TRACES OF PERMUTING GENERALIZED $N$-DERIVATIONS OF RINGS

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Received 03 November, 2015

Abstract. Let $n \geq 1$ be a fixed positive integer and $R$ be a ring. A permuting $n$-additive map $\Omega : R^n \to R$ is known to be permuting generalized $n$-derivation if there exists a permuting $n$-derivation $\Delta : R^n \to R$ such that $\Omega(x_1, x_2, \ldots, x_n) = \Delta(x_1, x_2, \ldots, x_n)$ holds for all $x_i \in R$. A mapping $\delta : R \to R$ defined by $\delta(x) = \Delta(x, x, \ldots, x)$ for all $x \in R$ is said to be the trace of $\Delta$. The trace $\omega$ of $\Omega$ can be defined in the similar way. The main result of the present paper states that if $R$ is a $n!/\mathbb{S}$-torsion free semi-prime ring which admits a permuting $n$-derivation such that the trace $\omega$ of $\Omega$ satisfies $\delta(x) = \Delta(x, x, \ldots, x)$ for all $x \in R$, then $\omega$ is commuting on $R$. Besides other related results it is also shown that in a $n!/\mathbb{S}$-torsion free prime ring if the trace $\omega$ of a permuting generalized $n$-derivation $\Omega$ is centralizing on $R$, then $\omega$ is commuting on $R$.

2010 Mathematics Subject Classification: 16W25; 16U80

Keywords: derivation, generalized derivation, permuting $n$-derivation, centralizing map, prime ring

1. Introduction

Throughout $R$ will denote an associative ring with center $Z(R)$. For any $x, y \in R$, $xy - yx$ denotes the commutator $[x, y]$. A ring $R$ is said to be prime (resp. semi-prime) if $aRb = \{0\}$ implies either $a = 0$ or $b = 0$ (resp. $aRa = \{0\}$ implies $a = 0$). Let $m \geq 1$ be a fixed positive integer. A map $f : R \to R$ is said to be centralizing (resp. commuting) on $R$ if $[f(x), x] \in Z(R)$ (resp. $[f(x), x] = 0$) holds for all $x \in R$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Following [4], an additive mapping $F : R \to R$ is said to be a generalized derivation on $R$ if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Suppose $n$ is a fixed positive integer and $R^n = R \times R \times \cdots \times R$. A map $\Delta : R^n \to R$ is said to be permuting if the relation $\Delta(x_1, x_2, \ldots, x_n) = \Delta(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ holds for all $x_i \in R$ and for every permutation $\{\pi(1), \pi(2), \ldots, \pi(n)\}$. The concept of derivation and symmetric bi-derivation was generalized by Park [7] as follows: a permuting map $\Delta : R^n \to R$ is said to be a permuting $n$-derivation if $\Delta$ is $n$-additive (i.e.; additive
in each coordinate) and \( \Delta(x_1, x_2, \cdots, x_i x'_i, \cdots, x_n) = x_i \Delta(x_1, x_2, \cdots, x_i, \cdots, x_n) \) + 
\Delta(x_1, x_2, \cdots, x_i, \cdots, x_n) x'_i \) holds for all \( x_i, x'_i \in R \). A 1-derivation is a derivation and a 2-derivation is a symmetric bi-derivation while a 3-derivation is known as permuting tri-derivation.

A well known result due to Posner [8] states that a prime ring \( R \) which admits a non-zero centralizing derivation is commutative. In fact, this result initiated the study of centralizing and commuting mappings in rings. Since then, several authors have done a great deal of work concerning commutativity of prime and semi-prime rings admitting different kinds of maps which are centralizing or commuting on some appropriate subsets of \( R \) (see [5, 6] and [7] for further references). Let \( R \) be a prime ring, a well known result due to Posner [8] states that a prime ring \( R \) is a permuting generalized derivation if there exists a permuting map, be the trace of \( \Delta \). Moreover, it can be easily seen that \( \Delta(x_1, x_2, \cdots, -x_i, \cdots, x_n) = -\Delta(x_1, x_2, \cdots, x_i, \cdots, x_n) \) for all \( x_i \in R, i = 1, 2, \cdots, n \).

