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BOUNDARY VALUE PROBLEMS FOR BAGLEY–TORVIK FRACTIONAL DIFFERENTIAL EQUATIONS AT RESONANCE

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Abstract. We investigate the nonlocal fractional boundary value problem $u'' = A\,{}^cD^\alpha u + f(t,u,{}^cD^\mu u,u')$, $u'(0) = u'(T)$, $\Lambda(u) = 0$, at resonance. Here, $\alpha \in (1,2)$, $\mu \in (0,1)$, f and $\Lambda : C[0,T] \to \mathbb{R}$ are continuous. We introduce a "three-component" operator $\mathcal{S}$ which first component is related to the fractional differential equation and remaining ones to the boundary conditions. Solutions of the problem are given by fixed points of $\mathcal{S}$. The existence of fixed points of $\mathcal{S}$ is proved by the Leray–Schauder degree method.

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1. INTRODUCTION

Let $T > 0$ be given and $J = [0,T]$. Denote by $\mathcal{A}$ the set of (generally nonlinear) functionals $\Lambda : C(J) \to \mathbb{R}$ which are

(a) continuous, $\Lambda(0) = 0$,

(b) increasing, that is, $x, y \in C(J)$, $x(t) < y(t)$ on $J \Rightarrow \Lambda(x) < \Lambda(y)$.

Remark 1. Let $\Lambda \in \mathcal{A}$ be linear. Then it follows from property (b) of Λ that Λ takes bounded sets into bounded sets. Hence Λ is a linear bounded functional.

Example 1. Let $p \in C(J)$ be positive, $n \in \mathbb{N}$, $0 \le t_0 < t_1 < \cdots < t_n \le T$, and $a_k > 0$, $k = 0,1,\ldots,n$. Then the functionals

$$\Lambda_1(x) = \max\{x(t) : t \in J\}, \quad \Lambda_2(x) = \min\{x(t) : t \in J\},$$

$$\Lambda_3(x) = \int_0^T p(s)(x(s))^{2n-1}\,\mathrm{d}s, \quad \Lambda_4(x) = \sum_{k=0}^{n} a_k x(t_k)$$

and their linear combinations with positive coefficients belong to the set $\mathcal{A}$.

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We discuss the fractional boundary value problem

$$u''(t) = A^cD^\alpha u(t) + f(t,u(t),{}^cD^\mu u(t),u'(t)), \tag{1.1}$$

$$u'(0) = u'(T), \quad \Lambda(u) = 0, \quad \Lambda \in \mathcal{A}. \tag{1.2}$$

Here, ${}^cD$ denotes the Caputo fractional derivative, $A \in \mathbb{R}$, $\alpha \in (1,2)$, $\mu \in (0,1)$, and the function f satisfies the condition:

(H) there exists $\Delta > 0$ such that $f \in C(\mathcal{D} \times [-\Delta,\Delta])$, where

$$\mathcal{D} = J \times [-\Delta T, \Delta T] \times [-\Delta K, \Delta K]), \quad K = \frac{T^{1-\mu}}{\Gamma(2-\mu)},$$

and

$$f(t,x,y,-\Delta) \le 0, \quad f(t,x,y,\Delta) \ge 0 \quad \text{for } (t,x,y) \in \mathcal{D}.$$

Equation 1.1 is the fractional differential equation of the Bagley-Torvik type. Its special case is the equation $u'' = A^cD^{3/2}u + au + \varphi(t)$. This equation with ${}^cD^{3/2}$ replaced by the Riemann–Liouville fractional derivative $D^{3/2}$ is called the Bagley–Torvik equation. Torvik and Bagley [22] used this equation in modelling the motion of a rigid plate immersing in a Newtonian fluid. Analytical and numerical solutions of the problem

$$u'' = AD^{3/2}u + au + \varphi(t), \quad u(0) = 0, \ u'(0) = 0,$$

are given in [13, 16, 18], while for the problem

$$u'' = A^cD^\alpha u + au + \varphi(t), \quad u(0) = u_0, \ u'(0) = u_1,$$

in [5, 6, 8, 11, 23]. The existence results for solutions of the generalized Bagley–Torvik equation (1.1) satisfying the boundary conditions $u'(0) = 0$, $u(T) + au'(T) = 0$ are given in [20]. Here, f is a Carathéodory function.

Definition 1. We say that $u \in C^2(J)$ is a solution of problem (1.1), (1.2) if u satisfies the boundary conditions (1.2) and (1.1) holds for $t \in J$.

