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A NOTE ON SOME DIOPHANTINE EQUATIONS

REFİK KESKİN AND MERVE GÜNEY DUMAN

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Abstract. Let $k \geq 3$ be an odd integer. In this paper we investigate all positive integer solutions of the equations $x^4 - kx^2y + y^2 = \mp A$, $x^4 - kx^2y + y^2 = \mp A(k^2-4)$, $x^4 - (k^2-4)y^2 = \mp 4A$, and $x^2 - (k^2-4)y^4 = \mp 4A$ with $A = \mp(k \mp 2)$. We show that if $k \equiv 1 (mod\, 8)$ and $k^2 - 4$ be a square-free integer, then the equation $x^4 - kx^2y + y^2 = (k-2)(k^2-4)$ has no positive integer solutions. Moreover, if $k^2 - 4$ be a square-free integer, then the equation $x^4 - kx^2y + y^2 = -(k+2)(k^2-4)$ has no positive integer solutions.

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1. Introduction

Let k and s be two nonzero integers with $k^2 + 4s > 0$. The generalized Fibonacci sequence is defined by $U_0(k,s) = 0$, $U_1(k,s) = 1$, and $U_{n+1}(k,s) = kU_n(k,s) + sU_{n-1}(k,s)$ for $n \geqslant 1$ and generalized Lucas sequence is defined by $V_0(k,s) = 2$, $V_1(k,s) = k$, and $V_{n+1}(k,s) = kV_n(k,s) + sV_{n-1}(k,s)$ for $n \geqslant 1$, respectively. Moreover, generalized Fibonacci and Lucas numbers for negative subscripts are defined as $U_{-n} = -(-s)^{-n}U_n$ and $V_{-n} = (-s)^{-n}V_n$ for $n \in \mathbb{N}$. Especially $U_{-n}(k,-1) = -U_n(k,-1)$ and $V_{-n}(k,-1) = V_n(k,-1)$ for every natural number n.

In the literature, there are many papers dealing with the number of the solutions of the equation $ax^4 + bx^2y + cy^2 - d = 0$. In [7], the author investigated all nonnegative integer solutions of the equations $x^4 - kx^2y + y^2 = 1,4$, under the assumption that k is even. If $k \neq 318$ and k is not a perfect square, then all nonnegative integer solutions of the equation $x^2 - kxy^2 + y^4 = 1$ are $(x,y) = (k,1)$, $(1,0)$, $(0,1)$. If $k = 338$, then all positive integer solutions of this equation are $(x,y) = (13051348805, 6214)$, $(114243, 6214)$. If k is a perfect square, then all positive integer solutions of this equation are $(x,y) = (1,\sqrt{k})$, $(k^2-1,\sqrt{k})$. Moreover, the author showed that if $k = 2v^2$ for some integer v, then all positive integer solutions of the equation $x^2 - kxy^2 + y^4 = 4$ are $(x,y) = (2,\sqrt{2k})$, $(2k^2-2,\sqrt{2k})$.

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Otherwise, the only nonnegative integer solution of this equation is $(x,y) = (2,0)$. In [2], wherein the author investigated all positive integer solutions of the equation $x^2 - kxy^2 + y^4 = c$ for $c \in \{\mp 1, \mp 2, \mp 4\}$, with the assumption that k is odd for the particular cases $c = 1$ or 4. He showed that the equation $x^2 - kxy^2 + y^4 = 4$ has no positive integer solutions. Moreover, he showed that if $y = k^2$ for some odd integer k, then the only integer solution of the equation $x^2 - kxy^2 + y^4 = 1$ is $y = 1$. Otherwise, all positive integer solutions of this equation are $y = 1$ or k. Moreover, he showed that the equation

$$x^2 - kxy^2 + y^4 = -1 \tag{1.1}$$

has no integer solutions when $k = 3$. Otherwise, the only integer solution of the equation (1.1) is $y = 1$. In addition, he showed that the equation

$$x^2 - kxy^2 + y^4 = -4 \tag{1.2}$$

has a solution only when $k = 3$ or 6. If $k = 3$, then all positive integer solutions of equation (1.2) are given by $(x,y) = (2,2)$ or $(10,2)$. If $k = 6$, then all positive integer solutions of equation (1.2) are given by $(x,y) = (1,1), (5,1), (29,13)$ or $(985,13)$.

Let k be an odd integer. In this paper, firstly, we solve the equations $x^2 = V_n(k,-1) \mp V_{n-1}(k,-1)$ and $x^2 = U_n(k,-1) \mp U_{n-1}(k,-1)$, respectively. We show that if $k \equiv 3 (mod 8)$, then the equation $x^2 = V_n(k,-1) + V_{n-1}(k,-1)$ has no solutions and if $k \equiv 1 (mod 8)$, then the equation $x^2 = V_n(k,-1) - V_{n-1}(k,-1)$ has no solutions. Lastly, we find all positive integer solutions of the following equations

$$x^4 - (k^2-4)y^2 = 4(k+2),$$
$$x^4 - (k^2-4)y^2 = -4(k-2),$$
$$x^2 - (k^2-4)y^4 = 4(k+2),$$
$$x^2 - (k^2-4)y^4 = -4(k-2),$$
$$x^4 - kx^2y + y^2 = -(k-2),$$

and

$$x^4 - kx^2y + y^2 = k+2.$$

We show that if $k \equiv 1 (mod 8)$, then the equation $x^4 - (k^2-4)y^2 = -4(k-2)$ has no integer solutions. Moreover, if $k^2 - 4$ is square-free, then we investigate all positive integer solutions of the equations

$$x^4 - kx^2y + y^2 = (k^2-4)(k-2)$$

and

$$x^4 - kx^2y + y^2 = -(k^2-4)(k+2).$$

We show that if $k \equiv 1 (mod 8)$ and $k^2 - 4$ is square-free, then the equation $x^4 - kx^2y + y^2 = (k-2)(k^2-4)$ has no positive integer solutions. Moreover, if $k^2 - 4$

is square-free, then we show that the equation $x^4 - kx^2y + y^2 = -(k+2)(k^2-4)$ has no positive integer solutions. Let $\left(\frac{a}{b}\right)$ represent Jacobi symbol. Then

$$\left(\frac{a^2}{k}\right) = 1$$

for every $a \in \mathbb{Z}$. Moreover, we have

$$\left(\frac{-1}{k}\right) = 1 \text{ if and only if } k \equiv 1,5(mod\,8), \tag{1.3}$$

$$\left(\frac{2}{k}\right) = 1 \text{ if and only if } k \equiv 1,7(mod\,8), \tag{1.4}$$

and

$$\left(\frac{-2}{k}\right) = 1 \text{ if and only if } k \equiv 1,3(mod\,8). \tag{1.5}$$

2. Preliminary results

From now on, $\square$ represents a perfect square. Now we give the following lemma from [5].

