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ON A GENERALIZATION OF NC-MCCOY RINGS

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Abstract. In the present paper we concentrate on a natural generalization of NC-McCoy rings that is called J-McCoy and investigate their properties. We prove that local rings are J-McCoy. For a ring R, $R[[x]]$ is J-McCoy if and only if R is J-McCoy. Also, for an abelian ring R, we show that R is J-McCoy if and only if eR is J-McCoy, where e is an idempotent element of R. Moreover, we give an example to show that the J-McCoy property does not pass $M_n(R)$, but $S(R,n), A(R,n), B(R,n)$ and $T(R,n)$ are J-McCoy.

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1. INTRODUCTION

Throughout this paper, R denotes an associative ring with identity. For a ring R, $N(R)$, $M_n(R)$ and e_{ij} denote the set of all nilpotent elements in R, the $n \times n$ matrix ring over R, and the matrix with (i,j)-entry 1 and elsewhere 0, respectively. Rege -Chhawchharia [8] called a noncommutative ring R right McCoy if whenever polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x] \setminus \{0\}$ satisfy $f(x)g(x) = 0$, there exists a nonzero element $r \in R$ such that $a_i r = 0$. Left McCoy rings are defined similarly. A number of papers have been written on McCoy property of rings (see, e.g., [1, 4, 6, 7, 9]). The name "McCoy" was chosen because McCoy [6] had noted that every commutative ring satisfies this condition. Victor Camillo, Tai Keun Kwak, and Yang Lee [2] called a ring R right nilpotent coefficient McCoy(simply, right NC-McCoy) if whenever polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x] \setminus \{0\}$ satisfy $f(x)g(x) = 0$, there exists a nonzero element $r \in R$ such that $f(x)r \in N(R)[x]$. Left NC-McCoy rings are defined analogously, and a ring R is called NC-McCoy if it is both left and right NC-McCoy. They proved for a reduced

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ring R and $n \geq 2$, $M_n(R)$ is neither right nor left NC-McCoy, but $T_n(R)$ is a NC-McCOy ring for $n \geq 2$. Moreover, it is shown that R is right NC-McCoy if the polynomial ring $R[x]$ is right NC-McCoy and the converse holds if $N(R)[x] \subset N(R[x])$.

Motivated by the above results, we investigate a generalization of the right NC-McCoy rings. The Jacobson radical is an important tool for studying the structure of noncommutative rings, and denoted by $J(R)$. A ring R is said to be right J-McCoy (respectively left J-McCoy) if for each pair of nonzero polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x] \setminus \{0\}$ with $f(x)g(x) = 0$, then there exists a nonzero element $r \in R$ such that $a_i r \in J(R)$ (respectively $rb_j \in J(R)$). A ring R is called J-McCoy if it is both left and right J-McCoy. It is clear that NC-McCoy rings are J-McCoy, but the converse is not always true. If R is J-semisimple (namely, $J(R) = 0$), then R is right J-McCoy if and only if R is right McCoy. Moreover, for Artinian rings, the concepts of NC-McCoy and J-McCoy rings are the same.

2. Results

Definition 1. A ring R is said to be right J-McCoy (respectively left J-McCoy) if for each pair of nonzero polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x] \setminus \{0\}$, $f(x)g(x) = 0$ implies that there exists a nonzero element $r \in R$ with $a_i r \in J(R)$ (respectively $rb_j \in J(R)$). A ring R is called J-McCoy if it is both left and right J-McCoy.

It is clear that NC-McCoy rings are J-McCoy, but the converse is not always true by the following example.

