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THE LATTICE OF REGULAR COEQUALITIES

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Abstract. For a coequality q we say that it is regular coequality on set X ordered under the anti-order α if there exists an anti-order θ on X/q such that the natural mapping $\pi : X \longrightarrow X/q$ is a reverse isotone surjection of anti-ordered sets. The lattice of regular coequalities is described.

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1. Introduction and preliminaries

This short investigation, in the framework of Bishop's constructive mathematics ([1–3] and [9]), is continuation of the author's previous papers [6–8]. Bishop's constructive mathematics is developed on Constructive Logic ([9]) - logic without the Law of Excluded Middle $P \vee \neg P$. Let us note that in Constructive Logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold, but the implication $P \Longrightarrow \neg\neg P$ holds even in Minimal Logic. We have to note that 'the crazy axiom' $\neg P \Longrightarrow (P \Longrightarrow Q)$ is included in the Constructive Logic. In Constructive Logic the 'Weak Law of Excluded Middle' $\neg P \vee \neg\neg P$ does not hold, too. It is interesting, that in Constructive Logic the following deduction principle

$$A \vee B, \neg A \vdash B$$

holds, but this is impossible to prove without 'the crazy axiom'. Bishop's Constructive Mathematics is consistent with Classical Mathematics.

A relational structure $(X, =, \neq)$, where the relation "$\neq$" is a binary relation on X, which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq z \Longrightarrow x \neq y \vee y \neq z,\ x \neq y \wedge y = z \Longrightarrow x \neq z$$

will be called a set. Following Heyting, the relation $\neq$ is called *apartness*. A relation q on X is a *coequality relation* on X if and only if it is consistent, symmetric and

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cotransitive ([4, 5]):

$$q \subseteq \neq,\ q^{-1} = q,\ q \subseteq q * q,$$

where "$*$ " is the *filled product* between relations (see [4]). Let β be a consitent relation on X. We put ${}^1\beta = \beta$ and ${}^n\beta = \beta * ... * \beta$ (n factors, $n \in N$). Then ([4]) the relation $c(\beta) = \bigcap_{n \in N} {}^n\beta$, the *cotransitive fulfillment* of β, is the maximal consistent and cotransitive relation on the set X under β.

A relation α on X is an *antiorder* ([6]) on X if and only if

$$\alpha \subseteq \neq,\ \alpha \subseteq \alpha * \alpha,\ \neq \subseteq \alpha \cup \alpha^{-1} (linearity).$$

A relation σ on X is a *quasi-antiorder* ([6]) on X if

$$\sigma \subseteq \neq,\ \sigma \subseteq \sigma * \sigma.$$

Let x be an element of X and let A be a subset of X. We use the notation $x \bowtie A$ if and only if $(\forall a \in A)(x \neq a)$, and $A^C = \{x \in X : x \bowtie A\}$. If σ is a quasi-antiorder on X, then the relation $q = \sigma \cup \sigma^{-1}$ is a coequality on X. Firstly, the relation $q^C = \{(x, y) \in X \times X : (x, y) \bowtie q\}$ is a equality relation on X compatible with q, in the following sense $(\forall a, b, c \in X)((a, b) \in q^C \wedge (b, c) \in q \Longrightarrow (a, c) \in q)$. We can construct the factor-set $X/(q^C, q) = \{aq^C : a \in X\}$ with

$$aq^C =_1 bq^C \Longleftrightarrow (a, b) \bowtie q,\ aq^C \neq_1 bq^C \Longleftrightarrow (a, b) \in q.$$

We can also construct the factor-set $X/q = \{aq : aX\}$ with

$$aq =_1 bq \Longleftrightarrow (a, b) \bowtie q,\ aq \neq_1 bq \Longleftrightarrow (a, b) \in q.$$

It is easy to check that $X/(q^C, q) \cong X/q$. The mapping $\pi : X \longrightarrow X/q$, defined by $\pi(a) = aq$ for any $a \in X$, is a strongly extensional surjection.

Secondly, note that the relation α^C is an order relation on set $(X, \neg \neq, \neq)$. If the relation $\neg\alpha$ is an order relation on $(X, =, \neq)$, when the apartness is tight, $\neg \neq \subseteq =$ ([5]), then the relation α is called *excise relation* on X. (The notion of anti-order relation is more general then notion of excise relation.)