Lemma 1. *Let k be odd. If $V_n(k,-1) = \square$ for some natural number n, then $n = 1$ and k is a perfect square.*

The following two theorems are given in [3].

Theorem 1. *Let $k \geq 3$ be an odd integer. If $V_n(k,-1) = k\square$ for some natural number n, then $n = 1$.*

Theorem 2. *Let $k \geq 3$ be an odd integer. If $V_n(k,-1) = q\square$ for some $q \mid k$ with $q > 1$ and $n \geq 1$, then $n = 1$.*

The following theorem is given in [6].

Theorem 3. *Let $n \in \mathbb{N} \cup \{0\}, m, r \in \mathbb{Z}$, and m be a nonzero integer. Then*

$$U_{2mn+r} \equiv U_r (mod\, U_m), \tag{2.1}$$

$$U_{2mn+r} \equiv (-1)^n U_r (mod\, V_m), \tag{2.2}$$

$$V_{2mn+r} \equiv V_r (mod\, U_m), \tag{2.3}$$

and

$$V_{2mn+r} \equiv (-1)^n V_r (mod\, V_m), \tag{2.4}$$

where $U_n = U_n(k,-1)$ and $V_n = V_n(k,-1)$.

When k is odd, it is seen that $V_{2^r} \equiv 7(mod\,8)$ and thus

$$\left(\frac{-1}{V_{2^r}}\right) = -1 \tag{2.5}$$

for $r \geq 1$. It can be easily seen that

$$\left(\frac{k+1}{V_{2^r}}\right) = \left(\frac{k-1}{V_{2^r}}\right) = 1 \tag{2.6}$$

for $r \geq 1$. Moreover, it can be shown that if $k \equiv 1,7(mod\,8)$, then

$$\left(\frac{k+2}{V_{2^r}}\right) = 1 \tag{2.7}$$

and

$$\left(\frac{k-2}{V_{2^r}}\right) = \begin{cases} -1 & \text{if } k \equiv 1,7(mod\,8), \\ 1 & \text{if } k \equiv 3(mod\,8). \end{cases} \tag{2.8}$$

From now on, instead of $U_n(k,-1)$ and $V_n(k,-1)$, we will write U_n and V_n, respectively. The following seven theorems are given in [4].

Theorem 4. *Let $k \geq 3$ be an integer and $k+2$ be not a perfect square. Then all positive integer solutions of the equation $x^2 - kxy + y^2 = k+2$ are given by $(x,y) = (U_{n+1} + U_n, U_n + U_{n-1})$ with $n \geq 1$.*

Theorem 5. *Let $k \geq 3$. Then all positive integer solutions of the equation $x^2 - kxy + y^2 = -(k-2)$ are given by $(x,y) = (U_{n+1} - U_n, U_n - U_{n-1})$ with $n \geq 0$.*

Theorem 6. *Let $k \geq 3$ an integer and $k+2$ be not a perfect square. Then all positive integer solutions of the equation $x^2 - (k^2-4)y^2 = 4(k+2)$ are given by $(x,y) = (V_n + V_{n-1}, U_n + U_{n-1})$ with $n \geq 1$.*

Theorem 7. *Let $k \geq 3$ an integer and $k+2$ be a perfect square. Then all positive integer solutions of the equation $x^2 - (k^2-4)y^2 = 4(k+2)$ are given by $(x,y) = (\sqrt{k+2}V_{n-2}, U_{n-2})$ with $n \geq 1$.*

Theorem 8. *Let $k \geq 3$. Then all positive integer solutions of the equation $x^2 - (k^2-4)y^2 = -4(k-2)$ are given by $(x,y) = (V_n - V_{n-1}, U_n - U_{n-1})$ with $n \geq 1$.*

Theorem 9. *Let $k \geq 3$ an integer and k^2-4 be square-free. Then all positive integer solutions of the equation $x^2 - kxy + y^2 = -(k+2)(k^2-4)$ are given by $(x,y) = (V_{n+1} + V_n, V_n + V_{n-1})$ with $n \geq 0$.*

Theorem 10. *Let $k \geq 3$ an integer and k^2-4 be square-free. Then all positive integer solutions of the equation $x^2 - kxy + y^2 = (k-2)(k^2-4)$ are given by $(x,y) = (V_{n+1} - V_n, V_n - V_{n-1})$ with $n \geq 1$.*

3. Some Theorems and Lemmas

From now on, we will assume that $n \geq 0$ and $k \geq 3$ is odd.

Lemma 2. *If* $k \equiv 3 (mod 8)$, *then* $\left(\frac{k^2-k-1}{V_{2^r}}\right) = -1$.

Proof. An induction method shows that

$$V_{2^r} \equiv \begin{cases} k-1 (mod k^2-k-1) & \text{if } r \text{ is even,} \\ -k (mod k^2-k-1) & \text{if } r \text{ is odd.} \end{cases}$$

Let r be even. Then we obtain

$$\begin{aligned} \left(\frac{k^2-k-1}{V_{2^r}}\right) &= (-1)^{\left(\frac{k^2-k-2}{2}\right)\left(\frac{V_{2^r}-1}{2}\right)} \left(\frac{V_{2^r}}{k^2-k-1}\right) \\ &= \left(\frac{k-1}{k^2-k-1}\right) = \left(\frac{2}{k^2-k-1}\right)\left(\frac{(k-1)/2}{k^2-k-1}\right) \\ &= (-1)^{\frac{(k^2-k-1)^2-1}{8}} (-1)^{\left(\frac{k^2-k-2}{2}\right)\left(\frac{k-3}{4}\right)} \left(\frac{k^2-k-1}{(k-1)/2}\right) \\ &= (-1)\left(\frac{-1}{(k-1)/2}\right) = (-1)(-1)^{\frac{k-3}{4}} = -1. \end{aligned}$$

Let r be odd. Then we have

$$\begin{aligned} \left(\frac{k^2-k-1}{V_{2^r}}\right) &= (-1)^{\left(\frac{k^2-k-2}{2}\right)\left(\frac{V_{2^r}-1}{2}\right)} \left(\frac{V_{2^r}}{k^2-k-1}\right) \\ &= \left(\frac{-k}{k^2-k-1}\right) = \left(\frac{-1}{k^2-k-1}\right)\left(\frac{k}{k^2-k-1}\right) \\ &= (-1)^{\frac{k^2-k-2}{2}} (-1)^{\left(\frac{k^2-k-2}{2}\right)\left(\frac{k-1}{2}\right)} \left(\frac{k^2-k-1}{k}\right) \\ &= \left(\frac{-1}{k}\right) = (-1)^{\left(\frac{k-1}{2}\right)} = -1. \end{aligned}$$

□

Since the proof of the following lemma is easy, we omit it.