Example 1. Let A be the 3 by 3 full matrix ring over the power series ring $F[[t]]$ over a field F. Let

$$B = \{M = (m_{ij}) \in A \mid m_{ij} \in tF[[t]] \text{ for } 1 \leq i, j \leq 2 \text{ and } m_{ij} = 0 \text{ for } i = 3 \text{ or } j = 3\}$$

and

$$C = \{M = (m_{ij}) \in A \mid m_{ij} \in F \text{ and } m_{ij} = 0 \text{ for } i \neq j\}.$$

Let R be the subring of A generated by B and C. Let $F = \mathbb{Z}_2$. Note that every element of R is of the form $(a + f_1)e_{11} + f_2 e_{12} + f_3 e_{21} + (a + f_4)e_{22} + ae_{33}$ for some $a \in F$ and $f_i \in tF[[t]]$ $(i = 1, 2, 3, 4)$. Consider two polynomials over R, $f(x) = te_{11} + te_{12}x + te_{21}x^2 + te_{22}x^3$ and $g(x) = -t(e_{21} + e_{22}) + t(e_{11} + e_{12})x \in R[x]$. Then $f(x)g(x) = 0$. If there exists $0 \neq r \in R$ such that $f(x)r \in N(R[x])$, then $r = 0$. Thus R is not right NC-McCoy.

Next we will show that R is right J-McCoy. Let $f(x) = \sum_{i=0}^{n} M_i x^i$ and $g(x) = \sum_{j=0}^{m} N_j x^j$ be nonzero polynomials in $R[x]$ such that $f(x)g(x) = 0$. Since $M_i = (a_i + f_{i1})e_{11} + f_{i2}e_{12} + f_{i3}e_{21} + (a_i + f_{i4})e_{22} + a_i e_{33}$ for some $a_i \in F$ and $f_{ij} \in tF[[t]]$ $(j = 0, 1, 2, 3, 4)$, then for $C = te_{11}$ we have $M_i C = (a_i + f_{i1})te_{11} + f_{i3}te_{21} \in J(R)$. Thus R is right J-McCoy ring.

Proposition 1. *Let R be a ring and I an ideal of R such that R/I is a right (resp. left) J-McCoy ring. If $I \subseteq J(R)$, then R is a right (resp. left) J-McCoy ring.*

Proof. Suppose that $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x] \setminus \{0\}$ such that $f(x)g(x) = 0$. Then $(\sum_{i=0}^{m} \bar{a_i} x^i)(\sum_{j=0}^{n} \bar{b_j} x^j) = \bar{0}$ in R/I . Thus there exists $\bar{c} \in R/I$ such that $\bar{a_i}\bar{c} \in J(R/I)$ and so $a_i c \in J(R)$. This means R is right J-McCoy ring. □

Corollary 1. *Let R be any local ring. Then R is J-McCoy.*

The following example shows that, if R is a right J-McCoy ring, then $R/J(R)$ is not necessary right J-McCoy.

Example 2. Let R denote the localization of the ring $\mathbb{Z}$ of integers at the prime ideal (3). Consider the quaternions $\mathbb{Q}$ over R, that is, a free R-module with basis $1, i, j, k$ and multiplication satisfying $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$. Then $\mathbb{Q}$ is a noncommutative domain, and so $J(\mathbb{Q}) = 3\mathbb{Q}$ and $\mathbb{Q}/J(\mathbb{Q})$ is isomorphic to the 2-by-2 full matrix ring over $\mathbb{Z}/(3)$. Thus $\mathbb{Q}$ is a right J-McCoy ring, but $\mathbb{Q}/J(\mathbb{Q})$ is not right J-McCoy.

Proposition 2. *Let R_k be a ring, where $k \in I$. Then R_k is right (resp. left) J-McCoy for each $k \in I$ if and only if $R = \prod_{k\in I} R_k$ is right (resp. left) J-McCoy.*

