = \min(\mu, \int_{\Omega_{w_0}^+} |f_0(s)w_0(s)|ds - \varepsilon)$. From (3.47) and the last inequality it is clear that, for any function h_1, we can choose $r > 0$ such that the inequality (3.26) will be true. In a similar manner one can prove (3.26) in the case when $\int_{C_{n_2,1}^-} |f_1(s,r)w_0(s)|ds \neq 0$.

□

Lemma 6. *Let $r > 0$, there exist functions $\alpha, f^-, f^+ \in L(I, R_+)$ such that the conditions*

$$\begin{aligned} f_1(t,x) &\le -f^-(t) \quad \text{for} \quad x \le -r, \\ f^+(t) &\le f_1(t,x) \quad \text{for} \quad x \ge r \end{aligned} \tag{3.51}$$

are satisfied,

$$\sup\{|f_1(t,x)| : x \in R\} = \alpha(t) \quad \text{for} \quad t \in I, \tag{3.52}$$

and there exist a nonzero solution w_0 of the problem (1.3), (1.4) *and $\varepsilon > 0$ such that*

$$\begin{aligned} &-\int_a^b (f^+(s)[w_0(s)]_- + f^-(s)[w_0(s)]_+)ds + \varepsilon||\alpha||_L \\ &\le -\int_a^b h_1(s)w_0(s)ds \\ &\le \int_a^b (f^-(s)[w_0(s)]_- + f^+(s)[w_0(s)]_+)ds - \varepsilon||\alpha||_L. \end{aligned} \tag{3.53}$$

Then, for every nonzero solution w of the problem (1.3), (1.4) *and functions $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$ fulfilling the conditions* (3.3), (3.24), *and* (3.25), *there exists $n_1 \in N$ such that the inequality* (3.26) *holds.*

Proof. First note that, for any nonzero solution w of the problem (1.3), (1.4), there exists $\beta \neq 0$ such that $w(t) = \beta w_0(t)$. Moreover, it is not difficult to verify that all the assumptions of Lemma 1 are satisfied for the function $w(t) = \beta w_0(t)$. From (3.1), (3.2), and (3.52) we get

$$\begin{aligned} \mathbb{M}_n(w) \ge &-\int_{A_{n,1} \cup B_{n,2}} \alpha(s)|w(s)|ds + \int_{B_{n,1}} f_1(s,u_n)w(s)ds \\ &+ \int_a^b h_1(s)w(s)ds. \end{aligned} \tag{3.54}$$

On the other hand, by the definition of the set $B_{n,1}$ we have

$$\operatorname{sgn} u_n(t) = \operatorname{sgn} w(t) \quad \text{for} \quad t \in B_{n,1}^+ \cup B_{n,1}^-. \tag{3.55}$$

Hence, by (3.1), (3.2), (3.10), (3.51), and (3.55), from (3.54) we obtain the estimate

$$\mathbb{M}_n(w) - \int_a^b h_1(s)w(s)ds$$

$$\geq -\int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds + +\int_{B_{n,1}^+} f^+(s)|w(s)|ds + \int_{B_{n,1}^-} f^-(s)|w(s)|ds$$
$$\geq -\int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds + \int_{C_{n,1}^+} f^+(s)|w(s)|ds + \int_{C_{n,1}^-} f^-(s)|w(s)|ds. \tag{3.56}$$

Now, note that $f^- \equiv 0$ and $f^+ \equiv 0$ if $f_1(t,x) \equiv 0$. Therefore by (3.7), (3.11), (3.15), and the inclusions $C_{n,1}^+ \subset \Omega_w^+$, $C_{n,1}^- \subset \Omega_w^-$, we see that there exist $\varepsilon > 0$ and $n_1 \in N$ such that

$$\frac{1}{3}\varepsilon||\alpha||_L \geq \int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w_0(s)|ds$$
$$\int_{\Omega_w^\pm} f^\pm(s)|w_0(s)|ds - \frac{1}{3}\varepsilon||\alpha||_L \leq \int_{C_{n,1}^\pm} f^\pm(s)|w_0(s)|ds \tag{3.57}$$

for $n \geq n_1$. By virtue of (3.56) and (3.57), we obtain

$$\frac{\mathsf{M}_n(w)}{|\beta|} \geq -\varepsilon||\alpha||_L + \int_{\Omega_w^+} f^+(s)|w_0(s)|ds$$
$$+ \int_{\Omega_w^-} f^-(s)|w_0(s)|ds + \sigma\int_a^b h_1(s)w_0(s)ds$$