Lemma 3. $V_{2^r} \equiv \begin{cases} k (mod k^2+k-1) & \textit{if } r \textit{ is even,} \\ -k-1 (mod k^2+k-1) & \textit{if } r \textit{ is odd.} \end{cases}$

By using the above lemma, we can give the following lemma.

Lemma 4. *If* $k \equiv 1,5 (mod 8)$, *then* $\left(\frac{k^2+k-1}{V_{2^r}}\right) = 1$.

Theorem 11. *Let* $k \equiv 1,5,7 (mod 8)$. *If the equation* $x^2 = V_n + V_{n-1}$ *has a solution, then* $n = 0$ *or* 1. *If* $k \equiv 3 (mod 8)$, *then we have two solutions* $(n,k) = (3,3)$, $(3,43)$.

Proof. Assume that $x^2 = V_n + V_{n-1}$ for some positive integer x. Firstly, let $k \equiv 1,5,7(mod\,8)$ and $n > 2$. We divide the reminder of the proof into four cases.

Case 1: Assume that $n = 4q-1$ with $q > 0$. Then $n = 2\cdot 2^r a - 1$ with a odd and $r \geq 1$. By (2.4) and (2.3), it follows that

$$x^2 = V_n + V_{n-1} \equiv -(V_{-1} + V_{-2}) \equiv -(k^2+k-2)(mod\,V_{2^r})$$

and

$$x^2 = V_n + V_{n-1} \equiv V_{-1} + V_{-2} \equiv k^2+k-2 \equiv -2(mod\,U_2).$$

The above congruences give $\left(\frac{-(k^2+k-2)}{V_{2^r}}\right) = 1$ and $\left(\frac{-2}{k}\right) = 1$. Since $\left(\frac{-2}{k}\right) = 1$, we obtain $k \equiv 1,3(mod\,8)$ by (1.5). Since $k \equiv 1,5,7(mod\,8)$, it follows that $k \equiv 1(mod\,8)$. By (2.6), (2.5), and (2.7), we get

$$1 = \left(\frac{-(k^2+k-2)}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{k+2}{V_{2^r}}\right)\left(\frac{k-1}{V_{2^r}}\right) = (-1)\cdot 1\cdot 1 = -1,$$

a contradiction.

Case 2: Assume that $n = 4q$ with $q \geq 1$. Then $n = 2\cdot 2^r a$ with a odd and $r \geq 1$. By (2.4), we get

$$x^2 = V_n + V_{n-1} \equiv -(V_0 + V_{-1}) \equiv -(k+2)(mod\,V_{2^r}).$$

This shows that $\left(\frac{-(k+2)}{V_{2^r}}\right) = 1$. Then by (2.5), it follows that $\left(\frac{-(k+2)}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{k+2}{V_{2^r}}\right) = (-1)\left(\frac{k+2}{V_{2^r}}\right) = 1$, i.e., $\left(\frac{k+2}{V_{2^r}}\right) = -1$. Moreover, since

$$x^2 = V_{4q} + V_{4q-1} \equiv V_0 + V_{-1} \equiv k+2 \equiv 2(mod\,U_2),$$

we obtain $\left(\frac{2}{U_2}\right) = \left(\frac{2}{k}\right) = 1$. By (1.4), we get $k \equiv 1,7(mod\,8)$. Then by (2.7), it follows that $\left(\frac{k+2}{V_{2^r}}\right) = 1$, which contradicts the fact that $\left(\frac{k+2}{V_{2^r}}\right) = -1$.

Case 3: Assume that $n = 4q+1$ with $q > 0$. Then $n = 2\cdot 2^r a + 1$ with a odd and $r \geq 1$. By (2.4) and (2.3), it follows that

$$x^2 = V_n + V_{n-1} \equiv -(V_1 + V_0) \equiv -(k+2)(mod\,V_{2^r})$$

and

$$x^2 = V_{4q+1} + V_{4q} \equiv V_1 + V_0 \equiv k+2 \equiv 2(mod\,U_2).$$

Both the congruences above give $\left(\frac{-(k+2)}{V_{2^r}}\right) = 1$ and $\left(\frac{2}{U_2}\right) = \left(\frac{2}{k}\right) = 1$. Then by (1.4) and (2.7), we have a contradiction.

Case 4: Assume that $n = 4q+2$ for some $q > 0$. Then $n = 2\cdot 2^r a + 2$ with a odd and $r \geq 1$. By (2.4) and (2.3), it follows that

$$x^2 = V_n + V_{n-1} \equiv -(V_2 + V_1) \equiv -(k^2+k-2)(mod\,V_{2^r})$$

and

$$x^2 = V_n + V_{n-1} \equiv V_2 + V_1 \equiv -2(mod\,U_2).$$

These show that $\left(\frac{-(k^2+k-2)}{V_{2^r}}\right) = 1$ and $\left(\frac{-2}{k}\right) = 1$. Since $\left(\frac{-2}{k}\right) = 1$, we obtain $k \equiv 1,3 (mod\, 8)$ by (1.5). It follows that $k \equiv 1 (mod\, 8)$ since $k \equiv 1,5,7 (mod\, 8)$. By (2.6), (2.5), and (2.7), we get

$$1 = \left(\frac{-(k^2+k-2)}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{k+2}{V_{2^r}}\right)\left(\frac{k-1}{V_{2^r}}\right) = (-1)\cdot 1\cdot 1 = -1,$$

a contradiction. Thus we get $n \leq 2$. If $n = 2$, then we obtain $x^2 = V_2 + V_1 = k^2 + k - 2$. The case that $k = 2$ is impossible since $k \geq 3$. Thus we get $n = 0$ or 1.

