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A. Dyki

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APPROXIMATION OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH DELAYED ARGUMENTS

A. DYKI

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Abstract. We discuss boundary value problems for first-order functional differential equations and investigate the approximation of extremal solutions of such problems. We use a monotone iterative method.

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1. INTRODUCTION

Let us consider the problem

$$\begin{cases} x'(t) = f(t, x(t), x(\alpha(t))), & t \in J := [0, T], \\ x(0) = r x(T), & r \in (0, 1], \end{cases} \tag{1.1}$$

where $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and the function $\alpha \in C(J, J)$ is such that $\alpha(t) \leq t$ for all $t \in J$.

In this paper we extend some results of paper [3] concerning the case where $r = 1$ and f satisfies a left-sided Lipshitz condition with suitable constants. We show the applicability of the monotone iterative method in obtaining monotone sequences approximating the extremal solutions of (1.1). We refer, e. g., to [2] for details about the monotone iterative method. Note that we use more general definition of lower and upper solution than the classical definition used, e. g., in [1].

We start by proving a comparison theorem used in the sequel. In Section 3, we prove the existence of a solution of the linear problem associated to (1.1). Finally, we prove the existence of monotone sequences approximating the extremal solutions of problem (1.1).

2. Comparison results

Lemma 1 ([1]). *Suppose that $\alpha \in C(J,J)$, $\alpha(t) \le t$ on J, $M \in C(J,\mathbb{R})$, $q \in C^1(J,\mathbb{R})$ is such that*

$$q'(t) \le -M(t)q(t) - N(t)q(\alpha(t)), \quad t \in J, \qquad q(0) \le 0,$$

and

$$\int_0^T N(t)e^{\int_{\alpha(t)}^t M(s)ds}dt \le 1,$$

where N is a nonnegative function integrable on J. Then $q(t) \le 0$ on J.

Theorem 1. *Assume that $r \in (0,1]$ and*

(1) *$x \in C^1(J,\mathbb{R})$, $\alpha \in C(J,J)$, $\alpha(t) \le t$, $t \in J$, $M \in C(J,\mathbb{R})$, N is integrable on J, $M(t) > 0$, $t \in J$ and $N(t) \ge 0$, $t \in J$,*
(2) *$x'(t) + M(t)x(t) + N(t)x(\alpha(t)) \le 0$, $t \in J$, if $x(0) \le rx(T)$,*
(3) *$x'(t) + M(t)x(t) + N(t)x(\alpha(t)) + \frac{M(t)t+N(t)\alpha(t)+1}{rT}[x(0) - rx(T)] \le 0$, $t \in J$, if $x(0) > rx(T)$,*
(4) *$\int_0^T N(t)e^{\int_{\alpha(t)}^t M(s)ds}dt \le 1$.*

Then $x(t) \le 0$, $t \in J$.

Proof. In case (1), we assume that $x \ge 0$ on J and $x \not\equiv 0$. Then $x'(t) \le 0$, $t \in J$, so x is nonincreasing and $x(t) \le x(0)$ $t \in J$ so $x(0) \ge x(T)$. On the other hand, we have $x(0) \le rx(T) \le x(T)$. Thus $x(0) = x(T)$ and x is a constant function, $x(t) = C > 0$. So $x' \equiv 0$ and $[M(t) + N(t)]C \le 0$. Hence $C = 0$ and finally $x(t) = 0$, $t \in J$. It is a contradiction.

Thus we can consider that x has some negative value. Note that if $x(0) \le 0$, then $x(t) \le 0$, $t \in J$, by Lemma 1. Assume that $x(0) > 0$, then $x(T) > 0$. Let us consider the function v defined by

$$v(t) = e^{\int_0^t M(s)ds}x(t), \quad t \in J.$$

We have $v(0) = x(0) > 0$ and $v(T) = e^{\int_0^T M(s)ds}x(T) > 0$. Since x has some negative value, there exists $t_* \in (0,T)$ such that

$$v(t_*) = \min_{t \in [0,T]} v(t) < 0.$$

Note that

$$v'(t) = e^{\int_0^t M(s)ds}M(t)x(t) + e^{\int_0^t M(s)ds}x'(t) \le -N(t)e^{\int_{\alpha(t)}^t M(s)ds}v(\alpha(t)).$$