Proof. Let each R_k be a right J-McCoy ring and $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x] \setminus \{0\}$ such that $f(x)g(x) = 0$, where $a_i = (a_i^{(k)})$, $b_j = (b_j^{(k)})$. If there exists $t \in I$ such that $a_i^{(t)} = 0$ for each $0 \le i \le m$, then we have $a_i c = 0 \in J(R)$ where $c = (0, 0, \ldots, 1_{R_t}, 0, \ldots, 0)$. Now suppose for each $k \in I$, there exists $0 \le i_k \le m$ such that $a_{i_k}^{(k)} \ne 0$. Since $g(x) \ne 0$, there exists $t \in I$ and $0 \le j_t \le n$ such that $b_{j_t}^{(t)} \ne 0$. Consider $f_t(x) = \sum_{i=0}^{m} a_i^{(t)} x^i$ and $g_t(x) = \sum_{i=0}^{n} b_j^{(t)} x^j \in R_t[x] \setminus \{0\}$. We have $f_t(x) g_t(x) = 0$. Thus there exists nonzero $c_t \in R_t$ such that $a_i^{(t)} c_t \in J(R_t)$, for each $0 \le i \le m$, since R_t is right J-McCoy ring. Therefore, $a_i(0, 0, \ldots, c_t, 0, \ldots, 0) \in \prod_{k\in I} J(R_k) = J(R)$, for each $0 \le i \le m$. Thus, R is right J-McCoy.
Conversely, suppose R is right J-McCoy and $t \in I$. Let $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j$ be nonzero polynomials in $R_t[x]$ such that $f(x)g(x) = 0$. Set

$$F(x) = \sum_{i=0}^{m} (0, 0, \ldots, 0, a_i, 0, \ldots, 0) x^i \ ,$$

$$G(x) = \sum_{j=0}^{n} (0, 0, \ldots, 0, b_j, 0, \ldots, 0) x^j \in R[x] \setminus \{0\}.$$

Hence $F(x)G(x) = 0$ and so there exists $0 \ne c = (c_i)$ such that $(0, 0, \ldots, 0, a_i, 0, \ldots, 0)c \in J(R) = \prod_{k\in I} J(R_k)$. Therefore, $a_i c_t \in J(R_t)$ and so R_t is right J-McCoy □

Corollary 2. *Let D be a ring and C a subring of D with $1_D \in C$. Let*

$$R(C,D) = \{(d_1,\dots,d_n,c,c,\dots) \mid d_i \in D, c \in C, n \geq 1\}$$

with addition and multiplication defined component-wise, $R(D,C)$ is a ring. Then D is right (resp. left) J-McCoy if and only if $R(D,C)$ is right (resp. left) J-McCoy.

Theorem 1. *The class of right (resp. left) J-McCoy rings is closed under direct limits with injective maps.*

Proof. Let $D = \{R_i, \alpha_{ij}\}$ be direct system of right J-McCoy rings R_i, for $i \in I$ and ring homomorphisms $\alpha_{ij} : R_i \longrightarrow R_j$ for each $i \leq j$ satisfying $\alpha_{ij}(1) = 1$, where I is a directed partially ordered set. Set $R = \varinjlim R_i$ be a direct limit of D with $L_i : R_i \longrightarrow R$ and $L_j\alpha_{ij} = L_i$ where every L_i is injective. We will show that R is an right J-McCoy ring. Take $a, b \in R$. Then $a = L_i(a_i), b = L_j(b_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \leq k, j \leq k$. Define

$$a + b = L_k(\alpha_{ik}(a_i) + \alpha_{jk}(b_j)) \text{ and } ab = L_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j))$$

where $\alpha_{ik}(a_i)$ and $\alpha_{jk}(b_j)$ are in R_k. Then R forms a ring with $0 = L_i(0)$ and $1 = L_i(1)$. Now let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$ be nonzero polynomials such that $f(x)g(x) = 0$. There is $k \in I$ such that $f(x), g(x) \in R_k[x]$. Hence we get $f(x)g(x) = 0$ in $R_k[x]$. Since R_k is right J-McCoy, there exist $0 \neq c_k$ in R_k such that $a_i c_k \in J(R_k)$. Put $c = L_k(c_k)$. Then $a_i c \in \varinjlim J(R_k) = J(R)$ with a nonzero c in R. Thus R is right J-McCoy ring. □

Theorem 2. *For a ring R, $R[[x]]$ is right (resp. left) J-McCoy if and only if R is right (resp. left) J-McCoy.*