Now, assume that $k \equiv 3 (mod\, 8)$. Let $n = 6q + r$ with $0 \leq r \leq 5$. Then we get

$$V_n = V_{6q+r} \equiv V_r (mod\, U_3)$$

by using (2.3). Since $8 \mid U_3$, it follows that $V_n \equiv V_r (mod\, 8)$ with $0 \leq r \leq 5$ and since $k \equiv 3 (mod\, 8)$, it can be easily seen that $V_n \equiv V_0, V_1, V_2, V_3, V_4, V_5 \equiv 2,3,7 (mod\, 8)$. Then we obtain $x^2 = V_n + V_{n-1} \equiv 5,2,1 (mod\, 8)$. Since $x^2 \equiv 0,1,4 (mod\, 8)$, we get $r = 3$ or 4.

Assume that $r = 3$. Then $n = 6q + 3$ with $q \geq 0$. Let q be odd. Then by (2.4), it follows that

$$x^2 = V_n + V_{n-1} = V_{6q+3} + V_{6q+2} \equiv -(V_3 + V_2) \equiv -V_2 \equiv -(k^2 - 2) (mod\, V_3).$$

Since $V_3 = k(k^2 - 3)$, we obtain

$$x^2 \equiv -(k^2 - 2) \equiv -k^2 + 2 \equiv -1 (mod\, k^2 - 3).$$

This is impossible since $\frac{k^2-3}{2} \equiv -1 (mod\, 4)$.

Let q be even and $q > 0$. Then $n = 6q + 3 = 2 \cdot 2^r a + 3$ with a odd and $r \geq 1$. By (2.4), it follows that

$$\begin{aligned} x^2 = V_n + V_{n-1} &= V_{2\cdot 2^r a+3} + V_{2\cdot 2^r a+2} \\ &\equiv -(V_3 + V_2) \equiv -(k+2)(k^2 - k - 1) (mod\, V_{2^r}). \end{aligned}$$

This shows that $\left(\frac{-(k+2)(k^2-k-1)}{V_{2^r}}\right) = 1$. However, this is impossible since

$$\left(\frac{-(k+2)(k^2-k-1)}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{k+2}{V_{2^r}}\right)\left(\frac{k^2-k-1}{V_{2^r}}\right) = (-1)(-1)(-1) = -1$$

by (2.5), (2.7), and Lemma 2. If $q = 0$, then we get $x^2 = V_3 + V_2 = k^3 + k^2 - 3k - 2$. But the integer points on $Y^2 = X^3 + X^2 - 3X - 2$ are easily determined by using MAGMA [1] to be $(X,Y) = (-2,0), (-1,\mp 1), (3,\mp 5), (2,\mp 2), (43,\mp 285)$. Then it follows that $k = 3$ or 43.

Now, assume that $r = 4$. Then $n = 6q + 4$ with $q \geq 0$. Thus we can write $n = 6(q+1) - 2 = 6t - 2$ and therefore,

$$V_n + V_{n-1} = V_{6t-2} + V_{6t-3} \equiv -(V_3 + V_2) \equiv -V_2 \equiv -(k^2 - 2) (mod\, V_3).$$

Since $V_3 = k(k^2-3)$, we obtain

$$x^2 \equiv -(k^2-2) \equiv -k^2+2 \equiv -1 (mod\, k^2-3).$$

This is impossible since $\frac{k^2-3}{2} \equiv -1 (mod\, 4)$. □

Theorem 12. *Let $k \equiv 3,5,7 (mod\, 8)$. If the equation $x^2 = V_n - V_{n-1}$ has a solution, then $n = 1$ or 2. Moreover, if $k \equiv 1 (mod\, 8)$, then the equation $x^2 = V_n - V_{n-1}$ has no solutions.*

Proof. Let $k \equiv 3,5,7 (mod\, 8)$. Assume that $x^2 = V_n - V_{n-1}$ for some positive integer x. Let $n > 2$. We divide the reminder of the proof into four cases.

Case 1: Assume that $n = 4q-1$ with $q > 0$. Then $n = 2 \cdot 2^r a - 1$ with a odd and $r \geq 1$. By (2.4) and (2.3), it follows that

$$x^2 = V_{4q-1} - V_{4q-2} \equiv -(V_{-1} - V_{-2}) \equiv k^2 - k - 2 (mod\, V_{2^r})$$

and

$$x^2 = V_{4q-1} - V_{4q-2} \equiv V_{-1} - V_{-2} \equiv 2 (mod\, U_2).$$

These show that $\left(\frac{k^2-k-2}{V_{2^r}}\right) = 1$ and $\left(\frac{2}{k}\right) = 1$. Since $\left(\frac{2}{k}\right) = 1$, it follows that $k \equiv 1,7 (mod\, 8)$ by (1.4). Since $k \equiv 3,5,7 (mod\, 8)$, we get $k \equiv 7 (mod\, 8)$. Then by using (1.4) and (2.8), we obtain

$$1 = \left(\frac{k^2-k-2}{V_{2^r}}\right) = \left(\frac{k-2}{V_{2^r}}\right)\left(\frac{k+1}{V_{2^r}}\right) = (-1) \cdot 1 = -1,$$

a contradiction.

Case 2: Assume that $n = 4q$ with $q \geq 1$. Then $n = 2 \cdot 2^r a$ with a odd and $r \geq 1$. By (2.4), we get

$$x^2 = V_{4q} - V_{4q-1} \equiv -(V_0 - V_{-1}) \equiv k - 2 (mod\, V_{2^r}).$$

This shows that $\left(\frac{k-2}{V_{2^r}}\right) = 1$. On the other hand, by using (2.3), we obtain

$$x^2 = V_{4q} - V_{4q-1} \equiv V_0 - V_{-1} \equiv -k+2 \equiv 2 (mod\, U_2),$$

which implies that $\left(\frac{2}{k}\right) = 1$. Then it follows that $k \equiv 1,7 (mod\, 8)$ by (1.4). Since $k \equiv 3,5,7 (mod\, 8)$, we have $k \equiv 7 (mod\, 8)$. Then by (2.8), we get $\left(\frac{k-2}{V_{2^r}}\right) = -1$, which contradicts the fact that $\left(\frac{k-2}{V_{2^r}}\right) = 1$.

