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SOME NEW HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR FUNCTIONS WHOSE nth DERIVATIVES ARE LOGARITHMICALLY RELATIVE $h-$PREINVEX

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Abstract. The authors have introduced the concept of logarithmically relative $h-$preinvex function which is a generalized form of previously known concepts [9, 11], and try to establish some new Hermite-Hadamard type integral inequalities for functions whose absolute values of nth derivatives are logarithmically relative $h-$preinvex.

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1. Introduction and preliminaries

In recent years, researchers have paid a lot of attention to study and investigate the theory of convex functions due to its extensive applications in different fields of pure and applied sciences. Weir et al. [15] had given a significant generalization of convex functions by introducing preinvex functions. Varosanec[12], introduced the $h-$convex functions. Noor et al. [9] have introduced the concept of logarithmically $h-$preinvex functions, generalizing the concept of logarithmically $s-$preinvex functions, logarithmically $P-$preinvex functions and logarithmically $Q-$preinvex functions.

There are many inequalities for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications. Hermit-Hadamard inequlaities for the convex functions and their variant forms are available in the literature [3–11, 13, 14].

Motivated by the works of Noor et al. [9], Fulga et al. [1], and using the concept of logarithmically relative $h-$preinvex functions, we have derived some new Hermite-Hadamard type inequalites.

The paper is arranged in such a way that after this Introduction,in Section 1, the authors have defined the logarithmically relative $h-$preinvex functions and its relating consequent concepts, in section 2, the authors have discussed their main results regarding Hermite-Hadamard type integral inequalities, using the concept of Young's inquality, Hölder's inequality, and power mean's inequality.

Let M be a nonempty closed set in $\mathbf{R}^n$. Let $f : M \to (0,\infty)$ be a continuous function and let $\theta : \mathbf{R^n} \times \mathbf{R^n} \to \mathbf{R}^n$ be a continuous bifunction.

Definition 1 ([1])**.** A set $M \subseteq \mathbf{R}^n$ is said to be a relative invex (or $g-$invex) with respect to the map θ, if there exists a function $g : \mathbf{R}^n \to \mathbf{R}^n$ such that,

$$g(a) + t\theta\,(g(b), g(a)) \in M;\ \forall\ a,b \in \mathbf{R}^n\ : g(a), g(b) \in M,\ t \in [0,1].$$

Definition 2. Let $M \subseteq \mathbf{R}^n$ be a relative invex set with respect to θ. Then $\theta-$relative path, $P_{g(a);g(c)}$, joining the points $g(a)$ and $g(c) = g(a) + \theta(g(b), g(a))$ is defined as:

$$P_{g(a);g(c)} = \{g(d) | g(d) = g(a) + t\theta\,(g(b), g(a));\ a,b \in M,\ t \in [0,1]\}.$$

Remark 1. For $g = I$, I an identity function, definition 2 coincides with definition of $\theta-$path [14].

Definition 3. Let $h : J \to \mathbf{R}$, where $(0,1) \subseteq J$ be an interval in $\mathbf{R}$; let $h \neq 0$ for all values of J; let $M \subseteq \mathbf{R}^n$ be a relative invex set with respect to θ. A function $f : M \to (0,\infty)$ is said to be logarithmically relative $h-$preinvex (or $\log-(g,h)$ preinvex) with respect to θ, if

$$f\,(g(a) + t\theta\,(g(b), g(a))) \leq [f(g(a))]^{h(1-t)}[f(g(b))]^{h(t)}, \tag{1.1}$$

for $a,b \in \mathbf{R}^n$ such that $g(a), g(b) \in M,\ t \in (0,1)$.

f is logarithmically relative $h-$preconcave (or $\log-(g,h)$ preconcave) with respect to θ whenever the inequality sign in (1.1) is reversed.

Remark 2. For $g(t) = I$, I an identity function, definition 3 coincides with definition of logarithmically $h-$preinvex function[9].

Example 1. Let $M = [-2,-1] \bigcup [1,2]$, then obviously M is relative invex for $g : \mathbf{R} \to \mathbf{R}^+ \bigcup \{0\}$ and $\theta : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ respectively defined by:

$$g(x) = \begin{cases} x^2 & \text{for } |x| \leq \sqrt{2} \\ 2 & \text{for } |x|\ > \sqrt{2}. \end{cases} \qquad \theta(x,y) = \begin{cases} x - y & \text{for } x.y > 0 \\ 2 - y & \text{for } x.y < 0. \end{cases}$$

Consider, $f : M \to (0,\infty)$ defined by

$$f(x) = \arctan(x + 2 + \sqrt{3})\ and\ h(t) = t^s,$$

where $s \leq 0$. Then f is logarithmically relative $h-$preinvex function with respect to θ.

Remark 3. For $x = \sqrt{2}$, $y = 1$, $t = \frac{1}{2}$, $s = 1$, the above function f is not logarithmically relative $h-$preinvex function.