Proof. Let R be a right J-McCoy ring. Since $R \cong \frac{R[[x]]}{<x>}$ and $<x> \subset J(R[[x]])$, then by Proposition 1, $R[[x]]$ is right J-McCoy. Conversely, assume that $R[[x]]$ is right J-McCoy. Let $f(y) = \sum_{i=0}^{n} a_i y^i$ and $g(y) = \sum_{j=0}^{m} b_j y^j$ be nonzero plynomials $\in R[y]$, such that $f(y)g(y) = 0$. Since $R[[x]]$ is right J-McCoy and $R \subseteq R[[x]]$, then there exists $0 \neq c(x) = c_0 + c_1 x + c_2 x + \dots \in R[[x]]$ such that $a_i c(x) \in J(R[[x]])$ and so $a_i c_i \in J(R[[x]]) \cap R \subseteq J(R)$ for all $i = 0, 1, \cdots, n$. Since $c(x)$ is nonzero, there exists $c_l \neq 0$ such that $a_i c_l \in J(R)$ for $i = 0, 1, \cdots, n$ and so R is J-McCoy. □

Theorem 3. *For a ring R, if $R[x]$ is right (resp. left) J-McCoy, then R is right (resp. left) J-McCoy . The converse holds if $J(R)[x] \subseteq J(R[x])$.*

Proof. Suppose that $R[x]$ is right J-McCoy. Let $f(y) = \sum_{i=0}^{n} a_i y^i$ and $g(y) = \sum_{j=0}^{m} b_j y^j$ be nonzero plynomials in $R[y]$, such that $f(y)g(y) = 0$. Since $R[x]$ is right J-McCoy and $R \subseteq R[x]$, then there exists $0 \neq c(x) = c_0 + c_1 x + \dots + c_k x^k \in R[x]$ such that $a_i c(x) \in J(R[x])$ and so $a_i c_i \in J(R[x]) \cap R \subseteq J(R)$ for all $i = 0, 1, \cdots, n$. Since $c(x)$ is nonzero, there exists $c_l \neq 0$ such that $a_i c_l \in J(R)$ for

$i = 0, 1, \cdots, n$ and so R is J-McCoy. Conversely, suppose that R is right J-McCoy and $f(y)g(y) = 0$ for nonzero polynomials $f(y) = f_0 + f_1 y + ... + f_m y^m$ and $g(y) = g_0 + g_1 y + ... + g_n y^n$ in $(R[x])[y]$. Take the positive integer k with $k = \sum_{i=0}^{m} deg f_i + \sum_{j=0}^{n} deg g_j$ where the degree of the zero polynomial is taken to be zero. Then $f(x^k)$ and $g(x^k)$ are nonzero polynomials in $R[x]$ and $f(x^k)g(x^k) = 0$, since the set of coefficients of the f_i's and g_j's coincide with the set of coefficients of $f(x^k)$ and $g(x^k)$. Since R is right J-McCoy, there exists a nonzero element $c \in R$ such that $a_i c \in J(R)$, for any coefficient a_i of $f_i(x)$. So $f_i c \in J(R)[x] \subseteq J(R[x])$. Thus $R[x]$ is right J-McCoy. □

Recall that a ring R is said to be abelian if every idempotent of it is central.

Proposition 3. *Let R be a right (resp. left) J-McCoy ring and e be an idempotent element of R. Then eRe is a right (resp. left) J-McCoy ring. The converse holds if R is an abelian ring.*

Proof. Consider $f(x) = \sum_{i=0}^{n} ea_i ex^i, g(x) = \sum_{j=0}^{m} eb_j ex^j \in (eRe)[x] \setminus \{0\}$ such that $f(x)g(x) = 0$. Since R is a right J-McCoy ring, there exists $s \in R$ such that $(ea_i e)s \in J(R)$. So $(ea_i e)ese \in eJ(R)e = J(eRe)$. Hence eRe is right J-McCoy. Now, assume that eRe is a right J-McCoy ring. Consider $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x] \setminus \{0\}$ such that $f(x)g(x) = 0$. Clearly, $ef(x)e, eg(x)e \in (eRe)[x]$ and $(ef(x)e)(eg(x)e) = 0$, since e is a central idempotent element of R. Then there exists $s \in eRe$ such that $(ea_i e)s = (a_i)s \in J(eRe) = eJ(R) \subset J(R)$. Hence, R is right J-McCoy. □

The following example shows that, if R is a right J-McCoy ring, then $M_n(R)$ is not necessary right J-McCoy for $n \geq 2$, i.e. the J-McCoy property is not Morita invariant.