Case 3: Assume that $n = 4q+1$ with $q > 0$. Then $n = 2 \cdot 2^r a + 1$ with a odd and $r \geq 1$. By (2.4) and (2.3), it follows that

$$x^2 = V_{4q+1} - V_{4q} \equiv -(V_1 - V_0) \equiv -(k-2) (mod\, V_{2^r})$$

and

$$x^2 = V_{4q+1} - V_{4q} \equiv V_1 - V_0 \equiv -2 (mod\, U_2).$$

These show that $\left(\frac{-(k-2)}{V_{2^r}}\right)=1$ and $\left(\frac{-2}{k}\right)=1$. Since $\left(\frac{-2}{k}\right)=1$, it follows that $k\equiv 1,3(mod\,8)$ by (1.5). Since $k\equiv 3,5,7(mod\,8)$, we get $k\equiv 3(mod\,8)$. Then by (2.5) and (2.8), we obtain

$$\left(\frac{-(k-2)}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{k-2}{V_{2^r}}\right)=-1,$$

which contradicts the fact that $\left(\frac{-(k-2)}{V_{2^r}}\right)=1$.

Case 4: Assume that $n=4q+2$ with $q>0$. Then $n=2\cdot 2^r a+2$ with a odd and $r\geq 1$. By (2.4) and (2.3), it follows that

$$x^2=V_{4q+2}-V_{4q+1}\equiv -(V_2-V_1)\equiv -(k^2-k-2)(mod\,V_{2^r})$$

and

$$x^2=V_{4q+2}-V_{4q+1}\equiv V_2-V_1\equiv -2(mod\,U_2).$$

The above congruences give $\left(\frac{-(k^2-k-2)}{V_{2^r}}\right)=1$ and $\left(\frac{-2}{k}\right)=1$. Since $\left(\frac{-2}{k}\right)=1$, it follows that $k\equiv 1,3(mod\,8)$ by (1.5). Since $k\equiv 3,5,7(mod\,8)$, we get $k\equiv 3(mod\,8)$. Then by using (1.4), (2.5), and (2.8), we obtain $1=\left(\frac{-(k^2-k-2)}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{k-2}{V_{2^r}}\right)\left(\frac{k+1}{V_{2^r}}\right)=-1$, a contradiction. Therefore, $n\leq 2$. If $n=0$, then we have $x^2=V_0-V_{-1}=2-k$, which is impossible. So, we get $n=1$ or 2.

Now, assume that $k\equiv 1(mod\,8)$. Let $n=6q+r$ with $0\leq r\leq 5$. Then by using (2.3), we get

$$V_n=V_{6q+r}\equiv V_r(mod\,U_3).$$

Since $8\mid U_3$, it follows that $V_n\equiv V_r(mod\,8)$ with $0\leq r\leq 5$. Moreover, since $k\equiv 1(mod\,8)$, it can be easily seen that $V_n\equiv V_0,V_1,V_2,V_3,V_4,V_5\equiv 2,1,7,6(mod\,8)$. Then we get $x^2=V_n-V_{n-1}\equiv 1,6,7(mod\,8)$. It can be seen that $r=0$ or 4.

Assume that $r=0$. Then $n=6q$ with $q\geq 0$. Let q be odd. Then it follows that

$$x^2=V_n-V_{n-1}=V_{6q}-V_{6q-1}\equiv -(V_0-V_{-1})\equiv k-2(mod\,V_3)$$

by (2.4). Since $V_3=k(k^2-3)$, we obtain

$$x^2\equiv k-2(mod\,k^2-3),$$

which implies that $x^2\equiv k-2\left(mod\,\frac{k^2-3}{2}\right)$. Therefore, $\left(\frac{k-2}{(k^2-3)/2}\right)=1$. Since $k\equiv 1(mod\,8)$, we get $\frac{k^2-3}{2}\equiv 3(mod\,4)$. These show that

$$\begin{aligned}1&=\left(\frac{k-2}{(k^2-3)/2}\right)=(-1)^{\left(\frac{k-3}{2}\right)\left(\frac{k^2-5}{4}\right)}\left(\frac{(k^2-3)/2}{k-2}\right)\\&=(-1)\left(\frac{2}{k-2}\right)\left(\frac{k^2-3}{k-2}\right)\end{aligned}$$

$$= (-1)\cdot 1\cdot\left(\frac{1}{k-2}\right) = -1,$$

a contradiction. Let q be even. Then $n = 6q = 2\cdot 2^r a$ with a odd and $r \geq 1$. By (2.4), it follows that

$$x^2 = V_n - V_{n-1} = V_{2\cdot 2^r a} - V_{2\cdot 2^r a-1} \equiv -(V_0 - V_{-1}) \equiv k-2 (mod\, V_{2^r}).$$

This shows that $\left(\frac{k-2}{V_{2^r}}\right) = 1$, which is impossible by (2.8).

Now, assume that $r = 4$. Then $n = 6q+4$ with $q \geq 0$. We can write $n = 6(q+1)-2 = 6t-2$. Let t be odd. Then it follows that

$$x^2 = V_n - V_{n-1} = V_{6t-2} - V_{6t-3} \equiv -(V_{-2} - V_{-3}) \equiv -V_2 \equiv -(k^2-2)(mod\, V_3)$$

by (2.4). Since $V_3 = k(k^2-3)$, we obtain

$$x^2 \equiv -(k^2-2) \equiv -1 (mod\, k^2-3),$$

which shows that $x^2 \equiv -1 \left(mod\, \frac{k^2-3}{2}\right)$. But this is impossible since $\frac{k^2-3}{2} \equiv 3 (mod\, 4)$. Let t be even. Then $n = 6t-2 = 2\cdot 2^r a - 2$ with a odd and $r \geq 1$. It follows that

$$\begin{aligned} x^2 &= V_n - V_{n-1} = V_{2\cdot 2^r a-2} - V_{2\cdot 2^r a-3} \\ &\equiv -(V_{-2} - V_{-3}) \equiv (k-2)(k^2+k-1)(mod\, V_{2^r}) \end{aligned}$$

by (2.4). This shows that $\left(\frac{(k-2)(k^2+k-1)}{V_{2^r}}\right) = 1$. Then by (2.8) and Lemma 4, we obtain

$$\left(\frac{k-2}{V_{2^r}}\right)\left(\frac{k^2+k-1}{V_{2^r}}\right) = -1,$$

which contradicts the fact that $\left(\frac{(k-2)(k^2+k-1)}{V_{2^r}}\right) = 1$. This completes the proof. □

Theorem 13. *If the equation $x^2 = U_{n+1} + U_n$ has a solution, then $n = 0$ or 1.*

Proof. Assume that $x^2 = U_{n+1} + U_n$ for some positive integer x. Now, assume that $n > 2$. We distinquish four cases.

