ON EQUIVALENCE OF TWO INTEGRABILITY METHODS

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Abstract. In this paper, we introduce the concept of \(| R, p |_{k,k} \geq 1\) integrability of improper integrals and by this definition we prove a theorem, that generalizes a theorem of Orhan [3].

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1. INTRODUCTION

Throughout this paper we assume that \( f \) is a real valued function which is continuous on \([0, \infty)\) and \( s(x) = \int_0^x f(t)dt \). By \( \sigma(x) \), we denote the Cesàro mean of \( s(x) \). The integral \( \int_0^\infty f(t)dt \) is said to be integrable \(| C, 1 |_{k,k} \geq 1\), in the sense of Flett [2], if

\[
\int_0^\infty x^{k-1} | \sigma'(x) |^k \, dx = \int_0^\infty \frac{|v(x)|^k}{x} \, dx \tag{1.1}
\]

is convergent. Here, \( v(x) = \frac{1}{2} \int_0^x t f(t)dt \) is called a generator of the integral \( \int_0^\infty f(t)dt \). Let \( p \) be a real valued, non-decreasing function on \([0, \infty)\) such that

\[
P(x) = \int_0^x p(t)dt, p(x) \neq 0, p(0) = 0.
\]

The Riesz mean of \( s(x) \) is defined by

\[
\sigma_p(x) = \frac{1}{P(x)} \int_0^x p(t)s(t)dt.
\]

We say that the integral \( \int_0^\infty f(t)dt \) is integrable \(| R, p |_{k,k} \geq 1\), if

\[
\int_0^\infty x^{k-1} | \sigma_p'(x) |^k \, dx \tag{1.2}
\]

is convergent. In the special case if we take \( p(x) = 1 \) for all values of \( x \), then \(| R, p |_{k} \) integrability reduces to \(| C, 1 |_{k} \) integrability of improper integrals.

Given any functions \( f, g \), it is customary to write \( g(x) = O(f(x)) \), if there exist \( \eta \)
and \( N \), for every \( x > N \), \( \left| \frac{s(x)}{f(x)} \right| \leq \eta \).

The difference between \( s(x) \) and its \( n \)th weighted mean \( \sigma_p(x) \), which is called the weighted Kronecker identity, is given by the identity

\[
s(x) - \sigma_p(x) = v_p(x),
\]

(1.3)

where

\[
v_p(x) = \frac{1}{P(x)} \int_0^x P(u) f(u) du.
\]

We note that if we take \( p(x) = 1 \), for all values of \( x \) then we have the following identity (see [1])

\[
s(x) - \sigma(x) = v(x).
\]

Since

\[
\sigma'_p(x) = \frac{p(x)}{P(x)} v_p(x),
\]

condition (1.3) can be rewritten as

\[
s(x) = v_p(x) + \int_0^x \frac{p(u)}{P(u)} v_p(u) du.
\]

(1.4)

In view of the identity (1.4), the function \( v_p(x) \) is called the generator function of \( s(x) \).

Condition (1.1) can also be written as

\[
\int_0^\infty x^{k-1} \left( \frac{p(x)}{P(x)} \right)^k |v_p(x)|^k dx
\]

(1.5)

is convergent. We note that for infinite series, an analogous definition was introduced by Orhan [3]. Using this definition, Orhan [3] proved the following theorem dealing with \( |R, p_n|_k \) and \( |R, q_n|_k \) summability methods.

**Theorem 1.** The \( |R, p_n|_k, (k \geq 1) \) summability implies the \( |R, q_n|_k, (k \geq 1) \) summability provided that

\[
nq_n = O(Q_n).
\]

(1.6)

\[
P_n = O(np_n).
\]

(1.7)

\[
Q_n = O(nq_n).
\]

(1.8)
The aim of this paper is to state Orhan’s theorem for $|R,p|_k$ and $|R,q|_k$ integrability of improper integrals. Now we shall prove the following theorem.

**Theorem 2.** Let $p$ and $q$ be real valued, non-decreasing functions on $[0, \infty)$ such that as $x \to \infty$

\[
xq(x) = O(Q(x)),
\]

\[
P(x) = O(xp(x)),
\]

\[
Q(x) = O(xq(x)).
\]

If $\int_0^\infty f(t)dt$ is integrable $|R,p|_k$, then it is also integrable $|R,q|_k, k \geq 1$.

### 3. Proof of the Theorem

Let $\sigma_p(x)$ and $\sigma_q(x)$ be the functions of $(R,p)$ and $(R,q)$ means of the integral $\int_0^\infty f(t)dt$. Since $\int_0^\infty f(t)dt$ is integrable $|R,p|_k$, we can write

\[
\int_0^\infty x^{k-1} \left( \frac{p(x)}{P(x)} \right)^k |v_p(x)|^k dx
\]

is convergent. Differentiating the equation (1.4), we have

\[
f(x) = v'_p(x) + \frac{p(x)}{P(x)} v_p(x).
\]

By definition, we obtain

\[
\sigma_q(x) = \frac{1}{Q(x)} \int_0^x q(t)s(t)dt = \frac{1}{Q(x)} \int_0^x (Q(x) - Q(t))f(t)dt
\]

and

\[
\sigma'_q(x) = \frac{q(x)}{Q^2(x)} \int_0^x Q(t)f(t)dt
\]

\[
= \frac{q(x)}{Q^2(x)} \int_0^x Q(t) \left[ v'_p(t) + \frac{p(t)}{P(t)} v_p(t) \right] dt
\]

\[
= \frac{q(x)}{Q^2(x)} \int_0^x Q(t)v'_p(t)dt + \frac{q(x)}{Q^2(x)} \int_0^x Q(t) \frac{p(t)}{P(t)} v_p(t)dt.
\]