Motivated by the concept of generalized derivation in ring, we introduce the notion of permuting generalized \( n \)-derivation in ring. Let \( n \geq 1 \) be a fixed positive integer. A permuting \( n \)-additive map \( \Omega : R^n \to R \) is known to be permuting generalized \( n \)-derivation if there exists a permuting \( n \)-derivation \( \Delta : R^n \to R \) such that \( \Omega(x_1, x_2, \cdots, x_i x'_i, \cdots, x_n) = \Omega(x_1, x_2, \cdots, x_i, \cdots, x_n)x'_i + x_i \Delta(x_1, x_2, \cdots, x_i, \cdots, x_n) \) holds for all \( x_i, x'_i \in R \). For an example of permuting generalized \( n \)-derivation, let \( n \geq 1 \) be a fixed positive integer and \( R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \right\} | a, b, c \in \mathbb{C} \) where \( \mathbb{C} \) is a complex field. Consider permuting \( n \)-derivation \( \Delta \) as above and define \( \Omega : R^n \to R \) such that

\[
\Omega \left( \begin{pmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & a_n & b_n \\ 0 & 0 & c_n \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & a_1 \cdots a_n \\ 0 & 0 & c_1 \cdots c_n \end{pmatrix}.
\]

Then \( \Omega \) is a permuting generalized \( n \)-derivation on \( R \) associated with a permuting \( n \)-derivation \( \Delta \) on \( R \).

Let \( \omega : R \to R \) such that \( \omega(x) = \Omega(x, x', \cdots, x) \). Then \( \omega \) is known as the trace of \( \Omega \). A permuting \( n \)-additive map \( \Lambda : R^n \to R \) is said to be a permuting left \( n \)-multiplier (resp. permuting right \( n \)-multiplier) if \( \Lambda(x_1, x_2, \cdots, x_i x'_i, \cdots, x_n) = \Lambda(x_1, x_2, \cdots, x_i, \cdots, x_n) x'_i \) (resp. \( \Lambda(x_1, x_2, \cdots, x_i x'_i, \cdots, x_n) = x_i \Lambda(x_1, x_2, \cdots, x'_i, \cdots, x_n) \)) holds for all \( x_i, x'_i \in R \). If \( \Lambda \) is both permuting left \( n \)-multiplier as well as right \( n \)-multiplier, then \( \Lambda \) is called a permuting \( n \)-multiplier.

Motivated by the results due to Posner [8], Vukman obtained some results concerning the trace of symmetric bi-derivation in prime ring (see [9, 10]). Ashraf [1] proved similar results for semi-prime ring. In the year 2009, Park [7] introduced the concept of symmetric permuting \( n \)-derivation and obtained some results related to
the commuting traces of permuting $n$-derivations in rings. Further, the first author together with Jamal and Parveen [2, 3] obtained commutativity of rings admitting $n$-derivations whose traces satisfy certain polynomial conditions.

The main objective of this paper is to find the analogous results for permuting generalized $n$-derivation in the setting of prime and semi-prime rings. In fact, our theorems present a wide generalization of the results obtained in [1], Theorem 2.1, [7], Theorem 2.3, [7], Theorem 2.5, [9], Theorem 1, [9], Theorem 2, [10], Theorem 2 etc.

2. Results

We begin with the following known results which are frequently used in our discussion.

**Lemma 1** (Lemma 2.4 in [7]). Let $n$ be a fixed positive integer and let $R$ be a $n!$-torsion free ring. Suppose that $y_1, y_2, \ldots, y_n \in R$ satisfy $\lambda y_1 + \lambda^2 y_2 + \cdots + \lambda^n y_n = 0 \ (or \ \in Z(R))$ for $\lambda = 1, 2, \ldots, n$. Then $y_i = 0 \ (or \ y_i \in Z(R))$ for all $i$.

**Lemma 2** (Theorem 2.3 in [7]). Let $n \geq 2$ be a fixed positive integer and $R$ be a non-commutative $n!$-torsion free prime ring. Suppose that there exists a permuting $n$-derivation $\Delta : R^n \to R$ such that the trace $\delta$ of $\Delta$ is commuting on $R$. Then we have $\Delta = 0$.

**Lemma 3** (Theorem 2.6 in [7]). Let $n \geq 2$ be a fixed positive integer and $R$ be a $n!$-torsion free prime ring. Suppose that there exists a non-zero permuting $n$-derivation $\Delta : R^n \to R$ such that the trace $\delta$ of $\Delta$ is centralizing on $R$ then $R$ is commutative.

