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ON THE FINITENESS OF BASS NUMBERS OF LOCAL COHOMOLOGY MODULES OVER NOETHERIAN REGULAR LOCAL RINGS

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Abstract. Let R be a commutative Noetherian regular local ring containing a field. Let I be an ideal of R and let $\ell \geq 0$ be an integer. In this paper it is shown that for every finitely generated R-module M and each integer $i \geq \ell$, the Bass numbers of the R-module $H^i_I(M)$ are finite, whenever, $\dim \operatorname{Supp}(H^i_I(R)) \leq 1$, for all $i \geq \ell$.

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1. INTRODUCTION

Let R denote a commutative Noetherian ring and let I be an ideal of R. In [10], Hartshorne defined an R-module L to be I*-cofinite*, if $\operatorname{Supp}(L) \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, L)$ is finitely generated module for all i. He posed the following question:

Is the category $\mathscr{M}(R,I)_{cof}$ of I-cofinite modules forms an Abelian subcategory of the category of all R-modules? That is, if $f : M \longrightarrow N$ is an R-homomorphism of I-cofinite modules, are $\ker f$ *and* $\operatorname{coker} f$ *I-cofinite?*

Hartshorne proved that if I is a prime ideal of dimension one in a complete regular local ring R, then the answer to his question is yes. On the other hand, in [7], Delfino and Marley extended this result to arbitrary complete local rings. Kawasaki [13] generalized the Delfino and Marley's result for an arbitrary ideal I of dimension one in a local ring R. Recently, Bahmanpour et al in [4] and Melkersson in [19] have removed the local assumption on R.

Recall that, we say that M is a *minimax* module if there is a finitely generated submodule N of M, such that M/N is Artinian. The interesting class of minimax modules was introduced by H. Zöshinger in [20] and he has given in [20] and [21] many equivalent conditions for a module to be minimax. Also, the R-module M is

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said to be I*-cominimax*, if support of M is contained in $V(I)$ and $\operatorname{Ext}^i_R(R/I, M)$ is minimax for all $i \geq 0$. The concept of the I-cominimax modules were introduced in [2] as a generalization of important notion of I-cofinite modules.

In this paper as a generalization the main results of [4] and [19] to the class of cominimax modules we prove the following:

Let $(R, \mathfrak{m})$ *be a Noetherian local ring and let* I *be an ideal of* R *of dimension one. Let* $\mathscr{M}(R, I)_{com}$ *denote the category of* I*-cominimax modules. Then* $\mathscr{M}(R, I)_{com}$ *forms an Abelian subcategory of the category of all* R*-modules.*

On the other hand, an important problem in commutative algebra is determining when the set of associated primes and the Bass numbers of the i^{th} local cohomology module $H^i_I(M)$ are finite. Lyubeznik conjectured that,

If R *is a regular ring and* I *is an ideal of* R*, then the local cohomology modules* $H^i_I(R)$ *have finitely many associated prime ideals for all* $i \geq 0$*,* (see [14, Remark 3.7(iii)]).

This conjecture have solved with an affirmative answer by Huneke and Sharp [12] and Lyubeznik in [14] and [15] for regular rings containing a field. Also, in [11] Huneke conjectured that,

For any ideal I *in a regular local ring* $(R, \mathfrak{m})$*, the Bass numbers*

$$\mu^j(\mathfrak{p}, H^i_I(R)) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^j_{R_\mathfrak{p}}(k(\mathfrak{p}), H^i_{IR_\mathfrak{p}}(R_\mathfrak{p}))$$

are finite for all i *and* j*, where* $k(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$*. In particular the injective resolution of the local cohomology has only finitely many copies of the injective hull of* $R/\mathfrak{p}$ *for any* $\mathfrak{p}$*.*

Huneke and Sharp [12] and Lyubeznik [14, 15] have shown that this conjecture holds for any regular local ring containing a field. But both of this conjectures are still open for the general case. Also, in some situations the second conjecture holds even for every finitely generated R-module instead of the ring R. (For example, see [1]).