Definition 4. A function $f : M \to (0,\infty)$ is said to be logarithmically relative $s-$preinvex (or $\log-(g,s)$ preinvex) with respect to θ, if

$$f\left(g(a)+t\theta\left(g(b),g(a)\right)\right) \le [f(g(a))]^{(1-t)^s}[f(g(b))]^{t^s},$$

for $a,b \in \mathbf{R}^n$ such that $g(a),g(b) \in M$, $t \in [0,1]$, $s \in (0,1]$.

Remark 4. (1) For $g(t) = I$, I an identity function, definition 4 coincides with definition of logarithmically $s-$preinvex function [9].
(2) For $h(t) = t^s$, definition 3 coincides with definition 4.

Definition 5. A function $f : M \to (0,\infty)$ is said to be logarithmically relative $P-$preinvex (or $\log-(g,P)$ preinvex) with respect to θ, if

$$f\left(g(a)+t\theta\left(g(b),g(a)\right)\right) \le [f(g(a))][f(g(b))],$$

for $a,b \in \mathbf{R}^n$ such that $g(a),g(b) \in M$, $t \in [0,1]$.

Remark 5. (1) For $g(t) = I$, I an identity function, definition 5 coincides with definition of logarithmically $P-$preinvex function [9].
(2) For $h(t) = 1$, definition 3 coincides with definition 5.

Definition 6. A function $f : M \to (0,\infty)$ is said to be logarithmically relative $Q-$preinvex (or $\log-(g,Q)$ preinvex) with respect to θ, if

$$f\left(g(a)+t\theta\left(g(b),g(a)\right)\right) \le [f(g(a))]^{\frac{1}{1-t}}[f(g(b))]^{\frac{1}{t}},$$

for $a,b \in \mathbf{R}^n$ such that $g(a),g(b) \in M$, $t \in (0,1)$.

Remark 6. (1) For $g(t) = I$, I an identity function, definition 6 coincides with definition of logarithmically $Q-$preinvex function[9].
(2) For $h(t) = \frac{1}{t}$, definition 3 coincides with definition 6.

Example 2. Let $M = [-2,-1]\bigcup[1,2]$, then obviously M is relative invex for $g : \mathbf{R} \to \mathbf{R}^+\bigcup\{0\}$ and $\theta : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ respectively defined by

$$g(x) = \begin{cases} x^2 & \text{for } |x| \le \sqrt{2} \\ 1 & \text{for } |x| > \sqrt{2}. \end{cases} \qquad \theta(x,y) = x-y.$$

Consider, $f : M \to (0,\infty)$ defined by $f(x) = \arctan(x+2+\epsilon)$ for $\epsilon > 0$ and $h(t) = \frac{1}{t}$. Then f is logarithmically relative $Q-$preinvex function with respect to θ.

Definition 7. The beta function is defined as:

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x,y > 0.$$

Definition 8. The incomplete beta function is defined as:

$$\beta_z(x,y) = \int_0^z t^{x-1}(1-t)^{y-1}dt, \quad 0 \le z \le 1; \quad x,y > 0.$$

2. RESULTS

For establishing some new Hermite-Hadamard type integral inequalities for $n-$times differentiable functions such that $|f^{(n)}|$ are logarithmically relative $h-$preinvex functions, we need the following result, which is a generalization of a result proved by Wang et al. [14, Lemma1].

Lemma 1 ([2]). *Let $A \subseteq \mathbf{R}$ be an open relative invex subset with respect to $\theta : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, with $a, b \in A$ and $\theta(g(a), g(b)) \neq 0$. If $f : \mathbf{R} \to \mathbf{R}$ is $n-$times differentiable function and $f^{(n)}$ is integrable on the $\theta-$relative path $P_{g(b);g(c)}$, then*

$$\begin{aligned}
&\frac{[\theta(g(a), g(b))]^n}{2^{n+2} \times n!} \int_0^1 (n-2t)t^{n-1} \left[f^{(n)} \left(g(b) + \frac{1-t}{2} \theta(g(a), g(b)) \right) \right. \\
&\qquad \left. + f^{(n)} \left(g(b) + \frac{2-t}{2} \theta(g(a), g(b)) \right) \right] dt \\
&= \frac{1}{2} \left[\frac{f(g(b)) + f(g(b) + \theta(g(a), g(b)))}{2} + f\left(\frac{2g(b) + \theta(g(a), g(b))}{2} \right) \right] \\
&\qquad - \frac{1}{\theta(g(a), g(b))} \int_{g(b)}^{g(b)+\theta(g(a),g(b))} f(g(x))\, dg(x) \\
&- \sum_{k=2}^{n-1} \frac{[\theta(g(a), g(b))]^k (k-1)}{2^{k+2} \times (k+1)!} \left[f^{(k)}(g(b)) + f^{(k)} \left(\frac{2g(b) + \theta(g(a), g(b))}{2} \right) \right]
\end{aligned} \tag{2.1}$$

where the above summation is zero for $n = 1, 2$

Before going to our main results, we make the following assumptions:

(1) $\widetilde{A_n} := \frac{|\theta(g(a), g(b))|^n}{n! \times 2^{n+2}}$

(2) $\widetilde{B}_{n,k}(t) := \left| f^{(n)}(g(b)) \right| \times \left| \frac{f^{(n)}(g(a))}{f^{(n)}(g(b))} \right|^{h(\frac{k-t}{2})}$

(3) $h(\frac{1+t}{2}) + h(\frac{1-t}{2}) = 1 = h(\frac{t}{2}) + h(\frac{2-t}{2})$.