For a given anti-ordered set $(X, =, \neq, \alpha)$ it is essential to know if there exists a coequality relation q on X such that X/q is an anti-ordered set. This plays an important role in the investigation of anti-ordered sets. The following question is natural: If $(X, =, \neq, \alpha)$ is an anti-ordered set and q a coequality on X, is the set X/q an anti-ordered set? A possible anti-order on X/q could be the relation Θ on X/q defined by the anti-order α on X, where $\Theta = \{(xq, yq) \in X/q \times X/q : (x, y) \in \alpha\}$. But it is not an anti-order, in general. The following question arises: Is there a coequality q on X for which X/q is anti-ordered set? The concept of quasi-antiorder relation was introduced in [6]. According to [6], if $(X, =, \neq, \alpha)$ is an anti-ordered set and σ a quasi-antiorder on X, then the relation q on X, defined by $q = \sigma \cup \sigma^{-1}$

is a coequality relation on X and the set X/q is an ordered set under anti-order Θ defined by $(xq, yq) \in \Theta \iff (x, y) \in \sigma$. So, according to the results in [6], each quasi-antiorder σ on an ordered set X under anti-order α induces a coequality relation $q = \sigma \cup \sigma^{-1}$ on X such that X/q is an ordered set under antiorder Θ. In [7] we prove that the converse of this statement also holds. If $(X, =, \neq, \alpha)$ is an anti-ordered set and q a coequality relation on X and if there exists an antiorder relation θ_1 on X/q such that $(X/q, =_1, \neq_1, \theta_1)$ is an ordered set under antiorder θ_1, then there exists a quasi-antiorder σ on X such that $q = \sigma \cup \sigma^{-1}$ and $\theta_1 = \Theta$. So, each coequality relation q on a set $(X, =, \neq, \alpha)$ such that X/q is an anti-ordered set induces a quasi-antiorder on X. This was the motivation of a new notion. For that we need the following notion: Let f be a strongly extensional mapping of anti-ordered sets from $(X, =, \neq, \alpha)$ into $(Y, =, \neq, \beta)$. For f we say that it is *reverse isotone* if

$$(\forall a, b \in X)((f(a), f(b)) \in \beta \Longrightarrow (a, b) \in \alpha)$$

holds. A coequality relation q on X is called *regular* if there is an antiorder "θ_1" on X/q satisfying the following conditions:

(1) $(X/q, =_1, \neq_1, \theta_1)$ is an anti-ordered set;
(2) The mapping $\pi : X \ni a \longmapsto aq \in X/q$ is an anti-order reverse isotone surjection.

We call the antiorder "θ_1" on X/q a *regular antiorder with respect* to a regular coequality q on X and the anti-order α.

It is obviously that the regular antiorder on X/q with respect to a regular coequality q and to the antiorder α on X is in general not unique. The following questions now naturally arise: Does there exist the maximal regular antiorder on X/q with respect to a regular coequality q on X? Are all coequalities on anti-ordered sets regular? Trying to find an answers for the above questions, in this note we give a description of the family of regular coequalities. In Theorem 1 and Theorem 2 we give necessary and sufficient conditions such that coequality on an anti-ordered set is regular. In Theorem 3 we give a construction of the maximal quasi-antiorder on the anti-ordered set X induced by a regular coequality q on X. The section "The lattice of regular coequalities" contains the main results of this paper. We prove that the family of all regular coequalities with respect to the one anti-order relation on ordered set is a complete lattice and describe that lattice.

For the necessary undefined notions, the reader is referred to books [1–3, 9] and to papers [4–8].

2. Regular anticongruences

In the following lemma we describe classes of a quasi-antiorder relation:

Lemma 1 ([7, Lemma 0]). *Let σ be a quasi-antiorder on set X. Then $x\sigma$ (σx) is a strongly extensional subset of X, such that $x \bowtie x\sigma$ $(x \bowtie \sigma x)$, for each $x \in X$.*

In order to obtain the relationship between regular anticongruence and quasi-antiorder on X, the following theorem is essential.