We recall that *the Riemann–Liouville fractional integral* I^γ of order $\gamma > 0$ of a function $x \colon J \to \mathbb{R}$ is defined as [10, 13, 16]

$$I^\gamma x(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} x(s)\,\mathrm{d}s,$$

and *the Caputo fractional derivative* ${}^cD^\gamma x$ of order $\gamma > 0$, $\gamma \notin \mathbb{N}$, of a function $x \colon J \to \mathbb{R}$ is given by the formula [10, 13]

$$^cD^\gamma x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)} \left(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} s^k \right) \mathrm{d}s,$$

where $n = [\gamma] + 1$, $[\gamma]$ means the integral part of γ and Γ is the Euler gamma function. If $x \in C^n(J)$ and $n-1 < \gamma < n$, then

$$^cD^\gamma x(t) = \int_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)} x^{(n)}(s)\,\mathrm{d}s = I^{n-\gamma}x^{(n)}(t).$$

In particular, if $x \in C^2(J)$ and $\alpha \in (1,2)$, $\mu \in (0,1)$, then

$$^cD^\alpha x(t) = \int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)} x''(s)\,\mathrm{d}s, \quad t \in J,$$

$$^cD^\alpha x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)} (x'(s)-x'(0))\,\mathrm{d}s = {}^cD^{\alpha-1}x'(t), \quad t \in J,$$

$$^cD^\mu x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{(t-s)^{-\mu}}{\Gamma(1-\mu)} (x(s)-x(0))\,\mathrm{d}s = I^{1-\mu}x'(t), \quad t \in J.$$

It is well known [10, 13] that $I^\gamma : C(J) \to C(J)$ for $\gamma \in (0,1)$. Therefore, if $x \in C^2(J)$, then ${}^cD^\alpha x, {}^cD^\mu x \in C(J)$ for $\alpha \in (1,2)$ and $\mu \in (0,1)$.

We will show that problem (1.1), (1.2) is at resonance. The linear function $x(t) = at + b$ is a solution of the problem $u'' - A{}^cD^\alpha u = 0$, $u'(0) = u'(T)$, for each $a, b \in \mathbb{R}$. Let us consider the set of all functions $at + b$ which are solutions of the equation $\Lambda(at+b) = 0$, where Λ is from (1.2).

If Λ is linear, then $b = -\frac{a\Lambda(t)}{\Lambda(1)}$. Hence $\left\{a\left(t - \frac{\Lambda(t)}{\Lambda(1)}\right) : a \in \mathbb{R}\right\}$ is the set of solutions to problem $u'' - A{}^cD^\alpha u = 0$, (1.2). This set is a one-dimensional linear subspace of $C^2(J)$.

Let Λ be nonlinear. If $a = 0$, then $b = 0$. Let $a \in \mathbb{R} \setminus \{0\}$. By our Lemma 1 (for $\lambda = 1$), there exists $\xi_a \in J$ such that $a\xi_a + b = 0$. Hence $b = -a\xi_a$ and the equality $\Lambda(a(t-\xi_a)) = 0$ is true. ξ_a is determined uniquely. If this is not true, then there exists $\rho_a \in J$, $\rho_a \neq \xi_a$, such that $\Lambda(a(t-\rho_a)) = 0$. Since $a(t-\xi_a) \neq a(t-\rho_a)$ for all $t \in J$, and therefore either $a(t-\xi_a) < a(t-\rho_a)$ or $a(t-\xi_a) > a(t-\rho_a)$ on J, it follows from property (b) of Λ that $\Lambda(a(t-\xi_a)) \neq \Lambda(a(t-\rho_a))$, which is impossible. Consequently, $u = 0$ and $\{a(t-\xi_a) : a \in \mathbb{R} \setminus \{0\}\}$ is the set of solutions to the problem $u'' - A{}^cD^\alpha u = 0$, (1.2). In contrast to previous case, this set is not a one-dimensional linear subspace of $C^2(J)$.

In order to show the solvability of problem (1.1), (1.2), we have to overcome troubles that derivatives are of fractional order, the problem is at resonance and finally that Λ in the boundary conditions (1.2) is generally a nonlinear functional. To this end, an auxiliary "three-component" operator $\mathcal{S}$ is introduced. Its first component is related to equation (1.1) and remaining ones to the boundary conditions (1.2). Solutions of (1.1), (1.2) are given by fixed points of $\mathcal{S}$. The existence of fixed points of $\mathcal{S}$ is proved by means of the Leray-Schauder degree method [7].

In the literature, see [1–4, 12, 14, 19] and references therein, existence results for fractional boundary value problems at resonance are usually proved by using the the coincidence degree theory due to Mawhin [15].

Our main result is as follows.