As stated in the beginning, there has been a great deal of work concerning centralizing and commuting mappings. The following result shows that if the trace $\delta$ of a permuting $n$-derivation $\Delta$ is centralizing on $R$ then it is commuting on $R$. In fact, we prove rather a more general result:

**Theorem 1.** Let $n \geq 2$ be a fixed positive integer and $R$ be a $(n+1)!$-torsion free semi-prime ring admitting a permuting $n$-derivation $\Delta$ such that the trace $\delta$ of $\Delta$ satisfies $[[\delta(x), x], x] = 0$ for all $x \in R$. Then $\delta$ is commuting on $R$.

**Proof.** From our hypothesis we have

$$[[\delta(x), x], x] = 0 \ \text{for all} \ x \in R. \quad (2.1)$$

An easy computation shows that the traces $\delta$ of $\Delta$ satisfies the following relations

$$\delta(x + y) = \delta(x) + \delta(y) + \sum_{r=1}^{n-1} \binom{n}{r} h_r(x, y) \text{ for all } x, y \in R$$

where $h_r(x, y) = \Delta(x, x, \ldots, x, y, y, \ldots, y)$. (n-r)-times $r$-times
Consider a positive integer $k, 1 \leq k \leq n + 1$. Replacing $x$ by $x + ky$ in equation (2.1), we obtain

$$kQ_1(x, y) + k^2 Q_2(x, y) + \ldots + k^{n+1} Q_{n+1}(x, y) = 0$$

for all $x, y \in R$, where $Q_i(x, y)$ denotes the sum of the terms in which $y$ appears $i$ times. By (2.1) and Lemma 1, we have for all $x, y \in R$,

$$[[\delta(x), x], y] + [[\delta(x), y], x] + n[[\Delta(x, x, \ldots, y), x], x] = 0.$$  \hspace{1cm} (2.2)

Replacing $y$ by $x$ in (2.2) we get

$$0 = [[\delta(x), x], xy] + [[\delta(x), xy], x] + n[[\Delta(xy, x, x, \ldots), x], x] + n[[\delta(x)y, x], x]$$

$$= [[\delta(x), x], xy] + [x[\delta(x), y], x] + n[[\delta(x)x, xy], x] + n[[\delta(x)y, x], x]$$

$$= [[\delta(x), x], x] + x[[\delta(x), x], x] + x[[\delta(x), y], x] + [x, x] [[\delta(x), y], x]$$

$$+ [\delta(x), x][y, x] + [[\delta(x), x], x]y + n[[\Delta(xy, x, x, \ldots), x], x]$$

$$+ n\delta(x)[y, x, x] + n\delta(x)[x, x] + n[[\delta(x), x], x]y + n[[\delta(x), x], x]y.$$  \hspace{1cm} (2.3)

Using (2.1) and (2.2) we find that

$$(2n + 1)[\delta(x), x][y, x] + n\delta(x)[y, x, x] = 0$$

for all $x, y \in R$.  \hspace{1cm} (2.3)

Similarly, replacing $y$ by $yx$ in (2.2), one can get

$$(2n + 1)[y, x][\delta(x), x] + n[[y, x], x] \delta(x) = 0$$

for all $x, y \in R$.  \hspace{1cm} (2.4)

Replacing $y$ by $yz$ in (2.3), we have

$$0 = (2n + 1)[\delta(x), x][y, x, x] + n\delta(x)[y, x, x, x]$$

$$= (2n + 1)[[\delta(x), x][y, x, x] + [\delta(x), x]y[x, x]] + n\delta(x)y[[x, x], x]$$

$$+ n\delta(x)[y, x, x][x, x] + n\delta(x)[y, x][x, x] + n\delta(x)[y, x, x][x, x]$$

$$= (2n + 1)[\delta(x), x][y, x, x] + n\delta(x)[y, x, x][x, x] + n\delta(x)[y, x, x][x, x].$$  \hspace{1cm} (2.5)

Using equation (2.3)

$$(2n + 1)[\delta(x), x][y, x, x] + n\delta(x)[y, x, x][x, x] + n\delta(x)[y, x, x][x, x] = 0.$$  \hspace{1cm} (2.5)