Theorem 1. *Let $A \subseteq \mathbf{R}$ be a relative invex subset with respect to $\theta : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, with $a, b \in A$ and $\theta(g(a), g(b)) \neq 0$. Suppose that $f : A \to (0, \infty)$ is $n-$times differentiable function and $f^{(n)}$ is integrable on the $\theta-$relative path $P_{g(b);g(c)}$, such that $|f^{(n)}|$ is logarithmically relative $h-$preinvex on A; let $q_k > 1$, $k = 1, 2$, then*

$$\begin{aligned}
&\left| \frac{1}{2} \left[\frac{f(g(b)) + f(g(b) + \theta(g(a), g(b)))}{2} + f\left(\frac{2g(b) + \theta(g(a), g(b))}{2} \right) \right] \right. \\
&\qquad - \frac{1}{\theta(g(a), g(b))} \int_{g(b)}^{g(b)+\theta(g(a),g(b))} f(g(u))\, dg(u)
\end{aligned}$$

$$
-\sum_{k=2}^{n-1} \frac{[\theta(g(a),g(b))]^k (k-1)}{(k+1)! \times 2^{k+2}} \left[f^{(k)}(g(b)) + f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2} \right) \right] \Bigg|
$$

$$
\leq \begin{cases} \widetilde{A_1} \sum_{k=1}^{2} \frac{1}{q_k} \left[\frac{(q_k-1)^2}{2q_k-1} + \int_0^1 \left(\widetilde{B}_{1,k}(t) \right)^{q_k} dt \right] & \textit{for } n=1 \\ \widetilde{A_n} \sum_{k=1}^{2} \frac{1}{q_k} \left[\frac{n^{\frac{nq_k}{q_k-1}+1} \times (q_k-1)}{2^{\frac{q_k(n-1)}{q_k-1}+1}} \beta_{\frac{2}{n}} \left(\frac{nq_k-1}{q_k-1}, \frac{2q_k-1}{q_k-1} \right) + \int_0^1 \left(\widetilde{B}_{n,k}(t) \right)^{q_k} dt \right] & \textit{for } n \geq 2. \end{cases} \tag{2.2}
$$

Proof. Since $g(b)+t\theta(g(a),g(b)) \in A$ for every $t \in (0,1)$, by Lemma 1 and Young's inequality, we have

$$
\Bigg| \frac{1}{2} \left[\frac{f(g(b)) + f(g(b)+\theta(g(a),g(b)))}{2} + f\left(\frac{2g(b)+\theta(g(a),g(b))}{2} \right) \right]
$$

$$
- \frac{1}{\theta(g(a),g(b))} \int_{g(b)}^{g(b)+\theta(g(a),g(b))} f(g(u))\, dg(u)
$$

$$
-\sum_{k=2}^{n-1} \frac{[\theta(g(a),g(b))]^k (k-1)}{(k+1)! \times 2^{k+2}} \left[f^{(k)}(g(b)) + f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2} \right) \right] \Bigg|
$$

$$
\leq \frac{|\theta(g(a),g(b))|^n}{n! \times 2^{n+2}} \int_0^1 \left| (n-2t)t^{n-1} \right| \left[\left| f^{(n)}\left(g(b) + \frac{1-t}{2}\theta(g(a),g(b)) \right) \right| \right.
$$

$$
\left. + \left| f^{(n)}\left(g(b) + \frac{2-t}{2}\theta(g(a),g(b)) \right) \right| \right] dt
$$

$$
\leq \frac{|\theta(g(a),g(b))|^n}{n! \times 2^{n+2}} \times \int_0^1 \left[\frac{q_1-1}{q_1} \left| (n-2t)t^{n-1} \right|^{\frac{q_1}{q_1-1}} \right.
$$

$$
+ \frac{1}{q_1} \left| f^{(n)}\left(g(b) + \frac{1-t}{2}\theta(g(a),g(b)) \right) \right|^{q_1} + \frac{q_2-1}{q_2} \left| (n-2t)t^{n-1} \right|^{\frac{q_2}{q_2-1}}
$$

$$
\left. + \frac{1}{q_2} \left| f^{(n)}\left(g(b) + \frac{2-t}{2}\theta(g(a),g(b)) \right) \right|^{q_2} \right] dt \tag{2.3}
$$

Using logarithmically relative $h-$preinvexity of $|f^{(n)}|$, we have

$$
\int_0^1 \left| f^{(n)}\left(g(b) + \frac{1-t}{2}\theta(g(a),g(b)) \right) \right|^{q_1} dt
$$

$$
\leq \int_0^1 \left(\left| f^{(n)}(g(b)) \right|^{h(\frac{1+t}{2})} \left| f^{(n)}(g(a)) \right|^{h(\frac{1-t}{2})} \right)^{q_1} dt
$$

$$
= \int_0^1 \left(\left| f^{(n)}(g(b)) \right| \left| \frac{f^{(n)}(g(a))}{f^{(n)}(g(b))} \right|^{h(\frac{1-t}{2})} \right)^{q_1} dt \tag{2.4}
$$