The integration of this from t_* to T yields

$$-v(t_*) < v(T) - v(t_*) \le -\int_{t_*}^T N(t)e^{\int_{\alpha(t)}^t M(s)ds}v(\alpha(t))dt$$

$$\leq -v(t_*)\int_{t_*}^{T} N(t)e^{\int_{\alpha(t)}^{t} M(s)\,ds}dt \leq -v(t_*).$$

So $-v(t_*) < -v(t_*)$, which is a contradiction. Thus $x(0) \leq 0$ and, in view of Lemma 1, we get $x(t) \leq 0$, $t \in J$.

In case (3), let us consider the function

$$w(t) = x(t) + \frac{t}{rT}[x(0) - rx(T)], \quad t \in J.$$

It yields that $w(0) = rw(T)$. Moreover,

$$\begin{aligned} w'(t) + M(t)w(t) + N(t)w(\alpha(t)) &= x'(t) + M(t)x(t) + N(t)x(\alpha(t)) \\ &+ \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[x(0) - rx(T)] \leq 0. \end{aligned}$$

This is included in case (2). Thus $w(t) \leq 0$, $t \in J$ and finally $x(t) \leq 0$, $t \in J$. □

3. Linear problem

Now we are going to prove the existence of solution of linear problem associated to (1.1). We will need that result in succeeding section.

Theorem 2. *Let* $\sigma \in C(J,\mathbb{R})$, $\alpha \in C(J,J)$, $\alpha(t) \leq t$, $t \in J$, $M, N \in C(J,\mathbb{R})$, $M(t) > 0$, $t \in J$, $N(t) \geq 0$, $t \in J$ *and*

$$\begin{cases} x'(t) + M(t)x(t) + N(t)x(\alpha(t)) = \sigma(t), & t \in J, \\ x(0) = rx(T), & r \in (0,1]. \end{cases} \tag{3.1}$$

Moreover, assume that there exist functions $y_0, z_0 \in C^1(J,\mathbb{R})$ *such that*

(1) $y_0 \leq z_0$ *on* J,

(2)

$$y_0'(t) + M(t)y_0(t) + N(t)y_0(\alpha(t)) \leq \sigma(t) - a(t), \quad t \in J,$$
$$z_0'(t) + M(t)z_0(t) + N(t)z_0(\alpha(t)) \geq \sigma(t) - b(t), \quad t \in J,$$

where

$$a(t) = \begin{cases} 0 & \text{if } y_0(0) \leq ry_0(T), \\ \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[y_0(0) - ry_0(T)] & \text{if } y_0(0) > ry_0(T), \end{cases}$$

and

$$b(t) = \begin{cases} 0 & \text{if } z_0(0) \geq rz_0(T), \\ \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[z_0(0) - rz_0(T)] & \text{if } z_0(0) < rz_0(T). \end{cases}$$

(3) $\int_0^T N(t)e^{\int_{\alpha(t)}^{t} M(s)ds}dt \leq 1$.

Then there exists, in sector $[y_0, z_0]_* = \{w \in C^1(J,\mathbb{R}) : y_0(t) \leq w(t) \leq z_0(t),\ t \in J\}$, *a unique solution of* (3.1).

Proof. First we prove the uniqueness of a solution. Let x_1, x_2 be solutions of (3.1). Put $v_1 = x_1 - x_2$ and $v_2 = x_2 - x_1$. Then

$$v_1(0) = x_1(0) - x_2(0) = r[x_1(T) - x_2(T)] = rv_1(T),$$
$$v_1'(t) + M(t)v_1(t) + N(t)v_1(\alpha(t)) = \sigma(t) - \sigma(t) = 0, \quad t \in J,$$

and

$$v_2(0) = rv_2(T),$$
$$v_2'(t) + M(t)v_2(t) + N(t)v_2(\alpha(t)) = 0, \quad t \in J.$$

In view of Theorem 1, we have $v_1 \le 0$ and $v_2 \le 0$. Hence $x_1 = x_2$.