for $n \geq n_1$, where $\sigma = \operatorname{sgn}\beta$. Now, by taking into account that

$$\int_{\Omega_w^\pm} l(s)|w_0(s)|ds = \int_{\Omega_{w_0}^\pm} l(s)|w_0(s)|ds = \int_a^b l(s)[w_0(s)]_\pm ds$$

if $\beta > 0$ and

$$\int_{\Omega_w^\pm} l(s)|w_0(s)|ds = \int_{\Omega_{w_0}^\mp} l(s)|w_0(s)|ds = \int_a^b l(s)[w_0(s)]_\mp ds$$

if $\beta < 0$ for an arbitrary $l \in L(I,R)$, from the last inequalities we get

$$\frac{\mathsf{M}_n(w)}{|\beta|} \geq -\varepsilon||\alpha||_L + \int_a^b (f^+(s)[w_0(s)]_+ + f^-(s)[w_0(s)]_-)ds$$
$$+ \int_a^b h_1(s)w_0(s)ds \quad \text{for} \quad n \geq n_1$$

if $\sigma = 1$, and

$$\frac{\mathsf{M}_n(w)}{|\beta|} \geq -\varepsilon||\alpha||_L + \int_a^b (f^+(s)[w_0(s)]_- + f^-(s)[w_0(s)]_+)ds$$
$$- \int_a^b h_1(s)w_0(s)ds \quad \text{for} \quad n \geq n_1$$

if $\sigma = -1$. From the last inequalities and (3.53) we immediately obtain (3.26). $\square$

Lemma 7. *Let problem* (1.3), (1.4) *has the nontrivial solution. Than there exists* $\varepsilon > 0$ *suth that the equation*

$$w''(t) = \lambda p(t)w \quad \textit{for} \quad t \in I, \tag{3.58}$$

under boundari conditions (1.4) *has only the trivial solution if* $\lambda \in]1, 1+\varepsilon]$.

Proof. Let G be the Green's function of the boundary value problem $u''(t) = 0,\ \ u(a) = 0,\ u'(b) = 0$, then problem (3.58), (1.4) is equivalent to the equation $w(t) = \lambda\Gamma(w)(t)$, where the operator $\Gamma : C(I;R) \to C(I;R)$ is defined by the equality $\Gamma(x)(t) = \int_a^b G(t,s)p(s)x(s)ds$. As it is well-known $\Gamma : C(I;R) \to C(I;R)$ is a compact operator, and then for every $r > 0$ the disc $|\lambda| \le r$, contains at most finite number of characteristic values [see [9], Capitol XIII, §3, Theorem 1]. From this fact the existence of $\varepsilon > 0$ such that the set $]1, 1+\varepsilon]$ does not contain the characteristic values of the equation $w(t) = \lambda\Gamma(w)(t)$, it follows. Consequently this equation, i.e., problem (3.58), (1.4) has only the trivial solution if $\lambda \in]1, 1+\varepsilon]$. $\square$

4. Proof of the main results

Proof of Theorem 1. Let $p_n(t) = (1 + (-1)^i/n)p(t)$ and for any $n \in N$, consider the problems

$$u_n''(t) = p_n(t)u_n(t) + f(t,u_n(t)) + h(t) \quad \text{for} \quad t \in I, \tag{4.1}$$

$$u_n(a) = 0, \quad u_n'(b) = 0. \tag{4.2}$$

and (3.58). In view of the condition (2.3) and the fact that $(-1)^i f(t,x)$ is non-decreasing in the second argument for $|x| \ge r$, we obtain

$$\lim_{n\to+\infty} \frac{1}{||z_n||_C} \int_a^b |f(s,z_n(s))|ds = 0 \tag{4.3}$$

for an arbitrary sequence $z_n \in C(I;R)$ with $\lim_{n\to+\infty} ||z_n||_C = +\infty$. Moreover, in view of Lemma 7, the problem (3.58) has only the zero solution for every $n \ge n_0$. Therefore, as it is well-known (see [12], Corollary 2.1, p. 2271), from the inequality (4.3) it follows that the problems (4.1), (4.2) has at least one solution, suppose u_n.