Similarly

$$\int_0^1 \left| f^{(n)}\left(g(b)+\frac{2-t}{2}\theta(g(a),g(b))\right)\right|^{q_2} dt$$
$$\leq \int_0^1 \left(\left|f^{(n)}(g(b))\right| \left|\frac{f^{(n)}(g(a))}{f^{(n)}(g(b))}\right|^{h\left(\frac{2-t}{2}\right)}\right)^{q_2} dt \quad (2.5)$$

Also,

$$\int_0^1 \left|(n-2t)t^{n-1}\right|^{\frac{q_k}{q_k-1}} dt = \begin{cases} \frac{q_k-1}{2q_k-1} & \text{for } n=1 \\ \frac{n^{\frac{nq_k}{q_k-1}+1}}{2^{\frac{q_k(n-1)}{q_k-1}+1}}\beta_{\frac{2}{n}}\left(\frac{nq_k-1}{q_k-1},\frac{2q_k-1}{q_k-1}\right) & \text{for } n\geq 2. \end{cases} \quad (2.6)$$

A combination of (2.3)-(2.6) yields the desired result. □

Corollary 1. *Under the conditions of theorem 1 for $q_1 = q_2$, we have*

$$\Bigg|\frac{1}{2}\left[\frac{f(g(b))+f(g(b)+\theta(g(a),g(b)))}{2}+f\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]$$
$$-\frac{1}{\theta(g(a),g(b))}\int_{g(b)}^{g(b)+\theta(g(a),g(b))} f(g(u))\,dg(u)$$
$$-\sum_{k=2}^{n-1}\frac{[\theta(g(a),g(b))]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}(g(b))+f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\Bigg|$$
$$\leq \begin{cases} \frac{\widetilde{A}_1}{q_2}\left[\frac{2(q_2-1)^2}{2q_2-1}+\int_0^1\sum_{k=1}^2\left(\widetilde{B}_{1,k}(t)\right)^{q_2}dt\right] & \textit{for } n=1 \\ \frac{\widetilde{A}_n}{q_2}\left[\frac{n^{\frac{nq_2}{q_2-1}+1}\times(q_2-1)}{2^{\frac{q_2(n-1)}{q_2-1}}}\beta_{\frac{2}{n}}\left(\frac{nq_2-1}{q_2-1},\frac{2q_2-1}{q_2-1}\right)+\int_0^1\sum_{k=1}^2\left(\widetilde{B}_{n,k}(t)\right)^{q_2}dt\right] & \textit{for } n\geq 2. \end{cases}$$

Theorem 2. *Let $A\subseteq \mathbf{R}$ be a relative invex subset with respect to $\theta:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$, with $a,b\in A$ and $\theta(g(a),g(b))\neq 0$. Suppose that $f:A\to(0,\infty)$ is $n-$times differentiable function and $f^{(n)}$ is integrable on the $\theta-$relative path $P_{g(b);g(c)}$, such that $|f^{(n)}|$ is logarithmically relative $h-$preinvex on A; let $q_3>1$, then*

$$\Bigg|\frac{1}{2}\left[\frac{f(g(b))+f(g(b)+\theta(g(a),g(b)))}{2}+f\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]$$
$$-\frac{1}{\theta(g(a),g(b))}\int_{g(b)}^{g(b)+\theta(g(a),g(b))} f(g(u))\,dg(u)$$
$$-\sum_{k=2}^{n-1}\frac{[\theta(g(a),g(b))]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}(g(b))+f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\Bigg|$$

$$\leq \begin{cases} \frac{\widetilde{A}_1}{q_3}\left[\frac{(q_3-1)^2}{2q_3-1}+\int_0^1\left(\sum_{k=1}^{2}\widetilde{B}_{1,k}(t)\right)^{q_3}dt\right] & \textit{for } n=1 \\ \frac{\widetilde{A}_n}{q_3}\left[\frac{n^{\frac{nq_3}{q_3-1}+1}\times(q_3-1)}{2^{\frac{q_3(n-1)}{q_3-1}+1}}\beta_{\frac{2}{n}}\left(\frac{nq_3-1}{q_3-1},\frac{2q_3-1}{q_3-1}\right)+\int_0^1\left(\sum_{k=1}^{2}\widetilde{B}_{n,k}(t)\right)^{q_3}dt\right] & \textit{for } n\geq 2. \end{cases} \tag{2.7}$$

Proof. Since $g(b)+t\theta(g(a),g(b))\in A$ for every $t\in(0,1)$, by Lemma 1, Young's inequality and the logarithmically relative $h-$preinvexity of $|f^{(n)}|$, we have