Case 1: Assume that $n = 4q-1$ with $q > 0$. Then by (2.1), we get

$$x^2 = U_{4q} + U_{4q-1} \equiv U_0 + U_{-1} \equiv -1 (mod\, U_2).$$

This shows that $\left(\frac{-1}{k}\right) = 1$. Then it follows that $k \equiv 1, 5 (mod\, 8)$ by (1.3).

Let $q = 3u-1$ with $u > 0$. Then $n = 12u-5$. By using (2.1), we obtain

$$x^2 = U_{12u-4} + U_{12u-5} \equiv U_{-4} + U_{-5} \equiv k-3 (mod\, U_3).$$

Since $8 \mid U_3$, we get $x^2 \equiv k-3 (mod\, 8)$. Moreover, since $k \equiv 1, 5 (mod\, 8)$, it follows that $x^2 \equiv -2, 2 (mod\, 8)$, which is impossible.

Let $q = 3u$ with $u > 0$. Then $n = 12u-1$. Here, we get

$$x^2 = U_{12u-1} + U_{12u} \equiv U_{-1} + U_0 \equiv -1 (mod\, U_3)$$

by (2.1). Since $8 \mid U_3$, it follows that $x^2 \equiv -1 (mod\, 8)$, which is impossible.

Let $q = 3u+1$ with $u \geq 0$. Then $n = 12u+3$. Here, we obtain

$$x^2 = U_{12u+4} + U_{12u+3} \equiv U_4 + U_3 \equiv U_4 \equiv -k (mod\, U_3)$$

by (2.1). Since $8 \mid U_3$, it follows that $x^2 \equiv -k (mod\, 8)$. Moreover, since $k \equiv 1,5 (mod\, 8)$, we obtain $x^2 \equiv -1, -5 (mod\, 8)$. However, this is impossible.

Case 2: Assume that $n = 4q$ with $q \geq 1$. Then $n = 2 \cdot 2^r a$ with a odd and $r \geq 1$. By (2.2), we get

$$x^2 = U_{4q+1} + U_{4q} \equiv -(U_1 + U_0) \equiv -1 (mod\, V_{2^r}),$$

which is impossible by (2.5).

Case 3: Assume that $n = 4q+1$ with $q > 0$. Then $n = 2 \cdot 2^r a + 1$ with a odd and $r \geq 1$. By (2.2), we get

$$x^2 = U_{4q+2} + U_{4q+1} \equiv -(U_2 + U_1) \equiv -(k+1) (mod\, V_{2^r}).$$

This shows that $\left(\frac{-(k+1)}{V_{2^r}}\right) = 1$. Since $\left(\frac{-1}{V_{2^r}}\right) = -1$ and $\left(\frac{-(k+1)}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{k+1}{V_{2^r}}\right)$, it follows that $\left(\frac{k+1}{V_{2^r}}\right) = -1$. This is impossible by (1.5).

Case 4: Assume that $n = 4q+2$. Then $n = 2 \cdot 2^r a + 2$ with a odd and $r \geq 1$. By (2.1), we obtain

$$x^2 = U_{4q+3} + U_{4q+2} \equiv U_3 + U_2 \equiv -1 (mod\, U_2),$$

i.e., $\left(\frac{-1}{k}\right) = 1$. It follows that $k \equiv 1,5 (mod\, 8)$ by (1.3). On the other hand, by (2.2), we get

$$x^2 = U_{4q+3} + U_{4q+2} \equiv -(U_3 + U_2) \equiv -(k^2+k-1) (mod\, V_{2^r}).$$

This shows that $\left(\frac{-(k^2+k-1)}{V_{2^r}}\right) = 1$. Since $\left(\frac{-(k^2+k-1)}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{k^2+k-1}{V_{2^r}}\right) = 1$ and $\left(\frac{-1}{V_{2^r}}\right) = -1$, we have $\left(\frac{k^2+k-1}{V_{2^r}}\right) = -1$. This is impossible by Lemma 4. Therefore, $n \leq 2$. If $n = 2$, then $x^2 = U_3 + U_2 = k^2 + k - 1$, which shows that $k = 1$. This is impossible since $k \geq 3$. Thus we get $n = 0$ or 1. □

Since the proof of the following theorem is similar to that of above theorem, we omit its proof.

Theorem 14. *If the equation $x^2 = U_{n+1} - U_n$ has a solution, then $n = 0$ or 1.*

4. Main Theorems

Now, we consider positive integer solutions of some fourth-order Diophantine equations.

Theorem 15. *Let $k+2$ be not a perfect square. If $k+1$ is a perfect square, then all positive integer solutions of the equation $x^4-kx^2y+y^2=k+2$ are given by $(x,y)=(\sqrt{k+1},1)$, $(\sqrt{k+1},k^2+k-1)$, $(1,k+1)$. If $k+1$ is not a perfect square, then the only positive integer solution of this equation is $(x,y)=(1,k+1)$.*

Proof. Assume that $x^4-kx^2y+y^2=k+2$ for some positive integers x and y. Then by Theorem 4, we obtain $x^2=U_{n+1}+U_n$, $y=U_n+U_{n-1}$ or $x^2=U_n+U_{n-1}$, $y=U_{n+1}+U_n$ with $n>0$. Let $x^2=U_{n+1}+U_n$. Then it follows that $n=0$ or 1 by Theorem 13. Since $n>0$, we have $n=1$. Then we have $x^2=U_{n+1}+U_n=U_2+U_1=k+1$ and $y=U_1+U_0=1$. If $k+1$ is a perfect square, then it follows that $x=\sqrt{k+1}$ and $y=1$. Now, let $x^2=U_n+U_{n-1}$. Then we obtain $n-1=0$ or 1 by Theorem 13. Assume that $n=1$. Then $x^2=U_1+U_0=1$ and $y=U_2+U_1=k+1$. Thus we have $x=1$ and $y=k+1$. Now, assume that $n=2$. Then we get $x^2=U_2+U_1=k+1$ and $y=U_3+U_2=k^2+k-1$. If $k+1$ is a perfect square, then it follows that $x=\sqrt{k+1}$ and $y=k^2+k-1$. Thus if $k+1$ is not a perfect square, then this equation has no positive integer solutions. Then the proof follows. □