Replacing $y$ by $\delta(x)$ in the above relation we find that

$$(2n + 1)[\delta(x), x][\delta(x), x] + n\delta(x)^2[[x, x], x] + 2n\delta(x)[\delta(x), x][x, x] = 0.$$  \hspace{1cm} (2.5)

From (2.3) we have $n\delta(x)^2[[y, x], x] = -(2n + 1)\delta(x)[\delta(x), x][y, x].$ Now using this relation in (2.5) we get

$$0 = (2n + 1)[\delta(x), x][\delta(x), x] - (2n + 1)\delta(x)[\delta(x), x][x, x]$$

$$+ 2n\delta(x)[\delta(x), x][x, x]$$

$$= ((2n + 1)[\delta(x), x][\delta(x) - \delta(x)][\delta(x), x])[x, x].$$  \hspace{1cm} (2.6)
Similarly using (2.4) one can easily obtain
\[ \{(2n + 1)\delta(x)\delta(x) - [\delta(x), x]\delta(x)\}[z, x] = 0. \tag{2.7} \]

Adding (2.6) and (2.7) we arrive at
\[ 2n\{[\delta(x), x]\delta(x) + 2n\delta(x)\delta(x)\}[z, x] = 0. \tag{2.8} \]

Since \(2n\) divides \((n + 1)\), we find that \(R\) is \(2n\)-torsion free and hence for all \(x, z \in R\),
\[ \{(\delta(x), x)\delta(x) + \delta(x)\delta(x)\}[z, x] = 0. \tag{2.8} \]

Using (2.8) in (2.6) we obtain
\[ (2n + 2)\delta(x, x)\delta(x) \delta(x) [z, x] = 0 \]
for all \(x, z \in R\). Since \(2(n + 1)\) divides \((n + 1)\), we find that \(R\) is \(2(n + 1)\)-torsion free and hence for all \(x, z \in R\),
\[ \{(\delta(x), x)\delta(x) = 0 \quad \text{for all} \quad x, z \in R \}. \tag{2.9} \]

Similarly application of (2.7) and (2.8) yields that
\[ \delta(x, x) = 0 \quad \text{for all} \quad x \in R. \tag{2.10} \]

Replacing \(x\) by \(kx\) in equation (2.10) where \(1 \leq k \leq 2n\) and implementing Lemma 1
\[ \delta(x, y) + n\delta(x)\delta(x, x, ..., y, x) + n\delta(x, x, ..., y)\delta(x, x) = 0. \tag{2.11} \]

Replacing \(y\) by \(yx\)
0 = \(\delta(x)[\delta(x), yx] + n\delta(x)[y\delta(x) + \Delta(x, x, ..., y)x, x] + n\delta(x)\delta(x, y)x]\delta(x, x) \]
= \(\delta(x)y[\delta(x), x] + n\delta(x)[y\delta(x), x] + n\delta(x)\delta(x, y][\delta(x), x] + n\delta(x)[y, x]\delta(x, x) + n\delta(x)\delta(x, x, ..., y)x]\]
= \(y\delta(x)[\delta(x), x] + n\delta(x)[y, x]\delta(x, x, ..., y)x\]
\[ \text{From (2.11) we have} \]
\[ -n\Delta(x, x, ..., y)[\delta(x), x]x = \delta(x)[\delta(x), y]x + n\delta(x)[\Delta(x, x, ..., y), x]x. \tag{2.12} \]

Using (2.10) and (2.12) in the above relation, we get
\[ 0 = (n + 1)\delta(x)y[\delta(x), x] + n\delta(x)[y, x]\delta(x) + n\Delta(x, x, ..., y)x[\delta(x), x] \]
\[ -n\Delta(x, x, ..., y)[\delta(x), x]x \]
= \((n + 1)\delta(x)y[\delta(x), x] + n\delta(x)[y, x]\delta(x) - n\Delta(x, x, ..., y)[[\delta(x), x]x]. \]

This gives that
\[ (n + 1)\delta(x)y[\delta(x), x] + n\delta(x)[y, x]\delta(x) = 0 \quad \text{for all} \quad x, y \in R. \tag{2.13} \]
Substituting $xy$ for $y$ in (2.13)

$$(n + 1)\delta(x)xy[\delta(x), x] + n\delta(x)x[y, x]\delta(x) = 0 \text{ for all } x, y \in R.$$  