Now we show that if x is a solution of (3.1), then $y_0 \le x \le z_0$. Put $w_1 = y_0 - x$ and $w_2 = x - z_0$. Then we have

$$w_1'(t) + M(t)w_1(t) + N(t)w_1(\alpha(t)) \le \sigma(t) - \sigma(t) = 0 \quad \text{if } w_1(0) \le rw_1(T),$$
$$\begin{aligned} w_1'(t) + M(t)w_1(t) + N(t)w_1(\alpha(t)) + \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[w_1(0) - rw_1(T)] \\ \le \sigma(t) - \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[y_0(0) - ry_0(T)] \\ - \sigma(t) + \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[y_0(0) - ry_0(T)] \\ = 0 \quad \text{if } w_1(0) > rw_1(T), \end{aligned}$$

and

$$w_2'(t) + M(t)w_2(t) + N(t)w_2(\alpha(t)) \le 0 \quad \text{if } w_2(0) \le rw_2(T)$$
$$\begin{aligned} w_2'(t) + M(t)w_2(t) + N(t)w_2(\alpha(t)) \\ + \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[w_2(0) - rw_2(T)] \le 0 \quad \text{if } w_2(0) > rw_2(T). \end{aligned}$$

In view of Theorem 1, $w_1 \le 0, w_2 \le 0$ on J. It shows that $y_0(t) \le x(t) \le z_0(t)$, $t \in J$.

Finally, we prove that problem (3.1) has the solution x. Let us consider two functions

$$\overline{y}_0(t) = \begin{cases} y_0(t) & \text{if } y_0(0) \le ry_0(T), \\ y_0(t) + \frac{t}{rT}[y_0(0) - ry_0(T)] & \text{if } y_0(0) > ry_0(T), \end{cases}$$

and

$$\overline{z}_0(t) = \begin{cases} z_0(t) & \text{if } z_0(0) \ge rz_0(T), \\ z_0(t) + \frac{t}{rT}[z_0(0) - rz_0(T)] & \text{if } z_0(0) < rz_0(T). \end{cases}$$

We have $y_0(t) \le \overline{y}_0(t)$ and $\overline{z}_0(t) \le z_0(t)$, $t \in J$. Moreover, $\overline{y}_0(0) \le r\overline{y}_0(T)$ and $\overline{z}_0(0) \ge r\overline{z}_0(T)$. Note that if $y_0(0) > ry_0(T)$, then $\overline{y}_0(0) = r\overline{y}_0(T)$ and if $z_0(0) < rz_0(T)$, then $\overline{z}_0(0) = r\overline{z}_0(T)$.

We show that $\overline{y}_0$ and $\overline{z}_0$ are classical lower and upper solutions, respectively, of (3.1) and that $\overline{y}_0 \le \overline{z}_0$. We have

$$\overline{y}_0'(t) + M(t)\overline{y}_0(t) + N(t)\overline{y}_0(\alpha(t)) = y_0'(t) + M(t)y_0(t) + N(t)y_0(\alpha(t)) \le \sigma(t),\ t \in J, \quad \text{if } y_0(0) \le r y_0(T)$$

and

$$\begin{aligned}\overline{y}_0'(t) + M(t)\overline{y}_0(t) &+ N(t)\overline{y}_0(\alpha(t)) \\ &= y_0'(t) + M(t)y_0(t) + N(t)y_0(\alpha(t)) \\ &\quad + \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[y_0(0) - r y_0(T)] \\ &\le \sigma(t) - \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[y_0(0) - r y_0(T)] \\ &\quad + \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[y_0(0) - r y_0(T)] \\ &= \sigma(t),\ t \in J, \quad \text{if } y_0(0) > r y_0(T).\end{aligned}$$

Similarly,

$$\overline{z}_0'(t) + M(t)\overline{z}_0(t) + N(t)\overline{z}_0(\alpha(t)) \ge \sigma(t), \quad t \in J.$$

Thus $\overline{y}_0$ is a classical lower and $\overline{z}_0$ a classical upper solution of (3.1).