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(g(b))+f(g(b)+\theta(g(a),g(b)))}{2}+f\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\right. \\
&\qquad -\frac{1}{\theta(g(a),g(b))}\int_{g(b)}^{g(b)+\theta(g(a),g(b))}f(g(u))\,dg(u) \\
&\left.-\sum_{k=2}^{n-1}\frac{[\theta(g(a),g(b))]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}(g(b))+f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\right| \\
&\leq \frac{|\theta(g(a),g(b))|^n}{n!\times 2^{n+2}}\int_0^1\left|(n-2t)t^{n-1}\right|\left[\left|f^{(n)}\left(g(b)+\frac{1-t}{2}\theta(g(a),g(b))\right)\right|\right. \\
&\qquad\left.+\left|f^{(n)}\left(g(b)+\frac{2-t}{2}\theta(g(a),g(b))\right)\right|\right]dt \\
&\leq \frac{|\theta(g(a),g(b))|^n}{n!\times 2^{n+2}}\times\int_0^1\left[\frac{q_3-1}{q_3}\left|(n-2t)t^{n-1}\right|^{\frac{q_3}{q_3-1}}\right. \\
&\qquad+\frac{1}{q_3}\left\{\left|f^{(n)}\left(g(b)+\frac{1-t}{2}\theta(g(a),g(b))\right)\right|\right. \\
&\qquad\left.\left.+\left|f^{(n)}\left(g(b)+\frac{2-t}{2}\theta(g(a),g(b))\right)\right|\right\}^{q_3}\right]dt \\
&\leq \frac{|\theta(g(a),g(b))|^n}{n!\times 2^{n+2}\times q_3}\times\int_0^1\left[(q_3-1)\left|(n-2t)t^{n-1}\right|^{\frac{q_3}{q_3-1}}\right. \\
&\qquad\left.+\left(\sum_{k=1}^{2}\left|f^{(n)}(g(b))\right|\left|\frac{f^{(n)}(g(a))}{f^{(n)}(g(b))}\right|^{h\left(\frac{k-t}{2}\right)}\right)^{q_3}\right]dt
\end{aligned}$$

□

Corollary 2. *Under the conditions of theorem 2 for $n=2$, we have*

$$\left|\frac{1}{2}\left[\frac{f(g(b))+f(g(b)+\theta(g(a),g(b)))}{2}+f\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\right. \\ \left.-\frac{1}{\theta(g(a),g(b))}\int_{g(b)}^{g(b)+\theta(g(a),g(b))}f(g(u))\,dg(u)\right|$$

$$\leq \frac{\widetilde{A_2}}{q_3}\left[2^{\frac{q_3}{q_3-1}}\times(q_3-1)\times\beta\left(\frac{2q_3-1}{q_3-1},\frac{2q_3-1}{q_3-1}\right)+\int_0^1\left(\sum_{k=1}^{2}\widetilde{B}_{2,k}(t)\right)^{q_3}dt\right]$$

Theorem 3. *Let $A \subseteq \mathbf{R}$ be a relative invex subset with respect to $\theta : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$, with $a,b \in A$ and $\theta(g(a),g(b)) \neq 0$. Suppose that $f : A \to (0,\infty)$ is $n-$times differentiable function and $f^{(n)}$ is integrable on the $\theta-$relative path $P_{g(b);g(c)}$, such that $|f^{(n)}|$ is logarithmically relative $h-$preinvex on A; let $q_3 > 1$, and $q_3 \geq r > 0$, then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(g(b))+f(g(b)+\theta(g(a),g(b)))}{2}+f\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\right.\\
&\qquad -\frac{1}{\theta(g(a),g(b))}\int_{g(b)}^{g(b)+\theta(g(a),g(b))}f(g(u))\,dg(u)\\
&\left.-\sum_{k=2}^{n-1}\frac{[\theta(g(a),g(b))]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}(g(b))+f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\right|\\
&\leq\begin{cases}\frac{\widetilde{A}_1}{q_3}\left[\frac{(q_3-1)^2}{2q_3-r-1}+\int_0^1|(n-2t)t^{n-1}|^r\left(\sum_{k=1}^2\widetilde{B}_{1,k}(t)\right)^{q_3}dt\right] & \text{for } n=1\\ \frac{\widetilde{A}_n}{q_3}\left[\frac{n^{\frac{n(q_3-r)}{q_3-1}+1}\times(q_3-1)}{2^{\frac{(q_3-r)(n-1)}{q_3-1}+1}}\,\beta_{\frac{2}{n}}\left(\frac{nq_3-nr+r-1}{q_3-1},\frac{2q_3-r-1}{q_3-1}\right)\right. & \\ \left.+\int_0^1|(n-2t)t^{n-1}|^r\left(\sum_{k=1}^2\widetilde{B}_{n,k}(t)\right)^{q_3}dt\right] & \text{for } n\geq 2.\end{cases}
\end{aligned}\tag{2.8}$$

Proof. Since $g(b)+t\theta(g(a),g(b)) \in A$ for every $t \in (0,1)$, by Lemma 1, Young's inequality and the logarithmically relative $h-$preinvexity of $|f^{(n)}|$, we have