Theorem 16. *If $k-1$ is a perfect square, then all positive integer solutions of the equation $x^4-kx^2y+y^2=-(k-2)$ are given by $(x,y)=(1,1)$, $(\sqrt{k-1},1)$, $(\sqrt{k-1},k^2-k-1)$, $(1,k-1)$. Moreover, if $k-1$ is not a perfect square, then all positive integer solutions of this equation are given by $(x,y)=(1,1)$ or $(1,k-1)$.*

Proof. Assume that $x^4-kx^2y+y^2=-(k-2)$ for some positive integers x and y. Then by Theorem 5, we obtain $x^2=U_{n+1}-U_n$, $y=U_n-U_{n-1}$ or $x^2=U_n-U_{n-1}$, $y=U_{n+1}-U_n$ with $n\geq 0$. Let $x^2=U_{n+1}-U_n$. Then it follows that $n=0$ or 1 by Theorem 14. Assume that $n=0$. Then we have $x^2=U_{n+1}-U_n=U_1-U_0=1$ and $y=U_0-U_{-1}=1$. Thus we have $x=1$ and $y=1$. Now, assume that $n=1$. Then we get $x^2=U_2-U_1=k-1$ and $y=U_1-U_0=1$. If $k-1$ is a perfect square, then it follows that $x=\sqrt{k-1}$ and $y=1$. Now, let $x^2=U_n-U_{n-1}$. Then we obtain $n=1$ or 2 by Theorem 14. Assume that $n=1$. Then $x^2=U_1-U_0=1$ and $y=U_2-U_1=k-1$. Thus we have $x=1$ and $y=k-1$. Now, assume that $n=2$. Then we get $x^2=U_2-U_1=k-1$ and $y=U_3-U_2=k^2-k-1$. If $k-1$ is a perfect square, then it follows that $x=\sqrt{k-1}$ and $y=k^2-k-1$. Thus if $k-1$ is not a perfect square, then this equation has no positive integer solutions other than $(1,1)$ and $(1,k-1)$. □

Theorem 17. *If $k\equiv 1,5,7(mod\,8)$, then the equation $x^4-(k^2-4)y^2=4(k+2)$ has a solution if and only if $k+2$ is a perfect square. Moreover, if $k\equiv 1,5,7(mod\,8)$ and $k+2$ is a perfect square, then the only positive integer solution of this equation is $(x,y)=(\sqrt{k+2},1)$. If $k\equiv 3(mod\,8)$, then the equation $x^4-(k^2-4)y^2=4(k+2)$ has positive integer solutions only when $k=3$ or 43.*

Proof. Assume that $x^4-(k^2-4)y^2=4(k+2)$ for some positive integers x and y. Firstly, let $k\equiv 1,5,7(mod\,8)$ and $k+2$ be not a perfect square. Then by Theorem 6, it follows that $x^2=V_n+V_{n-1}$ with $n\geq 1$. Therefore, by Theorem 11, it is seen that $n=0$ or 1. Since $n\geq 1$, we get $n=1$. Then it follows that $x^2=k+2$. But this is impossible since $k+2$ is not a perfect square. Let $k\equiv 1,5,7(mod\,8)$ and $k+2$ be a perfect square. Then by Theorem 7, it follows that $x^2=\sqrt{k+2}V_{n-2}$. Therefore, we obtain $V_{n-2}(\sqrt{k+2},-1)=\square,\sqrt{k+2}\square$ or $a\square$ with $a\mid\sqrt{k+2}$. If $V_{n-2}(\sqrt{k+2},-1)=\sqrt{k+2}\square$ or $a\square$ with $a\mid\sqrt{k+2}$, then it follows that $n=3$ by Theorem 2 and Theorem 1. Then it can be seen that $x=\sqrt{k+2}$ and $y=1$. If $V_{n-2}(\sqrt{k+2},-1)=\square$, we have $n=3$ and $k+2$ is a perfect square by Lemma 1. Thus if $k+2$ is a perfect square, then we obtain $x=\sqrt{k+2}$ and $y=1$. Now, let $k\equiv 3(mod\,8)$. Then $k+2$ is not a perfect square and it follows that $k=3$ or 43 by Theorem 6 and Theorem 11. When $k=3$, we get the solution $(x,y)=(5,11)$. When $k=43$, we get the solution $(x,y)=(285,1891)$. $\square$

Theorem 18. *If $k+1$ is not a perfect square, then the only positive integer solution of the equation $x^2-(k^2-4)y^4=4(k+2)$ is $(x,y)=(k+2,1)$. Moreover, if $k+1$ is a perfect square, then all positive integer solutions of this equation are given by $(x,y)=(k^2+k-2,\sqrt{k+1})$ or $(k+2,1)$.*

Proof. Assume that $x^2-(k^2-4)y^4=4(k+2)$ for some positive integers x and y. Then by Theorem 6, it follows that $x=V_n+V_{n-1}$, $y^2=U_n+U_{n-1}$ with $n>0$. Therefore, by Theorem 13, we have $n=1$ or 2. Let $n=1$. Then it follows that $x=V_1+V_0=k+2$ and $y^2=U_1+U_0=1$. Therefore, we obtain $(x,y)=(k+2,1)$. Let $n=2$. Then it is seen that $x=V_2+V_1=k^2-2+k$ and $y^2=U_2+U_1=k+1$. If $k+1$ is a perfect square, then we obtain $x=k^2+k-2$ and $y=\sqrt{k+1}$. $\square$

Theorem 19. *Let $k\equiv 3,5,7(mod\,8)$. If $k=3$, then all positive integer solutions of the equation $x^4-(k^2-4)y^2=-4(k-2)$ are given by $(x,y)=(2,2)$, $(1,1)$. If $k\neq 3$ and $k-2$ is a perfect square, then the only positive integer solution of the equation $x^4-(k^2-4)y^2=-4(k-2)$ is $(x,y)=(\sqrt{k-2},1)$. If $k\neq 3$ and $k-2$ is not a perfect square, then this equation has no positive integer solutions.*