Now consider the function $w = \overline{y}_0 - \overline{z}_0 \in C^1(J,\mathbb{R})$. We have

$$w'(t) + M(t)w(t) + N(t)w(\alpha(t)) \le 0, \quad t \in J$$

and $w(0) \le 0$. Lemma 1 yields $w \le 0$ on J, hence $\overline{y}_0 \le \overline{z}_0$.

Setting

$$\begin{cases} \overline{y}_{n+1}'(t) = \sigma(t) - M(t)\overline{y}_{n+1}(t) - N(t)\overline{y}_{n+1}(\alpha(t)), & t \in J, \\ \overline{y}_{n+1}(0) = r\overline{y}_n(T), \end{cases}$$

and

$$\begin{cases} \overline{z}_{n+1}'(t) = \sigma(t) - M(t)\overline{z}_{n+1}(t) - N(t)\overline{z}_{n+1}(\alpha(t)), & t \in J, \\ \overline{z}_{n+1}(0) = r\overline{z}_n(T), \end{cases}$$

and arguing similarly to the proof of [1, Theorem 3.1], we show that there exists a solution of (3.1). □

4. APPROXIMATION OF EXTREMAL SOLUTIONS OF (1.1)

In this section we develop monotone iterative technique for (1.1).

Theorem 3. *Let $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\alpha \in C(J,J)$, $\alpha(t) \le t$, $t \in J$, $M, N \in C(J,\mathbb{R})$, $M(t) > 0$, $t \in J$, $N(t) \ge 0$, $t \in J$. Moreover, assume that there exist functions $y_0, z_0 \in C^1(J,\mathbb{R})$ such that*

(1) $y_0 \le z_0$ *on* J,

(2) *y_0, z_0 are lower and upper solutions of (1.1), respectively, i. e.,*

$$y_0'(t) \le f(t, y_0(t), y_0(\alpha(t))) - a(t), \quad t \in J,$$
$$z_0'(t) \ge f(t, z_0(t), z_0(\alpha(t))) - b(t), \quad t \in J,$$

where

$$a(t) = \begin{cases} 0 & \text{if } y_0(0) \le r y_0(T), \\ \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[y_0(0) - r y_0(T)] & \text{if } y_0(0) > r y_0(T), \end{cases}$$

and

$$b(t) = \begin{cases} 0 & \text{if } z_0(0) \ge r z_0(T), \\ \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[z_0(0) - r z_0(T)] & \text{if } z_0(0) < r z_0(T), \end{cases}$$

(3) $f(t,u,v) - f(t,\bar{u},\bar{v}) \le M(t)[\bar{u} - u] + N(t)[\bar{v} - v]$, *if* $y_0(t) \le u \le \bar{u} \le z_0(t)$, $y_0(\alpha(t)) \le u \le \bar{u} \le z_0(\alpha(t))$,

(4) $\int_0^T N(t) e^{\int_{\alpha(t)}^t M(s)ds} dt \le 1$.

Then there exist monotone sequences $\{y_n\} \uparrow y$ and $\{z_n\} \downarrow z$ uniformly on J with $y_0 \le y_n \le z_n \le z_0$ for every $n \in \mathbb{N}$ and $y, z \in C^1(J, \mathbb{R})$. Functions y and z are extremal solutions of (1.1).

Proof. Let us consider the problem

$$\begin{cases} x'(t) + M(t)x(t) + N(t)x(\alpha(t)) = \sigma_u(t), & t \in J, \\ x(0) = r x(T), & r \in (0,1], \end{cases} \tag{4.1}$$

where $\sigma_u(t) = f(t, u(t), u(\alpha(t))) + M(t)u(t) + N(t)u(\alpha(t))$ for $u \in C(J, \mathbb{R})$, $y_0 \le u \le z_0$ on J. Note that, in view of Theorem 2, this problem has exactly one solution. Define operator $A\colon [y_0, z_0] \to [y_0, z_0]$ as $u \mapsto v$, where $Au = v$ is the unique solution of (4.1).