(2.14)

Left multiply (2.13) by $x$, we obtain

$$(n + 1)\delta(x)y[\delta(x), x] + nx\delta(x)[y, x]\delta(x) = 0 \text{ for all } x, y \in R.$$  

(2.15)

Combining (2.14) and (2.15), we get

$$(n + 1)[\delta(x), x]y[\delta(x), x] + n[\delta(x), x][y, x]\delta(x) = 0 \text{ for all } x, y \in R.$$  

(2.16)

Replacing $y$ by $yz$ in (2.4), we obtain

$$0 = (2n + 1)[yz, x]\delta(x, x) + n[yz, x]\delta(x)$$

$$= (2n + 1)[yz, x]\delta(x, x) + n[yz, x]\delta(x) + n[y, x]z\delta(x, x)$$

$$= (2n + 1)y[z, x]\delta(x, x) + (2n + 1)[y, x]z\delta(x, x) + ny[z, x]\delta(x)$$

$$+ n[y, x]z\delta(x) + n[y, x]z\delta(x) + n[y, x]z\delta(x).$$

Using (2.4) we get,

$$(2n + 1)[y, x]z\delta(x, x) + 2n[y, x]z\delta(x) + n[y, x]z\delta(x) = 0.$$  

Replacing $y$ by $\delta(x)$ in the above relation we get

$$(2n + 1)[\delta(x), x]z[\delta(x), x] + 2n[\delta(x), x]z[\delta(x), x] = 0 \text{ for all } x, z \in R.$$  

(2.17)

Combining equations (2.16) and (2.17) we find that

$$0 = (2n + 1)[\delta(x), x]z[\delta(x), x] - 2(n + 1)[\delta(x), x]z[\delta(x), x]$$

$$= [\delta(x), x]z[\delta(x), x] \text{ for all } x, z \in R.$$  

Since $R$ is semi-prime, we get $[\delta(x), x] = 0$, for all $x \in R$. □

**Theorem 2.** Let $n \geq 2$ be a fixed positive integer and $R$ be a $(n + 1)!$-torsion free semi-prime ring admitting a permuting $n$-derivation $\Delta$ such that the trace $\delta$ of $\Delta$ satisfies $[[\delta(x), x], x] \in Z(R)$ for all $x \in R$. Then $\delta$ is commuting on $R$.

**Proof.** Replace $x$ by $x + ky$ for $1 \leq k \leq n + 1$ in the given condition to find that

$$kQ_1(x, y) + k^2Q_2(x, y) + \ldots + k^{n+1}Q_{n+1}(x, y) \in Z(R) \text{ for all } x, y \in R,$$

where $Q_i(x, y)$ denotes the sum of the terms in which $y$ appears $i$ times. By Lemma 1, we have for all $x, y \in R$,

$$[[\delta(x), x], y] + [[\delta(x), y], x] + n[[\Delta(x, x, \ldots, y), x], x] \in Z(R).$$  

(2.18)

Again replacing $y$ by $xy$ in the above expression, we get

$$x([[\delta(x), x], y] + [[\delta(x), y], x] + n[[\Delta(x, x, \ldots, y), x], x]) + (n + 2)[[\delta(x), x], y]$$

$$+ (2n + 1)[\delta(x), x][y, x] + n\delta(x)[y, x, x] \in Z(R).$$
Combining (2.18) with the latter relation, we find that
\[(3n + 3)\langle\delta(x), x\rangle\langle y, x\rangle + (3n + 1)\langle\delta(x), x\rangle\langle y, x\rangle + n\delta(x)\langle\langle y, x\rangle, x\rangle = 0.\] (2.19)

Further replace \(y\) by \(\delta(x)\) in (2.19) to get
\[(6n + 4)\langle\delta(x), x\rangle\langle\delta(x), x\rangle = 0.\]

On commuting with \(x\), we find that
\[(6n + 4)\langle\delta(x), x\rangle\langle\delta(x), x\rangle^2 = 0.\] (2.20)

Next, on replacing \(y\) by \(\langle\delta(x), x\rangle\) in (2.19) and using the given condition, we have
\[(3n + 3)\langle\delta(x), x\rangle\langle\delta(x), x\rangle = 0.\] (2.21)