The main purpose of this note is to prove the following:

Let $(R, \mathfrak{m})$ *be a Noetherian regular local ring containing a field and let* I *be an ideal of* R*. Let* $\ell \geq 0$ *be an integer such that* $\dim \operatorname{Supp}(H^i_I(R)) \leq 1$*, for each* $i \geq \ell$*. Then for every finitely generated* R*-module* M *and for each integer* $i \geq \ell$*, the set* $\operatorname{Supp}(H^i_I(M))$ *is finite and the Bass numbers of the* R*-module* $H^i_I(M)$ *are finite.*

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and I will be an ideal of R. For any R-module M, the i^{th} local cohomology module of M with support in I is defined by

$$H^i_I(M) = \varinjlim_{n \geq 1} \operatorname{Ext}^i_R(R/I^n, M).$$

We refer the reader to [9] or [6] for the basic properties of local cohomology.

For any ideal $\mathfrak{a}$ of R, we denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. For any unexplained notation and terminology we refer the reader to [6] and [17].

2. Finiteness of Bass numbers of local cohomology modules

The following well known lemmata are crucial for the proof of Theorem 1.

Lemma 1. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and let* I *be an ideal of* R *with* $\dim R/I = 1$. *Assume that* M *is an* R*-module and* $n \geq 0$ *is an integer. Then the following conditions are equivalent:*

(1) $\mu^i(\mathfrak{p}, M)$ *is finite for all* $\mathfrak{p} \in V(I)$ *and for all* $0 \leq i \leq n$;
(2) *The* R*-module* $\operatorname{Ext}^i_R(R/I, M)$ *is minimax, for all integers* $0 \leq i \leq n$.

Proof. See [5, Theorem 2.3]. □

Recall that for an R-module M, the *cohomological dimension of* M *with respect to ideal* I *of* R, is defined as

$$\operatorname{cd}(I, M) := \max\{i \in \mathbb{Z} : H^i_I(M) \neq 0\}.$$

Lemma 2. *Let* R *be a Noetherian ring and let* M *be an* R*-module of finite dimension* d. *Then the followings are equivalent:*

(1) *For each* $\mathfrak{p} \in \operatorname{Spec}(R)$ *and any integer* $0 \leq i \leq d$, *the Bass numbers* $\mu^i(\mathfrak{p}, M)$ *are finite;*
(2) *For each* $\mathfrak{p} \in \operatorname{Spec}(R)$ *and any integer* $i \geq 0$, *the Bass numbers* $\mu^i(\mathfrak{p}, M)$ *are finite.*

Proof. (2)⇒(1) Is clear.

(1)⇒(2) Using localization we may assume $(R, \mathfrak{m}, k)$ is a local Noetherian ring and $\mathfrak{p} = \mathfrak{m}$ is the unique maximal ideal of R. As by [6, Corollary 10.2.8] the R-module $E_R(k)$, the injective hull of k, is Artinian it follows from the hypothesis and from the definition of local cohomology modules that for any $0 \leq i \leq d$ the R-module $H^i_{\mathfrak{m}}(M)$ is Artinian. But in view of [6, Theorem 6.1.2] we have $H^i_{\mathfrak{m}}(M) = 0$ for all integers $i > d$. Therefore for all integers $i \geq 0$ the R-modules $H^i_{\mathfrak{m}}(M)$ are Artinian and hence are $\mathfrak{m}$-cofinite. So the assertion follows from the [18, Proposition 3.9]. □

Lemma 3. *Let* R *be a Noetherian ring and let* $\mathcal{C}^1_B(R)$ *denote the category of all* R*-modules* M *with* $\dim \operatorname{Supp}(M) \leq 1$ *such that all Bass numbers of* M *are finite. Then* $\mathcal{C}^1_B(R)$ *forms an Abelian subcategory of the category of all* R*-modules.*

Proof. Let $M, N \in \mathcal{C}^1_B(R)$ and let $f : M \to N$ be an R-homomorphism. It is enough to prove that the R-modules $\ker(f)$ and $\operatorname{coker}(f)$ are in $\mathcal{C}^1_B(R)$.