Theorem 1. *Let* (H) *hold and let* $A > 0$. *Then problem* (1.1), (1.2) *has at least one solution.*

The paper is organized as follows. In Section 2 we state the results which are used in the next sections. Section 3 is devoted to auxiliary boundary value problems. To this end operators $\mathcal{Q}, \mathcal{S}, \mathcal{K}_\lambda$ and $\mathcal{H}_\lambda$ are introduced and their properties are given. In Section 4 Theorem 1 is proved. An example demonstrates our results.

Throughout the paper $\alpha \in (1,2)$, $\mu \in (0,1)$, $K = \frac{T^{1-\mu}}{\Gamma(2-\mu)}$ and $\|x\| = \max\{|x(t)| : t \in J\}$ is the norm in $C(J)$.

2. Preliminaries

This section contains the results that we will need in the next sections.

Lemma 1. *Let* $\Lambda \in \mathcal{A}$ *and let the equality*

$$\Lambda(x) + (\lambda - 1)\Lambda(-x) = 0$$

hold for some $x \in C(J)$ *and* $\lambda \in [0,1]$. *Then there exists* $\xi \in J$ *such that* $x(\xi) = 0$.

Proof. Assume that the statement is not true. Then either $x > 0$ or $x < 0$ on J. If $x > 0$ on J, then $\Lambda(x) > 0$, $\Lambda(-x) < 0$, and therefore $\Lambda(x) + (\lambda - 1)\Lambda(-x) > 0$, which is impossible. Similarly, $x < 0$ on J leads to a contradiction. □

The following maximal principle follows immediately from [17, Lemma 2.1] and [9, Lemma 2.7] and its proof.

Lemma 2 (Maximum principle)**.** *Let* $t_0 \in (0,T]$, $x \in C^1[0,t_0]$, $x(t) \le x(t_0)$ *for* $t \in [0,t_0]$, $x(0) < x(t_0)$ *and* $x'(t_0) = 0$. *Let* $\gamma \in (0,1)$. *Then*

$$^cD^\gamma x(t)|_{t=t_0} > 0.$$

Corollary 1. *Let* $t_0 \in (0,T]$, $x \in C^1[0,t_0]$, $x(t) \ge x(t_0)$ *for* $t \in [0,t_0]$, $x(0) > x(t_0)$ *and* $x'(t_0) = 0$. *Let* $\gamma \in (0,1)$. *Then*

$$^cD^\gamma x(t)|_{t=t_0} < 0.$$

Lemma 3 ([21])**.** *Let* $r \in C(J)$ *and* $\gamma \in (0,1)$. *Then the initial value problem*

$$x'(t) = A\,^cD^\gamma x(t) + r(t), \quad x(0) = a, \qquad A, a \in \mathbb{R},$$

has the unique solution

$$x(t) = a + \int_0^t r(s)\,ds + A\int_0^t \left(\int_0^s (s-\xi)^{-\gamma} E_{1-\gamma,1-\gamma}\left(A(s-\xi)^{1-\gamma}\right) r(\xi)\,d\xi\right) ds,$$

where

$$E_{1-\gamma,1-\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma((k+1)(1-\gamma))}, \quad z \in \mathbb{R},$$

is the classical Mittag-Leffler function.

Lemma 4 ([24, Lemma 2.2]). *Let $\rho \in (0,1)$ and let $E_{\rho,\rho}$ be the Mittag-Leffler function. Then*

$$E_{\rho,\rho}(z) > 0, \quad E'_{\rho,\rho}(z) > 0 \quad \textit{for } z \in \mathbb{R}.$$

We also need the following result.

Lemma 5. *Let $h \in C(J)$ and $A, c_1, c_2 \in \mathbb{R}$. Then the initial value problem*

$$u''(t) = A^cD^\alpha u(t) + h(t), \;\; u(0) = c_2, \; u'(0) = c_1, \tag{2.1}$$

has the unique solution

$$\begin{aligned} u(t) &= c_1 t + c_2 + \int_0^t (t-s)h(s)\,\mathrm{d}s \\ &\quad + A\int_0^t (t-s)\left(\int_0^s (s-\xi)^{1-\alpha} E_{2-\alpha,2-\alpha}\left(A(s-\xi)^{2-\alpha}\right)h(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s. \end{aligned} \tag{2.2}$$