Proof. Assume that $x^4-(k^2-4)y^2=-4(k-2)$ for some positive integers x and y. Then by Theorem 8, we obtain $x^2=V_n-V_{n-1}$ and $y=U_n-U_{n-1}$ with $n>0$. Since $k\equiv 3,5,7(mod\,8)$ and $x^2=V_n-V_{n-1}$, it follows that $n=1$ or 2 by Theorem 12. Let $n=1$. Then we have $x^2=V_1-V_0=k-2$ and $y=U_1-U_0=1$. If $k-2$ is a perfect square, then we obtain $(x,y)=(\sqrt{k-2},1)$. Otherwise this equation has no solutions. Let $n=2$. Then it follows that $x^2=V_2-V_1=k^2-2-k$ and $y=U_2-U_1=k-1$. Since $x^2=k^2-k-2$, we get $4x^2=4k^2-4k-8$, i.e., $(2k-1)^2-(2x)^2=9$. This shows that $k=3$. Hence, we get $x=2$ and $y=2$. $\square$

Theorem 20. *Let $k\equiv 1(mod\,8)$. Then the equation $x^4-(k^2-4)y^2=-4(k-2)$ has no integer solutions.*

Proof. Assume that $x^4-(k^2-4)y^2=-4(k-2)$ for some positive integers x and y. Then by Theorem 8, we obtain $x^2=V_n-V_{n-1}$ and $y=U_n-U_{n-1}$. This is impossible by Theorem 12. □

Theorem 21. *If $k-1$ is not a perfect square, then the only positive integer solution of the equation $x^2-(k^2-4)y^4=-4(k-2)$ is $(x,y)=(k-2,1)$. Moreover, if $k-1$ is a perfect square, then all positive integer solutions of the equation $x^2-(k^2-4)y^4=-4(k-2)$ are given by $(x,y)=(k^2-2-k,\sqrt{k-1})$, $(k-2,1)$.*

Proof. Assume that $x^2-(k^2-4)y^4=-4(k-2)$ for some positive integers x and y. Then by Theorem 8, we obtain $x=V_n-V_{n-1}$ and $y^2=U_n-U_{n-1}$ with $n>0$. Therefore, by Theorem 14, it follows that $n=1$ or 2. Let $n=1$. Then we have $x=V_1-V_0=k-2$ and $y^2=U_1-U_0=1$. Therefore, $(x,y)=(k-2,1)$. Now, let $n=2$. Then it follows that $x=V_2-V_1=k^2-2-k$ and $y^2=U_2-U_1=k-1$. Therefore, if $k-1$ is a perfect square, then it follows that $(x,y)=(k^2-k-2,\sqrt{k-1})$ is a solution. □

Theorem 22. *Let $k\equiv 1,5,7(mod\,8)$ and k^2-4 be square-free. Then the equation $x^4-kx^2y+y^2=-(k+2)(k^2-4)$ has no positive integer solutions.*

Proof. Assume that $x^4-kx^2y+y^2=-(k+2)(k^2-4)$ for some positive integers x and y. Then by Theorem 9, we obtain $x^2=V_{n+1}+V_n, y=V_n+V_{n-1}$ or $x^2=V_n+V_{n-1}, y=V_{n+1}+V_n$ with $n\geq 0$. Let $x^2=V_{n+1}+V_n$. Since $k\equiv 1,5,7(mod\,8)$, we obtain $n+1=0$ or 1 by Theorem 11. Since $n\geq 0$, we get $n=0$. Then we have $x^2=V_1+V_0=k+2$ and $y=V_0+V_{-1}=k+2$. This is impossible since k^2-4 is square-free. Let $x^2=V_n+V_{n-1}$. Then by Theorem 11, we obtain $n=0$ or 1. Thus it follows that $x^2=V_0+V_{-1}=V_1+V_0=k+2$. This is impossible since k^2-4 is square-free. □

From Theorem 9 and Theorem 11, we can give the following theorem.

Theorem 23. *Let $k\equiv 3(mod\,8)$ and k^2-4 be square-free. Then the equation $x^4-kx^2y+y^2=-(k+2)(k^2-4)$ has no positive integer solutions.*

Theorem 24. *Let $k\equiv 3,5,7(mod\,8)$ and k^2-4 be square-free. Then the equation $x^4-kx^2y+y^2=(k-2)(k^2-4)$ has positive integer solutions only when $k=3$ and in which case all positive integer solutions are given by $(x,y)=(2,1)$, $(1,4)$ or $(2,11)$.*

Proof. Assume that $x^4-kx^2y+y^2=(k-2)(k^2-4)$ for some positive integers x and y. Then by Theorem 10, we obtain $x^2=V_{n+1}-V_n$, $y=V_n-V_{n-1}$ or $x^2=V_n-V_{n-1}$, $y=V_{n+1}-V_n$ with $n\geq 1$. Assume that $x^2=V_{n+1}-V_n$. Then by Theorem 12, we obtain $n+1=1$ or 2. Since $n\geq 1$, we get $n=1$. Then it follows that $x^2=V_2-V_1=k^2-k-2$ and $y=V_1-V_0=k-2$. Since $x^2=k^2-k-2$, we get $4x^2=4k^2-4k-8$, i.e., $(2k-1)^2-(2x)^2=9$. Thus it follows that $k=3$. Then it can be seen that $(x,y)=(2,1)$. Now, assume that $x^2=V_n-V_{n-1}$. Then by Theorem

12, it follows that $n = 1$ or 2. Let $n = 1$. Then we get $x^2 = V_1 - V_0 = k - 2$ and $y = V_2 - V_1 = k^2 - 2 - k$. Thus $k - 2$ must be a perfect square. But this is impossible in the case that $k > 3$. Let $k = 3$. Then we obtain $x = 1$ and $y = 4$. Let $n = 2$. Then we obtain $x^2 = V_2 - V_1 = k^2 - k - 2$. It can be shown that $k = 3$. Then we have $(x, y) = (2, 11)$. This completes the proof. □

From Theorem 10 and Theorem 12, we can give the following theorem.

Theorem 25. *Let $k \equiv 1 (mod 8)$ and $k^2 - 4$ be square-free. Then the equation $x^4 - kx^2y + y^2 = (k-2)(k^2-4)$ has no positive integer solutions.*

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Authors' addresses

Refik Keskin
Sakarya University, Mathematics Department, TR 54187 Sakarya, Turkey
E-mail address: `rkeskin@sakarya.edu.tr`

Merve Güney Duman
Sakarya University, Mathematics Department, TR 54187 Sakarya, Turkey
E-mail address: `merveguneyduman@gmail.com`