First we show that A is well defined. Put $w = y_0 - v$. If $y_0(0) \le r y_0(T)$ then we have

$$w(0) = y_0(0) - v(0) \le r y_0(T) - r v(T) = r w(T)$$

and

$$\begin{aligned} w'(t) + M(t)w(t) + N(t)w(\alpha(t)) \\ &= y_0'(t) + M(t)y_0(t) + N(t)y_0(\alpha(t)) \\ &\quad - v'(t) - M(t)v(t) + N(t)v(\alpha(t)) \\ &\le f(t, y_0(y), y_0(\alpha(t))) - f(t, u(t), u(\alpha(t))) \\ &\quad + M(t)[y_0(t) - u(t)] + N(t)[y_0(\alpha(t)) - u(\alpha(t))] \\ &\le M(t)[u(t) - y_0(t)] + N(t)[u(\alpha(t)) - y_0(\alpha(t))] \\ &\quad + M(t)[y_0(t) - u(t)] + N(t)[y_0(\alpha(t)) - u(\alpha(t))] \\ &= 0. \end{aligned}$$

By Theorem 1, we have $w(t) \le 0$, $t \in J$. Similarly if $y_0(0) > ry_0(T)$ then

$$w(0) > rw(T)$$

and

$$w'(t) + M(t)w(t) + N(t)w(\alpha(t)) + \frac{M(t)t + N(t)\alpha(t) + 1}{rT}[w(0) - rw(T)] \le 0.$$

Hence $w \le 0$ on J. Analogously, we can show that $v \le z_0$. Thus A is well defined.

Now we prove that A is monotone increasing. Put $w = v_1 - v_2$, where $v_1 = Au_1$, $v_2 = Au_2$ and $u_1 \le u_2$. We have $w(0) = rw(T)$ and

$$\begin{aligned} w'(t) + M(t)w(t) + & N(t)w(\alpha(t)) \\ & = f(t, u_1(t), u_1(\alpha(t))) + M(t)u_1(t) + N(t)u_1(\alpha(t)) \\ & \quad - f(t, u_2(t), u_2(\alpha(t))) - M(t)u_2(t) - N(t)u_2(\alpha(t)) \\ & \le M(t)[u_2(t) - u_1(t)] + N(t)[u_2(\alpha(t)) - u_1(\alpha(t))] \\ & \quad + M(t)[u_1(t) - u_2(t)] + N(t)[u_1(\alpha(t)) - u_2(\alpha(t))] \\ & = 0. \end{aligned}$$

In view of Theorem 1, $v_1 \le v_2$. Since v_1 and v_2 were arbitrary, A is monotone increasing.

Define the sequences $\{y_n\}$ and $\{z_n\}$ as follows:

$$y_{n+1} = Ay_n, \qquad z_{n+1} = Az_n, \quad n \ge 0.$$

Using the mathematical induction we can show that these sequences have the properties

$$y_0 \le y_1 \le \cdots \le y_n \le z_n \le \cdots \le z_1 \le z_0, \quad n \ge 0,$$

because A is monotone increasing. Thus the sequence $\{y_n\}$ is increasing and $y_n \le z_0$, $n \ge 0$. Hence, there exists $\lim_{n\to\infty} y_n(t) = y(t)$ for $t \in J$. The convergence is uniform since $\{y_n\}$ is bounded in $C^1(J, \mathbb{R})$. Similarly $\{z_n\} \downarrow z$ uniformly on J. It is easy to see that y and z are extremal solutions of (1.1). □

Example 1. Let us consider the problem

$$\begin{cases} x'(t) = e^{-x(t)} - tx\left(\frac{1}{3}t\right) - \frac{1}{2}, & t \in [0,1], \\ x(0) = \frac{1}{3}x(1). \end{cases}$$

Put $y_0 = 0$ and $z_0 = 1$. All assumptions of Theorem 3 are satisfied with $M(t) = 1$ and $N(t) = t$. Thus there exist monotone sequences converging uniformly to the extremal solutions of above problem in the sector $[0,1]_*$.

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Author's address

A. Dyki
Gdańsk University of Technology, Department of Differential Equations, 11/12 G. Narutowicz St., 80-952 Gdańsk, Poland
E-mail address: adyki@wp.pl