Example 3. Let $\mathbb{Z}$ be the set of integers. It's clear that $\mathbb{Z}$ is J-McCoy, but $M_3(\mathbb{Z})$ is not right J-McCoy. For

$$f(x) = \begin{pmatrix} 1 & x & x^2 \\ x^3 & x^4 & x^5 \\ x^6 & x^7 & x^8 \end{pmatrix} \text{ and } g(x) = \begin{pmatrix} x & x & x \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

in $M_3(\mathbb{Z})[x]$, we have $f(x)g(x) = 0$. Assume to the contrary that $M_3(\mathbb{Z})$ is right J-McCoy, then there exists $c = (c_{ij}) \in M_3(\mathbb{Z})$ such that $(E_{ij}c) \in J(M_3(\mathbb{Z})) = M_3(J(\mathbb{Z})) = 0$ for $i, j = 1, 2, 3$. This implies $c = 0$, which is a contradiction.

Let R be a ring and σ denote an endomorphism of R with $\sigma(1) = 1$. In [3] the authors introduced skew triangular matrix ring as a set of all triangular matrices with addition point-wise and a new multiplication subject to the condition $E_{ij}r = \sigma^{j-i} E_{ij}$. So $(a_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = a_{ij}b_{ij} + a_{i,i+1}\sigma(b_{i+1,j}) + ... + a_{ij}\sigma^{j-i}(b_{jj})$, for each $i \leq j$ and denoted it by $T_n(R, \sigma)$. The subring of the skew triangular matrices with constant mail diagonal is denoted by $S(R, n, \sigma)$; and the subring of the skew triangular matrices with constant diagonals is denoted by $T(R,n,\sigma)$. We can denote $A=(a_{ij})\in T(R,n,\sigma)$ by $(a_{11},...,a_{1n})$. Then $T(R,n,\sigma)$ is a ring with addition point-wise and multiplication given by $(a_0,...,a_{n-1})(b_0,...,b_{n-1})=(a_0b_0,a_0*b_1+a_1*b_0,...,a_0*b_{n-1}+...+a_{n-1}*b_0)$, with $a_i*b_j=a_i\sigma^i(b_j)$, for each i and j. Therefore, clearly one can see that $T(R,n,\sigma)\cong R[x;\sigma]/(x^n)$ is the ideal generated by x^n in $R[x;\sigma]$. we consider the following two subrings of $S(R,n,\sigma)$, as follows (see [3]):

$$A(R,n,\sigma)=\sum_{j=1}^{[\frac{n}{2}]}\sum_{i=1}^{n-j+1}a_jE_{i,i+j-1}+\sum_{j=[\frac{n}{2}]+1}^{n}\sum_{i=1}^{n-j+1}a_{i,i+j-1}E_{i,i+j-1},$$

$$B(R,n,\sigma)=\{A+rE_{1k}\mid A\in A(R,n,\sigma)\text{ and } r\in R\}\quad n=2k\geq 4$$

In the special case, when $\sigma=id_R$, we use $S(R,n),A(R,n),B(R,n)$ and $T(R,n)$ instead of $S(R,n,\sigma),A(R,n,\sigma),B(R,n,\sigma)$ and $T(R,n,\sigma)$, respectively.