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(g(b))+f(g(b)+\theta(g(a),g(b)))}{2}+f\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\right.\\
&\qquad -\frac{1}{\theta(g(a),g(b))}\int_{g(b)}^{g(b)+\theta(g(a),g(b))}f(g(u))\,dg(u)\\
&\left.-\sum_{k=2}^{n-1}\frac{[\theta(g(a),g(b))]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}(g(b))+f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\right|\\
&\leq\frac{|\theta(g(a),g(b))|^n}{n!\times 2^{n+2}}\int_0^1\left|(n-2t)t^{n-1}\right|\left[\left|f^{(n)}\left(g(b)+\frac{1-t}{2}\theta(g(a),g(b))\right)\right|\right.\\
&\qquad\left.+\left|f^{(n)}\left(g(b)+\frac{2-t}{2}\theta(g(a),g(b))\right)\right|\right]dt\\
&\leq\frac{|\theta(g(a),g(b))|^n}{n!\times 2^{n+2}}\int_0^1\left[\frac{q_3-1}{q_3}\left|(n-2t)t^{n-1}\right|^{\frac{q_3-r}{q_3-1}}+\frac{1}{q_3}\left|(n-2t)t^{n-1}\right|^r\right.
\end{aligned}$$

$$\times\left\{\left|f^{(n)}\left(g(b)+\frac{1-t}{2}\theta(g(a),g(b))\right)\right| + \left|f^{(n)}\left(g(b)+\frac{2-t}{2}\theta(g(a),g(b))\right)\right|\right\}^{q_3}\Bigg]dt$$

$$\leq \frac{|\theta(g(a),g(b))|^n}{n!\times 2^{n+2}\times q_3}\times\int_0^1\Bigg[(q_3-1)\left|(n-2t)t^{n-1}\right|^{\frac{q_3-r}{q_3-1}} + \left|(n-2t)t^{n-1}\right|^r\times\left(\sum_{k=1}^{2}\left|f^{(n)}(g(b))\right|\left|\frac{f^{(n)}(g(a))}{f^{(n)}(g(b))}\right|^{h\left(\frac{k-t}{2}\right)}\right)^{q_3}\Bigg]dt$$

□

Corollary 3. *Under the conditions of theorem 3 for $r=q_3$, we have*

$$\Bigg|\frac{1}{2}\left[\frac{f(g(b))+f(g(b)+\theta(g(a),g(b)))}{2}+f\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right] - \frac{1}{\theta(g(a),g(b))}\int_{g(b)}^{g(b)+\theta(g(a),g(b))}f(g(u))\,dg(u) - \sum_{k=2}^{n-1}\frac{[\theta(g(a),g(b))]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}(g(b))+f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\Bigg|$$

$$\leq\begin{cases}\frac{\widetilde{A}_1}{q_3}\left[(q_3-1)+\int_0^1\left(|(n-2t)t^{n-1}|\sum_{k=1}^{2}\widetilde{B}_{1,k}(t)\right)^{q_3}dt\right] & \text{for } n=1\\ \frac{\widetilde{A}_n}{q_3}\left[\frac{n(q_3-1)}{2}\beta_{\frac{2}{n}}(1,1)+\int_0^1\left(|(n-2t)t^{n-1}|\sum_{k=1}^{2}\widetilde{B}_{n,k}(t)\right)^{q_3}dt\right] & \text{for } n\geq 2.\end{cases}$$

Theorem 4. *Let $A\subseteq\mathbf{R}$ be a relative invex subset with respect to $\theta:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$, with $a,b\in A$ and $\theta(g(a),g(b))\neq 0$. Suppose that $f:A\to(0,\infty)$ is $n-$times differentiable function and $f^{(n)}$ is integrable on the $\theta-$relative path $P_{g(b);g(c)}$, such that $|f^{(n)}|$ is logarithmically relative $h-$preinvex on A; let $q_3>1$, and then*

$$\Bigg|\frac{1}{2}\left[\frac{f(g(b))+f(g(b)+\theta(g(a),g(b)))}{2}+f\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right] - \frac{1}{\theta(g(a),g(b))}\int_{g(b)}^{g(b)+\theta(g(a),g(b))}f(g(u))\,dg(u) - \sum_{k=2}^{n-1}\frac{[\theta(g(a),g(b))]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}(g(b))+f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2}\right)\right]\Bigg|$$

$$\leq \begin{cases} \widetilde{A}_1 \sum_{k=1}^{2} \left(\int_0^1 \left(\widetilde{B}_{1,k}(t) \right)^{q_3} dt \right)^{\frac{1}{q_3}} \left(\frac{q_3-1}{2q_3-1} \right)^{1-\frac{1}{q_3}} & \textit{for } n=1 \\ \widetilde{A}_n \sum_{k=1}^{2} \left(\int_0^1 \left(\widetilde{B}_{n,k}(t) \right)^{q_3} dt \right)^{\frac{1}{q_3}} \left[\frac{n^{\frac{nq_3}{q_3-1}+1}}{2^{\frac{q_3(n-1)}{q_3-1}+1}} \beta_{\frac{2}{n}} \left(\frac{nq_3-1}{q_3-1}, \frac{2q_3-1}{q_3-1} \right) \right]^{1-\frac{1}{q_3}} & \textit{for } n \geq 2. \end{cases} \tag{2.9}$$