Now combine (2.20) and (2.21) to get \(2\langle\delta(x), x\rangle\langle\delta(x), x\rangle^2 = 0.\)

Since, \(R\) is \((n+1)!\)-torsion free and also the center of semi-prime ring is free from nilpotent element, we have \(\langle\delta(x), x\rangle\langle\delta(x), x\rangle = 0.\) From Theorem 1, \(\delta\) is commuting on \(R\). □

Combining Theorem 2 with Lemma 3, we can prove the following:

**Corollary 1.** Let \(n \geq 2\) be a fixed positive integer and \(R\) be a \((n + 1)!\)-torsion free semi-prime ring admitting a non-zero permuting \(n\)-derivation \(\Delta\) such that the trace \(\delta\) satisfies \(\langle\delta(x), x\rangle\langle\delta(x), x\rangle = Z(R)\) for all \(x \in R\). Then \(R\) is commutative.

**Theorem 3.** Let \(n \geq 1\) be a fixed positive integer and \(R\) be a non-commutative \(n\)!-torsion free prime ring admitting a permuting generalized \(n\)-derivation \(\Xi\) with associated \(n\)-derivation \(\delta\) such that the trace \(\omega\) of \(\Xi\) is commuting on \(R\). Then \(\Xi\) is a left \(n\)-multiplier on \(R\).

**Proof.** Our hypothesis yields that
\[\langle\omega(x), x\rangle = 0\] for all \(x \in R\). (2.22)

It can be easily seen that
\[\omega(x + y) = \omega(x) + \omega(y) + \sum_{r=1}^{n-1} \binom{n}{r} p_r(x, y)\] for all \(x, y \in R\)

where \(p_r(x, y) = \Omega(x, \ldots, x, y, y, \ldots, y)\).

Substituting \(x + \lambda y\), where \(\lambda (1 \leq \lambda \leq n)\) is a positive integer, in place of \(x\) in the above equation we obtain
\[0 = \langle\omega(x + \lambda y), x + \lambda y\rangle\]
\[= \langle\omega(x) + \omega(\lambda y) + \sum_{r=1}^{n-1} \binom{n}{r} p_r(x, \lambda y), x + \lambda y\rangle.\]
Using (2.22), we have
\[
0 = \lambda \{[\omega(x), y] + \binom{n}{1} [p_1(x, y), x] + \lambda^2 \{ \binom{n}{1} [p_1(x, y) y] \\
+ \binom{n}{2} [p_2(x, y), x] + \cdots + \lambda^n \{ [\omega(y), x] \\
+ \binom{n}{n-1} [p_{n-1}(x, y), x] \} \text{ for all } x, y \in R.
\]
Implementing Lemma 1 we get
\[
0 = [\omega(x), y] + \binom{n}{1} [p_1(x, y), x] = [\omega(x), y] + n[\Omega(x, x, \ldots, y), x].
\]
Replacing \( y \) by \( yx \) we obtain
\[
0 = y[\omega(x), x] + [\omega(x), y] x + n[\Omega(x, x, \ldots, y) x + y\Delta(x, x, \ldots, x), x] = [\omega(x), y] x + n[y, x] \delta(x) + n[\Omega(x, x, \ldots, y), x] x = n[y, x] \delta(x) + ny[\delta(x), x].
\]
Again replacing \( y \) by \( z y \) for any \( z \in R \) we have \([z, x] y \delta(x) = 0 \) for all \( x, y, z \in R \). Since \( R \) is prime we find that for any \( x \not\in Z(R), \delta(x) = 0 \). Now for any \( y \in Z(R) \) and \( x \not\in Z(R), x + \lambda y \not\in Z(R) \). Hence,
\[
0 = \delta(x + \lambda y) = \lambda p_1(x, y) + \cdots + \lambda^{n-1} p_{n-1}(x, y) + \lambda^n \delta(y).
\]
Using Lemma 2 we obtain \( \delta(y) = 0 \) for any \( y \in Z(R) \). Thus, \( \delta(x) = 0 \) for all \( x \in R \). Lemma 1 yields that \( \Delta = 0 \). This implies that \( \Omega \) acts as a left \( n \)-multiplier.