Proposition 4. *Let R be a ring. Then S is right J-McCoy ring, for $n\geq 2$, where S is one of the rings $T_n(R,\sigma),S(R,n,\sigma),T(R,n,\sigma),A(R,n,\sigma)$ or $B(R,n,\sigma)$.*

Proof. Let $f(x)=A_0+A_1x+...+A_px^p, g(x)=B_0+B_1x+...+B_qx^q$ be elements of $S[x]$ satisfying $f(x)g(x)=0$ where the $(1,1)-th$ entry of A_i is $a_{11}^{(i)}$. Then $A_iE_{1n}=a_{11}^{(i)}E_{1n}\in J(S)$ and the proof is complete. □

Let R and S be two rings, and Let M be an (R,S)-bimodule. This means that M is a left R-module and a right S-module such that $(rm)s=r(ms)$ for all $r\in R$, $m\in M$, and $s\in S$. Given such a bimodule M we can form

$$T=\begin{pmatrix} R & M\\ 0 & S\end{pmatrix}=\left\{\begin{pmatrix} r & m\\ 0 & s\end{pmatrix}: r\in R, m\in M, s\in S\right\}$$

and define a multiplication on T by using formal matrix multiplication:

$$\begin{pmatrix} r & m\\ 0 & s\end{pmatrix}\begin{pmatrix} r' & m'\\ 0 & s'\end{pmatrix}=\begin{pmatrix} rr' & rm'+ms'\\ 0 & ss'\end{pmatrix}.$$

This ring construction is called triangular ring T.

Proposition 5. *Let R and S be two rings and T be the triangular ring $T=\begin{pmatrix} R & M\\ 0 & S\end{pmatrix}$ (where M is an (R,S)-bimodule). Then the rings R and S are right (resp. left) J-McCoy if and only if T is right (resp. left) J-McCoy.*

Proof. Assume that R and S are two right J-McCoy rings. Take $I=\begin{pmatrix} 0 & M\\ 0 & 0\end{pmatrix}$, then $T/I\simeq\begin{pmatrix} R & 0\\ 0 & S\end{pmatrix}$. Let

$$f(x)=\begin{pmatrix} r_0 & 0\\ 0 & s_0\end{pmatrix}+\begin{pmatrix} r_1 & 0\\ 0 & s_1\end{pmatrix}x+\cdots+\begin{pmatrix} r_n & 0\\ 0 & s_n\end{pmatrix}x^n,$$

$$g(x)=\begin{pmatrix} r'_0 & 0\\ 0 & s'_0\end{pmatrix}+\begin{pmatrix} r'_1 & 0\\ 0 & s'_1\end{pmatrix}x+\cdots+\begin{pmatrix} r'_m & 0\\ 0 & s'_m\end{pmatrix}x^m\in T[x]$$

satisfy $f(x)g(x)=0$. Define

$$f_r(x) = r_0 + r_1x + \cdots + r_nx^n,\ g_r(x) = r'_0 + r'_1x + \cdots + r'_mx^m \in R[x]$$

and

$$f_s(x) = s_0 + s_1x + \cdots + s_nx^n,\ g_s(x) = s'_0 + s'_1x + \cdots + s'_mx^m \in S[x].$$

From $f(x)g(x) = 0$, we have $f_r(x)g_r(x) = f_s(x)g_s(x) = 0$. Since R and S are right J-McCoy rings, then there exists $c \in R$ and $d \in S$ such that $r_ic \in J(R)$ and $s_id \in J(S)$ for any $1 \le i \le n$ and $1 \le j \le m$. Hence if we put $I = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ then T/I is right J-McCoy and so T is right J-McCoy by Proposition 1. Conversely, let T be a right J-McCoy ring, $f_r(x) = r_0 + r_1x + \cdots + r_nx^n$, $g_r(x) = r'_0 + r'_1x + \cdots + r'_mx^m \in R[x]$, such that $f_r(x)g_r(x) = 0$, and $f_s(x) = s_0 + s_1x + \cdots + s_nx^n$, $g_s(x) = s'_0 + s'_1x + \cdots + s'_mx^m \in S[x]$, such that $f_s(x)g_s(x) = 0$. Let

$$f(x) = \begin{pmatrix} r_0 & 0 \\ 0 & s_0 \end{pmatrix} + \begin{pmatrix} r_1 & 0 \\ 0 & s_1 \end{pmatrix}x + \cdots + \begin{pmatrix} r_n & 0 \\ 0 & s_n \end{pmatrix}x^n \text{ and}$$

$$g(x) = \begin{pmatrix} r'_0 & 0 \\ 0 & s'_0 \end{pmatrix} + \begin{pmatrix} r'_1 & 0 \\ 0 & s'_1 \end{pmatrix}x + \cdots + \begin{pmatrix} r'_m & 0 \\ 0 & s'_m \end{pmatrix}x^m \in T[x].$$