Theorem 1 ([7, Theorem 1]). *Let $(X,=,\neq,\alpha)$ be an anti-ordered set, let q be a coequality on X. The following are equivalent:*

(1) *q is regular.*
(2) *there exists a quasi-antiorder σ on X, such that $q = \sigma \cup \sigma^1$.*

Theorem 2 ([7, Corollary 2]). *Let $(X,=,\neq,\alpha)$ be an anti-ordered set and let q be a coequality on X. The following are equivalent:*

(1) *q is regular;*
(2) *there exists an anti-ordered set $(T,=,\neq,\theta)$ and a strongly extensional reverse isotone mapping $\varphi : S \longrightarrow T$ such that $q = \{(a,b) \in X \times X : \varphi(a) \neq \varphi(b)\}$.*

Recall that, by Lemma 1, any class aq of coequality relation q, generated by the element $a \in X$, is strongly extensional subset of X. Besides, we have the following assertion, which is crucial for the characterization of regular coequality on an anti-ordered set $(X,=,\neq,\alpha)$: If q is a regular coequality relation on an anti-ordered set X, then for every q- class aq in X we have

$$((x,y) \bowtie \alpha \wedge (y,z) \bowtie \alpha \wedge x,z \bowtie aq) \Longrightarrow y \bowtie aq$$

for any $x,y,z,a \in X$. If q is a regular coequality on a set X, then there exists an anti-order relation θ on X/q such that the natural mapping $\pi : X \longrightarrow X/q$ is a strongly extensive reverse isotone surjection. Besides, there exists a quasi-antiorder σ under α, defined by $(x,y) \in \sigma \Longleftrightarrow (xq,yq) \in \theta$ such that $\sigma \cup \sigma^{-1} = q$. Let t be an arbitrary element of aq. Then $(a,t) \in q = \sigma \cup \sigma^{-1}$. Thus $(a,t) \in \sigma$ or $(t,a) \in \sigma$. Hence, we have

$$\begin{aligned}
(a,t) \in \sigma &\Longrightarrow ((a,x) \in \sigma \subseteq q \vee (x,y) \in \sigma \subseteq \alpha \vee (y,t) \in \sigma \subseteq q \subseteq\neq) \\
&\Longrightarrow t \neq y; \\
(t,a) \in \sigma &\Longrightarrow ((t,y) \in \sigma \subseteq\neq \vee (y,z) \in \sigma \subseteq \alpha \vee (z,a) \in \sigma \subseteq q \\
&\Longrightarrow t \neq y.
\end{aligned}$$

So, in both cases, we have that $t \in aq \Longrightarrow t \neq y$. Therefore, $y \bowtie aq$.

We also have

$$((x,y) \bowtie \alpha \wedge (y,z) \bowtie \alpha \wedge y \in aq) \Longrightarrow x \in aq \vee z \in aq$$

for any $x,y,a \in X$. Indeed, if $x,y,z,a \in X$ such that $(x,y) \bowtie \alpha$ and $(y,z) \bowtie \alpha$ and $x \in aq$, then $(a,y) \in q = \sigma \cup \sigma^{-1} \Longrightarrow ((a,y) \in \sigma \vee (y,a) \in \sigma)$. Thus, we have

$$\begin{aligned}
&((a,y) \in \sigma \vee (y,a) \in \sigma) \Longrightarrow \\
&((a,x) \in \sigma \subseteq q \vee (x,y) \in \sigma \subseteq \alpha) \vee ((y,z) \in \sigma \subseteq \alpha \vee (z,a) \in \sigma \subseteq q) \Longrightarrow \\
&x \in aq \vee z \in aq.
\end{aligned}$$

Let q be a regular coequality relation on the anti-ordered set $(X,=,\neq,\alpha)$. Then there exists anti-order θ on X/q such that the natural mapping $\pi : X \longrightarrow X/q$ is reverse isotone. Hence, by [7], there exists a quasi-antiorder σ under α such that $q=\sigma\cup\sigma^{-1}$ and $\theta \subseteq \{(aq,bq)\in X/q\times X/q : (a,b)\in\sigma\}$. In the following theorem we show that there exists such maximal quasi-antiorder τ under α and we prove that there exists such construction of that relation.

Theorem 3 ([7, Theorem 3])**.** *Let q be a regular coequality relation on anti-ordered set $(X,=,\neq,\alpha)$. Then there exists the maximal quasi-antiorder relation τ under α such that $q=\tau\cup\tau^{-1}$ and $\theta\subseteq\{(aq,bq)\in X/q\times X/q : (a,b)\in\tau\}$. That relation is exactly the following relation $c(q\cap\alpha)=\bigcap_{n\in N}{}^{n}(q\cap\alpha)$.*

At the end of this consideration, we give the following assertion:

Theorem 4 ([7, Corollary 4])**.** *Let q be a regular coequality on anti-ordered set $(X,=,\neq,\alpha)$. Then there exists the maximal antiorder relation on X/q. That relation is exactly the following relation $\{(aq,bq)\in X/q\times X/q : (a,b)\in c(q\cap\alpha)\}$.*

3. THE LATTICE OF REGULAR COEQUALITIES

Let $(X,=,\neq,\alpha)$ be anti-ordered set. We denote by $\Re(X,\alpha)$ the family of all regular coequality relations on X with respect to α and $\Im(X,\alpha)$ denotes the family of all quasi-antiorder relation on X included in α .