Now, the exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow M \longrightarrow \operatorname{im}(f) \longrightarrow 0,$$

implies that for each $\mathfrak{p} \in \operatorname{Spec}(R)$ and any $0 \leq i \leq 1$ the Bass numbers $\mu^i_R(\mathfrak{p}, \ker(f))$ are finite. So, by Lemma 2 it follows that $\ker(f)$ is in $\mathcal{C}^1_B(R)$. Now the reminder section of the proof follows from the exact sequences

$$0 \longrightarrow \ker(f) \longrightarrow M \longrightarrow \operatorname{im}(f) \longrightarrow 0,$$

and

$$0 \longrightarrow \operatorname{im}(f) \longrightarrow N \longrightarrow \operatorname{coker}(f) \longrightarrow 0.$$

□

Now, we are ready to state and prove our first main result.

Theorem 1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R with* $\dim R/I = 1$. *Let $\mathscr{M}(R, I)_{com}$ denote the category of I-cominimax modules over R. Then $\mathscr{M}(R, I)_{com}$ forms an Abelian subcategory of the category of all R-modules.*

Proof. The assertion follows from Lemma 1 and 3. □

The following lemma plays a key role in the proof of Proposition 1.

Lemma 4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R. Let M be a finitely generated R-module of dimension d. Let $\ell \geq 0$ be an integer such that* $\operatorname{cd}(I, M) \geq \ell$. *Assume that, for each $i \geq \ell$, the set* $\operatorname{Supp}(H_I^i(M))$ *is finite and the Bass numbers of the R-module $H_I^i(M)$ are finite. Then, for each $i \geq \ell$, the R-module $H_I^i(M)$ is J-cominimax, where,*

$$J := \bigcap_{\mathfrak{p} \in \operatorname{Supp}(\oplus_{i \geq \ell} H_I^i(M))} \mathfrak{p}.$$

Proof. From the hypothesis $\operatorname{cd}(I, M) \geq \ell$ it follows that $\operatorname{Supp}(\oplus_{i \geq \ell} H_I^i(M)) \neq \varnothing$. Also since by the hypothesis, for each $\ell \geq 0$, the set $\operatorname{Supp}(H_I^i(M))$ is finite and by Grothendiek's Vanishing Theorem, for every $i > d$, we have $H_I^i(M) = 0$, it follows that the set $\operatorname{Supp}(\oplus_{i \geq \ell} H_I^i(M))$ is finite and hence $\dim(R/J) \leq 1$. Now if $\dim(R/J) = 0$, then we have $J = \mathfrak{m}$ and so the assertion is clear. So, we may assume $\dim(R/J) = 1$. But in this situation the assertion follows immediately from Lemma 1. (Note that $\operatorname{Supp}(H_I^i(M)) \subseteq V(J)$, for each $i \geq \ell$). □

The following lemma and its corollary is needed in the proof of Proposition 1.

Lemma 5. *Let R be a Noetherian ring and let I be an ideal of R. Let M and N be two finitely generated R-modules and let $\ell \geq 0$ be an integer. If* $\operatorname{Supp}(N) \subseteq \operatorname{Supp}(M)$, *then we have*

$$\bigcup_{i \geq \ell} \operatorname{Supp}(H_I^i(N)) \subseteq \bigcup_{i \geq \ell} \operatorname{Supp}(H_I^i(M)).$$

Proof. Let $\mathfrak{p} \in \cup_{i \geq \ell} \operatorname{Supp}(H_I^i(N))$. Then there exists an integer $j \geq \ell$ such that $(H_I^j(N))_{\mathfrak{p}} \neq 0$, which implies that $H_{IR_{\mathfrak{p}}}^j(N_{\mathfrak{p}}) \neq 0$. Therefore, it follows from the definition that $\operatorname{cd}(IR_{\mathfrak{p}}, N_{\mathfrak{p}}) \geq j$. Now as $\operatorname{Supp}(N_{\mathfrak{p}}) \subseteq \operatorname{Supp}(M_{\mathfrak{p}})$ it follows from [8, Theorem 2.2], that $\operatorname{cd}(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) \geq \operatorname{cd}(IR_{\mathfrak{p}}, N_{\mathfrak{p}}) \geq j \geq \ell$. So, if we have $\operatorname{cd}(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) = t$, then we have $H_{IR_{\mathfrak{p}}}^t(M_{\mathfrak{p}}) \neq 0$. Therefore, $(H_I^t(M))_{\mathfrak{p}} \neq 0$ and hence $\mathfrak{p} \in \cup_{i \geq \ell} \operatorname{Supp}(H_I^i(M))$. □