Then $f_r(x)g_r(x) = 0$ and $f_s(x)g_s(x) = 0$ implies that $f(x)g(x) = 0$. Since T is a right J-McCoy ring then there exists $\begin{pmatrix} c & m \\ 0 & d \end{pmatrix} \in T$ such that $\begin{pmatrix} r_i & 0 \\ 0 & s_i \end{pmatrix}\begin{pmatrix} c & m \\ 0 & d \end{pmatrix} \in J(T) = \begin{pmatrix} J(R) & M \\ 0 & J(S) \end{pmatrix}$. Thus $r_ic \in J(R)$ and $s_id \in J(S)$ for any i, j. This shows that R and S are right J-McCoy. □

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Corollary 3. *A ring R is right (resp. left) J-McCoy if and only if the trivial extension $T(R, R)$ is a right (resp. left) J-McCoy ring.*

Let S denote a multiplicatively closed subset of a ring R consisting of central regular elements. Let RS^{-1} be the localization of R at S. Then we have:

Theorem 4. *For a ring R, if R is right (resp. left) J-McCoy, then RS^{-1} is right (resp. left) J-McCoy.*

Proof. Suppose that R is right J-McCoy. Let $f(x) = \sum_{i=0}^{n} a_ic_i^{-1}x^i$, $g(x) = \sum_{j=0}^{m} b_jd_j^{-1}$
x^j be nonzero elements in $(RS^{-1})[x]$ such that $f(x)g(x) = 0$. Let $a_ic_i^{-1} = c^{-1}a'_i$ and $b_jd_j^{-1} = d^{-1}b'_j$ with c, d regular elements in R. So $f'(x)g'(x) = 0$ such that $f'(x) = \sum_{i=0}^{n} a'_ix^i$ and $g'(x) = \sum_{j=0}^{m} b'_jx^j \in R[x]\setminus\{0\}$. Since R is right J-McCoy, there exists $r \in R\setminus\{0\}$ such that $a'_ir \in J(R)$ for each i, equivalently we have $1 - ta'_ir$ is left invertible in R for each $t \in R$. So $c^{-1}w^{-1}(1 - tw^{-1}a_ic_i^{-1}rcw) = c^{-1}w^{-1} -$

$tw^{-1}a_i c_i^{-1} r$ is left invertible in RS^{-1}, for each $tw^{-1} \in RS^{-1}$ and so $a_i c_i^{-1} rcw \in J(RS^{-1})$. Thus RS^{-1} is right J-McCoy. □

Corollary 4. *For a ring R , let $R[x]$ be a right (resp. left) J-McCoy ring. Then $R[x,x^{-1}]$ is a right (resp. left) J-McCoy ring.*

Proof. Let $\Delta = \{1, x, x^2, \ldots\}$. Then clearly Δ is multiplicatively closed subset of $R[x]$. Since $R[x,x^{-1}] = \Delta^{-1}R$, it follows that $R[x,x^{-1}]$ is right J-McCoy. □

A ring R is called right (left) quasi-duo if every maximal right (left) ideal of R is two-sided. It is clear that a ring R is right (left) quasi-duo if and only if $R/J(R)$ is right (left) quasi-duo. Also $R/J(R)$ is a reduced ring in case it is right (left) quasi-duo.

Proposition 6. *If R is a right (left) quasi-duo ring, then R is right (resp. left) J-McCoy, the converse does not hold in general.*

Proof. Since R is right quasi-duo ring, then $R/J(R)$ is reduced by [7] and so R is right J-McCoy by Proposition 1. But there exists a right J-McCoy ring R which is not right (left) quasi-duo. For instance, take any right primitive domain R that is not division ring (e.g. the free algebra $R = Q < x, y >$). Then $R/J(R) = R$ is right J-McCoy, but R is not right quasi-duo by [5]. □

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