Theorem 5. *Let X be an anti-ordered set. Then $\Re(X,\alpha)$ is a complete lattice.*

Proof. Let $\{q_k\}_{k\in K}$ be a family of regular coequality relations on X.
(1) Then $\bigcup_k q_k$ is a regular coequality relation on X. If fact, if θ_k is an anti-order relation on X/q_k with respect to q_k and α, then $\bigcup_k \theta_k$ is an anti-order relation on $X/(\bigcup_k q_k)$ with respect to $\bigcup_k q_k$ and α.
(2) For each k there exists a quasi-antiorder relation σ_k on X under α such that $q_k=\sigma_k\cup(\sigma_k)^{-1}$. Then $\tau=c(\bigcap_k\sigma_k)$ is the maximal quasi-antiorder relation under $(\bigcap_k\sigma_k\subseteq)\alpha$. Thus, the relation $q=\tau\cup\tau^{-1}$ is a coequality relation on X and the relation $\theta=\pi\circ\tau\circ\pi^{-1}$ is an anti-order relation on X/q . So, the relation q is a regular coequality relation on X with respect to and α. □

Let us note that family $\Im(X,\alpha)$ is a completely lattice. Indeed, in the following theorem we give prove this fact:

Theorem 6. *If $\{\tau_k\}_{k\in J}$ is a family of quasi-antiorders on a set $(X,=,\neq,\alpha)$, then $\bigcup_{k\in J}\tau_k$ and $c(\bigcap_{k\in J}\tau_k)$ are quasi-antiorders in X. So, the family $\Im(X,\alpha)$ is a completely lattice.*

Proof. Let $\{\tau_k\}_{k\in J}$ be a family of quasi-antiorders on a set $(X,=,\neq)$ under α and let (x,z) be an arbitrary elements of X such that $(x,z)\in\bigcup_{k\in J}\tau_k$. Then, there exists k in J such that $(x,z)\in\tau_k$. Hence, for every $y\in X$ we have $(x,y)\in\tau_k(y,z)\in\tau_k$.

So, $(x,y) \in \bigcup_{k\in J} \tau_k \vee (y,z) \in \bigcup_{k\in J} \tau_k$. On the other hand, for every k in J holds $\tau_k \subseteq \alpha$. From this we have $\bigcup_{k\in J} \tau_k \subseteq \alpha$.
It is clear that the relation $c(\bigcap_{k\in J} \tau_k)$ is the maximal quasi-antiorder relation under $\bigcap_{k\in J} \tau_k (\subseteq \alpha)$. □

As end of this consideration we establish connection between lattices $\Re(X,\alpha)$ and $\Im(X,\alpha)$

Theorem 7. *The mapping*

$$\varphi : \Im(X,\alpha) \longrightarrow \Re(X,\alpha),$$

defined by $\varphi(\tau) = \tau \cup \tau^{-1}$ *, is a strongly extensional surjective function. Relations* $\varepsilon = Ker\varphi$ *and* $\omega = Antiker\varphi = \{(\tau,\sigma) \in \Im(X,\alpha) \times \Im(X,\alpha) : \tau \cup \tau^{-1} \neq \sigma \cup \sigma^{-1}\}$ *are compatible equality and diversity relation on* $\Im(X,\alpha)$*, and the following isomorphism* $\Im(X,\alpha)/(\varepsilon,\omega) \cong \Re(X,\alpha)$ *exists.*