Corollary 1. *Let R be a Noetherian ring and let I be an ideal of R. Let M a finitely generated R-modules and $\ell \geq 0$ be an integer. Then*

$$\bigcup_{i\geq \ell} \operatorname{Supp}(H^i_I(M)) \subseteq \bigcup_{i\geq \ell} \operatorname{Supp}(H^i_I(R)).$$

Proof. Since $\operatorname{Supp}(M) \subseteq \operatorname{Spec}(R) = \operatorname{Supp}(R)$, the assertion follows immediately from Lemma 5. □

The following proposition is crucial for the proof of Theorem 2.

Proposition 1. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R. Let $\ell \leq \operatorname{cd}(I,R)$ be an integer such that for each $i \geq \ell$, the set $\operatorname{Supp}(H^i_I(R))$ is finite and the Bass numbers of the R-module $H^i_I(R)$ are finite. Then the following statements hold:*

(1) *For every finitely generated R-module M with finite projective dimension and each integer $i \geq \ell$, the R-module $H^i_I(M)$ is J-cominimax, where,*

$$J := \bigcap_{\mathfrak{p}\in \operatorname{Supp}(\oplus_{i\geq \ell} H^i_I(R))} \mathfrak{p}.$$

(2) *For every finitely generated R-module M with finite projective dimension and for each integer $i \geq \ell$, the set $\operatorname{Supp}(H^i_I(M))$ is finite and the Bass numbers of the R-module $H^i_I(M)$ are finite.*

Proof. (1) For every finitely generated R-module M, by the Grothendiek's Vanishing Theorem we have $H^i_I(M) = 0$, for each integer $i > \dim(R)$ and so we have $\ell \leq \dim(R)$. Now, we argue by induction on $t := \dim(R) - \ell$. If $t = 0$, then for every finitely generated R-module M (even not necessary with finite projective dimension), using Grothendiek's Vanishing Theorem and [18, Proposition 5.1], it is straightforward to see that, the R-module $H^{\dim(R)}_I(M)$ is I-cofinite and hence is I-cominimax. Now, if $H^{\dim(R)}_I(M) \neq 0$, then the assertion follows from Corollary 1 and [2, Corollary 2.8]. (Note that in this situation we have $\{\mathfrak{m}\} = \operatorname{Supp}(H^{\dim(R)}_I(M)) = V(J) \subseteq V(I)$). Now, let $t > 0$ and inductively, the assertion has been proved for all smaller values of t for all finitely generated R-modules with finite projective dimension. Then by inductive hypothesis, for every finitely generated R-module M with finite projective dimension and each $i \geq \ell + 1$, the R-module $H^i_I(M)$ is J_1-cominimax, where,

$$J_1 := \bigcap_{\mathfrak{p}\in \operatorname{Supp}(\oplus_{i\geq \ell+1} H^i_I(R))} \mathfrak{p}.$$

Then as by the hypothesis the set $\operatorname{Supp}(H^\ell_I(R))$ is finite, it follows that $\dim(R/\mathfrak{q}) \leq 1$ for every $\mathfrak{q} \in \operatorname{Supp}(H^\ell_I(R))$ and so using Lemma 1, it follows that, for every finitely generated R-module M with finite projective dimension and each $i \geq \ell+1$, the R-module $H_I^i(M)$ is J-cominimax, where,

$$J := \bigcap_{\mathfrak{p} \in \mathrm{Supp}(\oplus_{i \geq \ell} H_I^i(R))} \mathfrak{p}.$$