Proof. Since $g(b)+t\theta(g(a),g(b)) \in A$ for every $t \in (0,1)$, by Lemma 1, Hölder's inequality and the logarithmically relative $h-$preinvexity of $|f^{(n)}|$, we have

$$\Bigg| \frac{1}{2}\left[\frac{f(g(b))+f(g(b)+\theta(g(a),g(b)))}{2} + f\left(\frac{2g(b)+\theta(g(a),g(b))}{2} \right) \right] - \frac{1}{\theta(g(a),g(b))} \int_{g(b)}^{g(b)+\theta(g(a),g(b))} f(g(u))\,dg(u)$$
$$- \sum_{k=2}^{n-1} \frac{[\theta(g(a),g(b))]^k (k-1)}{(k+1)! \times 2^{k+2}} \left[f^{(k)}(g(b)) + f^{(k)}\left(\frac{2g(b)+\theta(g(a),g(b))}{2} \right) \right] \Bigg|$$
$$\leq \frac{|\theta(g(a),g(b))|^n}{n! \times 2^{n+2}} \int_0^1 \left| (n-2t)t^{n-1} \right| \left[\left| f^{(n)}\left(g(b) + \frac{1-t}{2}\theta(g(a),g(b)) \right) \right| + \left| f^{(n)}\left(g(b) + \frac{2-t}{2}\theta(g(a),g(b)) \right) \right| \right] dt$$

$$\leq \frac{|\theta(g(a),g(b))|^n}{n! \times 2^{n+2}} \left(\int_0^1 \left| (n-2t)t^{n-1} \right|^{\frac{q_3}{q_3-1}} dt \right)^{1-\frac{1}{q_3}}$$
$$\times \left\{ \left[\int_0^1 \left| f^{(n)}\left(g(b) + \frac{1-t}{2}\theta(g(a),g(b)) \right) \right|^{q_3} dt \right]^{\frac{1}{q_3}} + \left[\int_0^1 \left| f^{(n)}\left(g(b) + \frac{2-t}{2}\theta(g(a),g(b)) \right) \right|^{q_3} dt \right]^{\frac{1}{q_3}} \right\}$$
$$\leq \frac{|\theta(g(a),g(b))|^n}{n! \times 2^{n+2}} \left(\int_0^1 \left| (n-2t)t^{n-1} \right|^{\frac{q_3}{q_3-1}} dt \right)^{1-\frac{1}{q_3}}$$
$$\times \sum_{k=1}^{2} \left(\int_0^1 \left(\left| f^{(n)}(g(b)) \right| \left| \frac{f^{(n)}(g(a))}{f^{(n)}(g(b))} \right|^{h\left(\frac{k-t}{2}\right)} \right)^{q_3} dt \right)^{\frac{1}{q_3}}$$

□

Corollary 4. *Under the conditions of theorem 4 for $n=2$, we have*

$$\left|\frac{1}{2}\left[\frac{f\left(g(b)\right)+f\left(g(b)+\theta\left(g(a),g(b)\right)\right)}{2}+f\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right.$$
$$\left.-\frac{1}{\theta\left(g(a),g(b)\right)}\int_{g(b)}^{g(b)+\theta(g(a),g(b))} f\left(g(u)\right)dg(u)\right|$$
$$\leq 2\widetilde{A}_2\sum_{k=1}^{2}\left(\int_0^1 \left(\widetilde{B}_{2,k}(t)\right)^{q_3}dt\right)^{\frac{1}{q_3}}\left[\beta\left(\frac{2q_3-1}{q_3-1},\frac{2q_3-1}{q_3-1}\right)\right]^{1-\frac{1}{q_3}}$$

Theorem 5. *Let $A\subseteq \mathbf{R}$ be a relative invex subset with respect to $\theta:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$, with $a,b\in A$ and $\theta(g(a),g(b))\neq 0$. Suppose that $f:A\to(0,\infty)$ is $n-$times differentiable function and $f^{(n)}$ is integrable on the $\theta-$relative path $P_{g(b);g(c)}$, such that $|f^{(n)}|$ is logarithmically relative $h-$preinvex on A; let $q_3\geq 1$, then*

$$\left|\frac{1}{2}\left[\frac{f\left(g(b)\right)+f\left(g(b)+\theta\left(g(a),g(b)\right)\right)}{2}+f\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right.$$
$$-\frac{1}{\theta\left(g(a),g(b)\right)}\int_{g(b)}^{g(b)+\theta(g(a),g(b))} f\left(g(u)\right)dg(u)$$
$$\left.-\sum_{k=2}^{n-1}\frac{\left[\theta\left(g(a),g(b)\right)\right]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}\left(g(b)\right)+f^{(k)}\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right|$$
$$\leq\begin{cases}\dfrac{\widetilde{A}_1\sum_{k=1}^{2}\left(\int_0^1|(n-2t)t^{n-1}|\left(\widetilde{B}_{1,k}(t)\right)^{q_3}dt\right)^{\frac{1}{q_3}}}{2^{1-\frac{1}{q_3}}} & \text{for } n=1\\ \widetilde{A}_n\sum_{k=1}^{2}\left(\int_0^1|(n-2t)t^{n-1}|\left(\widetilde{B}_{n,k}(t)\right)^{q_3}dt\right)^{\frac{1}{q_3}}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q_3}} & \text{for } n\geq 2.\end{cases}$$