Integrating by parts of the first statement, we have

\[
\sigma'_q(x) = \frac{q(x)}{Q^2(x)} \left[ Q(x)v_p(x) - \int_0^x q(t)v_p(t)dt \right] + \frac{q(x)}{Q^2(x)} \int_0^x Q(t) \frac{p(t)}{P(t)} v_p(t)dt
\]

\[
= \frac{q(x)}{Q(x)} v_p(x) + \frac{q(x)}{Q^2(x)} \int_0^x Q(t) \frac{p(t)}{P(t)} v_p(t)dt - \frac{q(x)}{Q^2(x)} \int_0^x Q(t)v_p(t)dt
\]

\[= \sigma_{q,1}(x) + \sigma_{q,2}(x) + \sigma_{q,3}(x), \text{ say}.
\]
To complete the proof of the theorem, it is sufficient to show that
\[
\int_0^m x^{k-1} |\sigma_{q,r}(x)|^k \, dx = O(1) \text{ as } m \to \infty, \text{ for } r = 1, 2, 3.
\]
Using conditions (2.1) and (2.2), we have
\[
\int_0^m x^{k-1} |\sigma_{q,1}(x)|^k \, dx = \int_0^m x^{k-1} \left( \frac{q(x)}{Q(x)} \right)^k |v_p(x)|^k \, dx
\]
\[
= \int_0^m x^{k-1} \left( \frac{q(x)}{Q(x)} \right)^k \left( \frac{p(x)}{P(x)} \right)^k |v_p(x)|^k \, dx
\]
\[
= O(1) \int_0^m x^{k-1} \left( \frac{p(x)}{P(x)} \right)^k |v_p(x)|^k \, dx
\]
\[
= O(1) \int_0^m x^{k-1} |\sigma_p'(x)|^k \, dx
\]
by virtue of the hypotheses of Theorem 2.
Applying Hölder’s inequality with \( k > 1 \), we get
\[
\int_0^m x^{k-1} |\sigma_{q,2}(x)|^k \, dx =
\]
\[
= O(1) \int_0^m x^{k-1} \left( \frac{q(x)}{Q^2(x)} \right)^k \left( \int_0^x \frac{Q(t)p(t)}{P(t)} |v_p(t)| \, dt \right)^k \, dx
\]
\[
= O(1) \int_0^m \frac{q(x)}{Q^{k+1}(x)} \left( \int_0^x \frac{Q(t)p(t)}{P(t)} |v_p(t)| \, dt \right)^k \, dx
\]
\[
= O(1) \int_0^m \frac{q(x)}{Q^2(x)} \left( \int_0^x \left( \frac{Q(t)}{q(t)} \right)^k q(t) \left( \frac{p(t)}{P(t)} \right)^k |v_p(t)|^k \, dt \right)^k \, dx
\]
\[
x \left( \frac{1}{Q(x)} \int_0^x q(t) \, dt \right)^{k-1} \, dx
\]
\[
= O(1) \int_0^m t^k q(t) \left( \frac{p(t)}{P(t)} \right)^k |v_p(t)|^k \, dt \int_0^m \frac{q(x)}{Q^2(x)} \, dx
\]
\[
= O(1) \int_0^m t^k q(t) \left( \frac{p(t)}{Q(t)} \right)^k |v_p(t)|^k \, dt
\]
\[
= O(1) \int_0^m t^{k-1} \left( \frac{p(t)}{P(t)} \right)^k |v_p(t)|^k \, dt
\]
\[
= O(1) \int_0^m t^{k-1} |\sigma_p'(t)|^k \, dt
\]
by virtue of the hypotheses of Theorem 2.
Finally, again by Hölder’s inequality with $k > 1$, we have

\[ \int_0^m x^{k-1} | \sigma_{q,3}(x) |^k dx = O(1) \int_0^m x^{k-1} \left( \left( \frac{q(x)}{Q(x)} \right)^k \left( \int_0^x q(t) \, v_p(t) |^k \, dt \right) \right) dx \]

\[ = O(1) \int_0^m \frac{q(x)}{Q(x)} \left( \int_0^x q(t) \, v_p(t) \, |^k \, dt \right) dx \]

\[ = O(1) \int_0^m \frac{q(t)}{Q(t)} \, v_p(t) \, |^k \, dt \]

\[ = O(1) \text{ as } m \to \infty \]

by virtue of the hypotheses of Theorem 2.
This completes the proof of the theorem.

**Theorem 3.** Let $p$ and $q$ be real valued, non-decreasing functions on $[0, \infty)$ such that as $x \to \infty$

\[ xp(x) = O(P(x)), \quad (3.1) \]
\[ Q(x) = O(xq(x)), \quad (3.2) \]
\[ P(x) = O(xp(x)). \quad (3.3) \]

If $\int_0^\infty f(t) dt$ is integrable $| R, q \,|_k$, then it is also integrable $| R, p \,|_k, k \geq 1.$

**Proof.** In Theorem 2 if we take $p(x) = q(x)$ and $q(x) = p(x)$, then we get Theorem 3. □

**Theorem 4.** Let $p$ and $q$ be real valued, non-decreasing functions on $[0, \infty)$ such that as $x \to \infty$

\[ xp(x) = O(P(x)), \quad (3.4) \]
\[ P(x) = O(xp(x)), \quad (3.5) \]
\[ xq(x) = O(Q(x)), \quad (3.6) \]
\[ Q(x) = O(xq(x)). \quad (3.7) \]

Then the $| R, p \,|_k$ integrability of $\int_0^\infty f(t) dt$ is equivalent to the $| R, q \,|_k$ integrability of $\int_0^\infty f(t) dt$, where $k \geq 1.$
REFERENCES


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