Assume that

$$\lim_{n\to+\infty} ||u_n||_C = +\infty \tag{4.4}$$

and $v_n(t) = u_n(t)||u_n||_C^{-1}$, then the conditions

$$v_n(a) = 0 \qquad v_n'(b) = 0, \tag{4.5}$$

$$||v_n||_C = 1 \tag{4.6}$$

are fulfilled, and

$$v_n''(t) = p_n(t)v_n(t) + \frac{1}{||u_n||_C}(f(t,u_n(t)) + h(t)). \tag{4.7}$$

Hence, by the conditions (4.3) and (4.6), from (4.7) we get the existence of $r_0 > 0$ such that $||v_n'||_C \leq r_0$. Consequently, in view of (4.6) by the Arzela-Ascoli lemma, without loss of generality we can assume that there exists a function $w \in \widetilde{C}'(I,R)$ such that $\lim_{n\to+\infty} v_n^{(i)}(t) = w^{(i)}(t)$ $(i = 0,1)$ uniformly on I. From the last equality and (4.4) there follows the existence of an increasing sequence $\{\alpha_k\}_{k=1}^{+\infty}$ of a natural numbers, such that $||u_{\alpha_k}||_C \geq 2rk$ and $||v_{\alpha_k}^{(i)} - w^{(i)}||_C \leq 1/2k$ for $k \in N$. Without loss of generality we can suppose that $u_n \equiv u_{\alpha_n}$ and $v_n \equiv v_{\alpha_n}$. In this case we see that u_n and v_n are the solutions of the problems (4.1), (4.2) and (4.7), (4.5) respectively, and the inequalities

$$||u_n||_C \geq 2rn, \quad ||v_n^{(i)} - w^{(i)}||_C \leq 1/2n \quad \text{for} \quad n \in N \tag{4.8}$$

are fulfilled.

From (4.7), by virtue of (4.5), (4.8) and (2.3), we obtain that w is a solution of the problem (1.3), (1.4). Multiplying the equations (4.1) and (1.3) respectively by w and $-u_n$, and therefore integrating their sum from a to b, in view of conditions (4.2) and (1.4), we obtain

$$(-1)^{i+1}\frac{||u_n||_C}{\alpha_n}\int_a^b p(s)w(s)v_n(s)ds = \int_a^b (h(s) + f(s,u_n(s)))w(s)ds \tag{4.9}$$

for $n \geq n_0$, where in view of conditions (4.8) the equality

$$\lim_{n\to+\infty}\int_a^b p(s)w(s)v_n(s)ds = \int_a^b p(s)w^2(s)ds$$

holds. On the other hand multiplying equation (1.3) by w, and therefore integrating from a to b, in view of condition (1.4), we obtain

$$\int_a^b p(s)w^2(s)ds = \int_a^b w''(s)w(s)ds = -\int_a^b w'^2(s)ds < 0,$$

and from (4.9) by the last two relations we get

$$(-1)^i\int_a^b (h(s) + f(s,u_n(s)))w(s)ds > 0. \tag{4.10}$$

for $n \in N \geq n_0$. Now note that, in view the conditions (2.1), (2.2), (2.4), (4.2), and (4.8), all the assumptions of Lemma 4 with $f_1(t,x) = (-1)^i f(t,x)$, $h_1(t) = (-1)^i h(t)$ are satisfied. Therefore, the inequality (3.26) is true, which contradicts (4.10). This contradiction proves that (4.4) does not hold and thus there exists $r_1 > 0$ such that $||u_n||_C \leq r_1$ for $n \in N$. Consequently, from (4.1) and (4.2) it is clear that there exists $r_1' > 0$ such that $||u_n'||_C \leq r_1'$ and $|u_n''(t)| \leq \sigma(t)$ for $t \in I$, $n \in N$, where $\sigma(t) = 2|p(t)|r_1 + |h(t)| + \gamma_{r_1}(t)$. Hence, by Arzela-Ascoli lemma, without loss of generality we can assume that there exists a function $u_0 \in \widetilde{C}'(I;R)$ such that $\lim_{n\to+\infty} u_n^{(i)}(t) = u_0^{(i)}(t)$ $(i = 0,1)$ uniformly on I. Therefore, it follows from (4.1) and (4.2) that u_0 is a solution of the problem (1.1), (1.2). □

Proof of Theorem 2. The proof is the same as the proof of Theorem 1. The only difference is that we use Lemma 5 instead of Lemma 4. □

Proof of Theorem 3. From (2.13) it is clear that, for an arbitrary sequence $z_n \in C(I;R)$ such that $\lim_{n\to+\infty} ||z_n||_C = +\infty$, the equality (4.3) holds. From (4.3) and Lemma 6, analogously as in the proof of Theorem 1, we show that the problem (1.1), (1.2) has at least one solution. □

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