Proof. (1) The mapping φ is a well-defined strongly extensional function: If τ is a quasi-antiorder relation on X, then $q(\tau) = \tau \cup \tau^{-1}$ is a coequality relation on X. Then, there exists an anti-order relation θ on X/q defined by $(aq,bq) \in \theta \Longleftrightarrow (a,b) \in \tau$ and the natural mapping $\pi : X \longrightarrow X/q(\tau)$ is reverse isotone. This means that $\varphi(\tau) = \tau \cup \tau^{-1} = q \in \Re(X,\alpha)$. Let σ and τ be elements of $\Im(X,\alpha)$ such that $\tau\varepsilon = \sigma\varepsilon$. Then $(\tau,\sigma) \in \varepsilon$ and $\varphi(\tau) = \tau \cup \tau^{-1} = \sigma \cup \sigma^{-1} = \varphi(\sigma)$. Suppose that $\varphi(\tau) = \tau \cup \tau^{-1} \neq \sigma \cup \sigma^{-1} = \varphi(\sigma)$ for some $\sigma,\tau \in \Im(X,\alpha)$. Then there exists an element $(x,y) \in X \times X$ such that $((x,y) \in \tau \cup \tau^{-1}$ and $(x,y) \bowtie \sigma \cup \sigma^{-1})$ or $((x,y) \in \sigma \cup \sigma^{-1}$ and $(x,y) \bowtie \tau \cup \tau^{-1})$. In the first case, we have:
$((x,y) \in \tau \vee (x,y) \in \tau^{-1}) \wedge (x,y) \bowtie \sigma \wedge (x,y) \bowtie \sigma^{-1} \Longrightarrow$
$((x,y) \in \tau \wedge (x,y) \bowtie \sigma) \vee ((x,y) \in \tau^{-1} \wedge (x,y) \bowtie \sigma^{-1}) \Longleftrightarrow$
$((x,y) \in \tau \wedge (x,y) \bowtie \sigma) \vee ((y,x) \in \tau \wedge (y,x) \bowtie \sigma) \Longrightarrow \tau \neq \sigma.$

In the second case we derive similar implication analogously.
(2) φ is an injective function. In fact: let τ and σ be elements of $\Im(X,\alpha)$ such that $\varphi(\tau) = \tau \cup \tau^{-1} = \sigma \cup \sigma^{-1} = \varphi(\sigma)$. Then, $(\tau,\sigma) \in \varepsilon$ and $\tau\varepsilon = \sigma\varepsilon$.
(3) φ is an embedding. Indeed, let τ and σ be elements of $\Im(X,\alpha)$ such that $\tau\varepsilon \neq \sigma\varepsilon$, i.e. such that $(\tau,\sigma) \in \omega$. It means $\varphi(\tau) = \tau \cup \tau^{-1} \neq \sigma \cup \sigma^{-1} = \varphi(\sigma)$.
(4) φ is a surjective function: Let q be a regular coequality relation on X with respect to α , i.e. let q be a coequality relation on X such that there exists an anti-order θ on X/q and the natural mapping $\pi : X \longrightarrow X/q$ is reverse isotone. Then, there exists a quasi-antiorder $\sigma(\subseteq \alpha)$ on X such that $\sigma \cup \sigma^{-1} = q$. Thus, $\sigma \in \Im(X,\alpha)$ and $\varphi(\sigma\varepsilon) = \sigma \cup \sigma^{-1} = q$. □

REFERENCES

[1] E. Bishop, *Foundations of constructive analysis*. New York-Toronto: McGraw-Hill Book Co., 1967.

[2] D. Bridges and F. Richman, *Varieties of constructive mathematics*, ser. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 1987, vol. 97.

[3] R. Mines, F. Richman, and W. Ruitenburg, *A course in constructive algebra*, ser. Universitext. New York: Springer-Verlag, 1988.

[4] D. A. Romano, "On construction of maximal coequality relation and its applications," *Proceedings of the VIII international conference on logic and computer science: theoretical foundations of computer science, Lira '97*, pp. 225–230, 1997.

[5] D. A. Romano, "Some relations and subsets generated by principal consistent subset of semigroup with apartness," *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.*, vol. 13, pp. 7–25, 2002.

[6] D. A. Romano, "A note on the quasi-antiorder in a semigroup," *Novi Sad J. Math.*, vol. 37, no. 1, pp. 3–8, 2007.

[7] D. A. Romano, "On regular anti-congruence in anti-ordered semigroups," *Publ. Inst. Math. (Beograd) (N.S.)*, vol. 81, no. 95, pp. 95–102, 2007.

[8] D. A. Romano, "A remark on coequality relation in anti-ordered set," *Int. J. Contemp. Math. Sci.*, vol. 3, no. 1-4, pp. 43–48, 2008.

[9] A. S. Troelstra and D. van Dalen, *Constructivism in mathematics. Vol. I.*, ser. Studies in Logic and the Foundations of Mathematics. Amsterdam: North-Holland Publishing Co., 1988, vol. 121.

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