Proof. By using the weighted power mean inequality and the similar techniques used in theorem 4, we can prove this theorem. □

Corollary 5. *Under the conditions of theorem 5 for $q_3=1$, we have*

$$\left|\frac{1}{2}\left[\frac{f\left(g(b)\right)+f\left(g(b)+\theta\left(g(a),g(b)\right)\right)}{2}+f\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right.$$
$$-\frac{1}{\theta\left(g(a),g(b)\right)}\int_{g(b)}^{g(b)+\theta(g(a),g(b))} f\left(g(u)\right)dg(u)$$
$$\left.-\sum_{k=2}^{n-1}\frac{\left[\theta\left(g(a),g(b)\right)\right]^k(k-1)}{(k+1)!\times 2^{k+2}}\left[f^{(k)}\left(g(b)\right)+f^{(k)}\left(\frac{2g(b)+\theta\left(g(a),g(b)\right)}{2}\right)\right]\right|$$
$$\leq\begin{cases}\widetilde{A}_1\sum_{k=1}^{2}\int_0^1|(n-2t)t^{n-1}|\,\widetilde{B}_{1,k}(t)dt & \text{for } n=1\\ \widetilde{A}_n\sum_{k=1}^{2}\int_0^1|(n-2t)t^{n-1}|\,\widetilde{B}_{n,k}(t)dt & \text{for } n\geq 2.\end{cases}$$

Remark 7. If $g(t) = I$, where I is an identity function, then our results coincide the results for logarithmically $h-$preinvex functions.

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REFERENCES

[1] C. Fulga and V. Preda, "Nonlinear programming with E-preinvex and local E-preinvex functions," *Eur. J. Oper. Reseach.*, vol. 192, no. 3, pp. 737–743, 2009, doi: 10.1016/j.ejor.2007.11.056.

[2] S. Hussain and S. Rafeeq, "Some new Hermite-Hadamard type inequalities for functions whose $n-$th derivatives are $\alpha-$relative preinvex," *Submitted*.

[3] U. S. Kirmaci and M. E. Özdemir, "On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula," *Appl. Math. Comp.*, vol. 153, no. 2, pp. 361–368, 2004, doi: 10.1016/S0096-3003(03)00637-4.

[4] M. A. Latif, "On Hermite-Hadamard type integral inequalities for $n-$times differentiable preinvex functions with applications," *Stud. Univ. Babes-Bolyai Math.*, vol. 58, no. 3, pp. 325–343, 2013.

[5] M. A. Latif and S. S. Dragomir, "Some weighted integral inequalities for differentiable preinvex and prequasiinvex functions with applications," *J. Inequal. Appl.*, no. 575, 2013, doi: 10.1186/1029-242X-2013-575.

[6] M. A. Noor, "Invex equilibrium problems," *J. Math. Anal. Appl.*, vol. 302, no. 2, pp. 463–475, 2005, doi: 10.1016/j.jmaa.2004.08.014.

[7] M. A. Noor, "Hermite-Hadamard integral inequalities for log-preinvex functions," *J. Math. Anal. Approx. Theory*, no. 2, pp. 126–131, 2007.

[8] M. A. Noor, "Hadamrd integral inequalities for product of two preinvex function," *Nonl. Anal. forum*, no. 14, pp. 167–173, 2009.

[9] M. A. Noor, K. I. Noor, M. U. Awan, and F. Qi, "Integral inequalities of Hermite-Hadamard type for logarithmically $h-$preinvex functions," *Cogent Math.*, vol. 2, no. 1035856, pp. 1–10, 2015, doi: 10.1080/23311835.2015.1035856.

[10] C. E. M. Pearce and J. Pecaric, "Inequalities for differentiable mappings with application to sprcial means and quadrature formulae," *Appl. Math. Lett.*, vol. 13, no. 2, pp. 51–55, 2000, doi: 10.1016/S0893-9659(99)00164-0.

[11] M. Z. Sarikaya, N. Alp, and H. Bozkurt, "On Hermite-Hadamard type Integral inequalities for preinvex and log-preinvex functions," *Contemp. Anal. Appl. Math.*, vol. 1, no. 2, pp. 237–252, 2013.

[12] S. Varosanec, "On $h-$convexity," *J. Math. Anal. Appl.*, vol. 326, no. 1, pp. 303–311, 2007, doi: 10.1016/j.jmaa.2006.02.086.

[13] S.-H. Wang and F. Qi, "Hermite-Hadamrd type inequalities for $n-$times differentiable and preinvex functions," *J. Inequal. Appl.*, 2014, doi: 10.1186/1029-242X-2014-49.

[14] Y. Wang, M.-M. Zheng, and F. Qi, "Integral inequalities of Hermite-Hadamard type for functions whose derivatives are $\alpha-$preinvex," *J. Inequal. Appl.*, no. 97, pp. 1–10, 2014, doi: 10.1186/1029-242X-2014-97.

[15] T. Weir and B. Mond, "Preinvex functions in multiobjective optimization," *J. Math. Anal. Appl.*, vol. 136, no. 1, pp. 29–38, 1988, doi: 10.1016/0022-247X(88)90113-8.

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