**Theorem 4.** Let \( n \geq 2 \) be a fixed positive integer and \( R \) be an \( n! \)-torsion free semi-prime ring admitting a permuting generalized \( n \)-derivation \( \Omega \) with associated \( n \)-derivation \( \Delta \) such that the trace \( w \) of \( \Omega \) is centralizing on \( R \). Then \( w \) is commuting on \( R \).

**Proof.** It is given that \([w(x), x] \in Z(R) \) for all \( x \in R \). Using the similar arguments as used in Theorem 2, we obtain
\[
[w(x), x] + n[\Omega(x, x, \ldots, y), x] \in Z(R) \text{ for all } x, y \in R. \quad (2.23)
\]
Replacing \( y \) by \( yx \), we obtain
\[
y[w(x), x] + [w(x), y] x + n[\Omega(x, x, \ldots, y), x] x + ny[\delta(x), x] + n[y, x] \delta(x) \in Z(R).
\]

Now in view of (2.23), we find that
\[
0 = [y, x][w(x), x] + n[y, x][\delta(x), x] + ny[[\delta(x), x], x]
+ n[y, x][\delta(x), x] + n[[y, x], x]\delta(x) \text{ for all } x, y \in R. \tag{2.24}
\]
Again replace \( y \) by \( w(x)y \) to get
\[
0 = w(x)[y, x][w(x), x] + [w(x), x]y[w(x), x] + nw(x)[y, x][\delta(x), x]
+ n[w(x), x]y[\delta(x), x] + nw(x)y[[\delta(x), x], x] + n w(x)[y, x][\delta(x), x]
+ n[w(x), x]y[\delta(x), x] + n[w(x), x][y, x]\delta(x) + nw(x)[y, x, x]\delta(x)
+ n[w(x), x][y, x]\delta(x) + n[[w(x), x], x]y\delta(x).
\]
Using (2.24) and the given condition, we find that
\[
[w(x), x]y[w(x), x] + 2n[w(x), x]y[\delta(x), x] + 2n[w(x), x][y, x]\delta(x) = 0. \tag{2.25}
\]
Further, replacing \( y \) by \( [w(x), x]^2 \) in (2.25) and using the given condition, we have
\[
[w(x), x]^4 + 2n[w(x), x]^3[\delta(x), x] = 0 \text{ for all } x \in R. \tag{2.26}
\]
Again, replace \( y \) by \( yz \) in (2.25) and use (2.25), to get
\[
2n[w(x), x][y, x]z\delta(x) = 0 \text{ for all } x, y, z \in R. \tag{2.27}
\]
Next, we replace \( y \) by \( w(x) \) and \( z \) by \([w(x), x] \) to find that \( 2n[w(x), x]^3\delta(x) = 0 \) for all \( x \in R \). On commuting the latter relation with \( x \) and using the given condition, we have
\[
2n[w(x), x]^3[\delta(x), x] = 0 \text{ for all } x \in R. \tag{2.28}
\]
From (2.26) and (2.28), we find that \([w(x), x]^4 = 0\). Since the center of a semi-prime ring does not contain any nilpotent element, we get \([w(x), x] = 0\). \( \square \)

**Corollary 2.** Let \( n \geq 2 \) be a fixed positive integer and \( R \) be a non-commutative \( n!\)-torsion free semi-prime ring admitting a permuting generalized \( n \)-derivation \( \Delta \) with associated \( n \)-derivation \( \delta \) such that the trace \( w \) of \( \Omega \) is centralizing on \( R \). Then \( \Omega \) is a left \( n \)-multiplier on \( R \).

**Proof.** By Theorem 3 and Theorem 2, we get the required result. \( \square \)

In conclusion, if we look at Theorem 2 closely, it is tempting to conjecture as follows:

**Conjecture 1.** Let \( R \) be a semi-prime ring with suitable torsion restrictions and \( \Delta \) be a non-zero permuting \( n \)-derivation. Suppose that for some integer \( m \geq 1 \), \( \delta_m(x) \in Z(R) \) for all \( x \in R \) where \( \delta_{k+1}(x) = \delta_k(x, x) \) for \( k > 1 \) and \( \delta_1(x) = \delta(x) \) stands for the trace of \( \Delta \). Then \( \delta_1(x) = 0 \) for all \( x \in R \).

**Acknowledgement**

The authors are indebted to the referee for his/her useful suggestions and valuable comments.
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