(Note that $\dim(R/J) \leq 1$). Next, let M be an arbitrary non-zero finitely generated R-module with finite projective dimension. Then, we argue on $s := \mathrm{projdim}_R(M)$ that, the R-module $H_I^\ell(M)$ is J-cominimax. For $s = 0$, the assertion follows from Lemma 4. Now let $s > 0$ and the result has been proved for all finitely generated R-modules with finite projective dimension smaller than s. Let M be a finitely generated R-module with $\mathrm{projdim}_R(M) = s$. Then there is an exact sequence

$$0 \to K \to F \to M \to 0, \tag{2.1}$$

for some finitely generated free R-modules F and some finitely generated R-modules K with $\mathrm{projdim}_R(K) = s-1$. Then by inductive hypothesis of the second inductive argument, the R-modules $H_I^\ell(K)$ and $H_I^\ell(F)$ are J-cominimax. Moreover, by the inductive hypothesis of the first inductive argument, the R-modules $H_I^{\ell+1}(K)$ and $H_I^{\ell+1}(F)$ are J-cominimax. On the other hand, the exact sequence 2.1 induces the following exact sequence

$$H_I^\ell(K) \xrightarrow{f} H_I^\ell(F) \to H_I^\ell(M) \to H_I^{\ell+1}(K) \xrightarrow{g} H_I^{\ell+1}(F). \tag{2.2}$$

Now the exact sequence 2.2 yields the exact sequence

$$0 \to \mathrm{Coker}(f) \to H_I^\ell(M) \to \mathrm{Ker}(g) \to 0. \tag{2.3}$$

On the other hand, the exact sequence 2.3 induces the exact sequence

$$0 \to \mathrm{Hom}_R(R/J, \mathrm{Coker}(f)) \to \mathrm{Hom}_R(R/J, H_I^\ell(M)) \to \mathrm{Hom}_R(R/J, \mathrm{Ker}(g))$$

$$\to \mathrm{Ext}_R^1(R/J, \mathrm{Coker}(f)) \to \mathrm{Ext}_R^1(R/J, H_I^\ell(M)) \to \mathrm{Ext}_R^1(R/J, \mathrm{Ker}(g)) \to \cdots,$$

which using [3, Lemma 2.1] and Theorem 1, implies that the R-module $H_I^\ell(M)$ is J-cominimax. This completes the second inductive step. Now the inductive step of the first inductive argument is complete, too.

(2) The assertion follows immediately from (i), using Lemma 1 and Corollary 1. □

Now we are ready to state and prove our second main result.

Theorem 2. *Let $(R, \mathfrak{m})$ be a Noetherian local regular ring containing a field and let I be an ideal of R. Let $0 \leq \ell \leq \mathrm{cd}(I,R)$ be an integer such that $\dim \mathrm{Supp}(H_I^i(R)) \leq 1$, for each $i \geq \ell$. Then the following statements hold:*

(1) *For every finitely generated R-module M and each integer $i \geq \ell$, the R-module $H_I^i(M)$ is J-cominimax, where,*

$$J := \bigcap_{\mathfrak{p}\in \operatorname{Supp}(\oplus_{i\geq \ell} H_I^i(R))} \mathfrak{p}.$$

(2) *For every finitely generated R-module M and for each integer $i \geq \ell$, the set $\operatorname{Supp}(H_I^i(M))$ is finite and the Bass numbers of the R-module $H_I^i(M)$ are finite.*

Proof. In view of [12] and [14, 15], the Bass numbers of the R-module $H_I^i(R)$ are finite, for all i. Moreover, in view of [12] and [14, 16], the set $\operatorname{Ass}_R(H_I^i(R))$ is finite, for all i. In particular, if $\dim \operatorname{Supp}(H_I^i(R)) \leq 1$ then the set $\operatorname{Supp}(H_I^i(R))$ is finite. Now, as $(R,\mathfrak{m})$ is a regular ring, it follows that any finitely generated R-module has finite projective dimension. Now, we can conclude the assertion immediately from Proposition 1. □

Corollary 2. *Let $(R,\mathfrak{m})$ be a Noetherian regular local ring of dimension $d \geq 1$, containing a field. Then for any finitely generated R-module M, the Bass numbers of the local cohomology module $H_I^i(M)$ are finite, for all integers $i \geq d-1$.*

Proof. In view of the main results of [12], [14] and [15], the set $\operatorname{Ass}_R(H_I^i(R))$ is finite, for all integers $i \geq d-1$. Also, in view of Grothendiek's Vanishing Theorem we have $\dim \operatorname{Supp}(H_I^i(R)) \leq 1$, for all integers $i \geq d-1$. Now, it is clear that $\dim \operatorname{Supp}(H_I^i(R)) \leq 1$, for each $i \geq d-1$ and hence the assertion follows from Theorem 2. □

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