Proof. Since ${}^cD^\alpha x(t) = {}^cD^{\alpha-1}x'(t)$ for $t \in J$ and $x \in C^2(J)$, the equation of (2.1) can be written as

$$u''(t) = A^cD^{\alpha-1}u'(t) + h(t). \tag{2.3}$$

Hence, by Lemma 3 (for $r = h$ and with x and γ replaced by u' and $\alpha - 1$),

$$\begin{aligned} u'(t) &= c_1 + \int_0^t h(s)\,\mathrm{d}s \\ &\quad + A\int_0^t \left(\int_0^s (s-\xi)^{1-\alpha} E_{2-\alpha,2-\alpha}\left(A(s-\xi)^{2-\alpha}\right)h(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s, \end{aligned}$$

where $u'(0) = c_1$. Consequently, $u(t) = c_2 + \int_0^t u'(s)\,\mathrm{d}s$ is the unique solution of problem (2.1) and (2.2) follows. □

3. Operators

In this section auxiliary operators are introduced and their properties are proved. The most important of these operators is an operator $\mathcal{S}$ by which the solvability of problem (1.1), (1.2) is proved in Section 4.

Let

$$\chi_1(x) = \begin{cases} \Delta T & \text{for } x > \Delta T, \\ x & \text{for } |x| \le \Delta T, \\ -\Delta T & \text{for } x < -\Delta T, \end{cases} \qquad \chi_2(y) = \begin{cases} \Delta K & \text{for } y > \Delta K, \\ y & \text{for } |y| \le \Delta K, \\ -\Delta K & \text{for } y < -\Delta K, \end{cases}$$

where Δ and K are from (H). Let

$$\tilde{f}(t,x,y,z) = f(t,\chi_1(x),\chi_2(y),z) \quad \text{for } (t,x,y,z) \in J \times \mathbb{R}^2 \times [-\Delta,\Delta]$$

and

$$f^*(t,x,y,z) = \begin{cases} \tilde{f}(t,x,y,\Delta) + \dfrac{z-\Delta}{z} & \text{if } z > \Delta, \\ \tilde{f}(t,x,y,z) & \text{if } |z| \le \Delta, \\ \tilde{f}(t,x,y,-\Delta) - \dfrac{z+\Delta}{z} & \text{if } z < -\Delta. \end{cases}$$

Under condition (H), $f^* \in C(J \times \mathbb{R}^3)$,

$$f^*(t,x,y,-\Delta) \le 0, \quad f^*(t,x,y,\Delta) \ge 0 \quad \text{for } (t,x,y) \in J \times \mathbb{R}^2,$$

$$\left.\begin{aligned} f^*(t,x,y,z) < 0 \quad \text{for } (t,x,y,z) \in J \times \mathbb{R}^2 \times (-\infty,-\Delta), \\ f^*(t,x,y,z) > 0 \quad \text{for } (t,x,y,z) \in J \times \mathbb{R}^2 \times (\Delta,\infty), \end{aligned}\right\} \tag{3.1}$$

and

$$|f^*(t,x,y,z)| \le E \quad \text{for } (t,x,y,z) \in J \times \mathbb{R}^3, \tag{3.2}$$

where

$$E = 1 + \max\Big\{|f(t,x,y,z)| : (t,x,y) \in \mathcal{D},\, z \in [-\Delta,\Delta]\Big\}.$$

Consider the fractional differential equation

$$u''(t) = A\,{}^cD^\alpha u(t) + f^*(t,u(t),{}^cD^\mu u(t),u'(t)) \tag{3.3}$$

associated to equation (1.1). Keeping in mind Lemma 5 define operators $\mathcal{Q}: C^1(J) \to C(J)$ and $\mathcal{S}: C^1(J) \times \mathbb{R}^2 \to C^1(J) \times \mathbb{R}^2$ by the formulae

$$\begin{aligned}(\mathcal{Q}x)(t) &= f^*(t,x(t),{}^cD^\mu x(t),x'(t)) \\ &\quad + A\int_0^t (t-s)^{1-\alpha} E_{2-\alpha,2-\alpha}\left(A(t-s)^{2-\alpha}\right) f^*(s,x(s),{}^cD^\mu x(t),x'(s))\,\mathrm{d}s,\end{aligned}$$

$$\mathcal{S}(x,c_1,c_2) = \left(c_1t + c_2 + \int_0^t (t-s)(\mathcal{Q}x)(s)\,\mathrm{d}s,\; c_1 + \int_0^T (\mathcal{Q}x)(s)\,\mathrm{d}s,\; c_2 + \Lambda(x)\right),$$

where Λ is from (1.2).

Lemma 6. *Let (H) hold. If (x,c_1,c_2) is a fixed point of the operator $\mathcal{S}$, then x is a solution of problem (3.3), (1.2) and $x'(0) = c_1$, $x(0) = c_2$.*

Proof. Let (x,c_1,c_2) be a fixed point of the operator $\mathcal{S}$, that is, $\mathcal{S}(x,c_1,c_2) = (x,c_1,c_2)$. Then

$$x(t) = c_1t + c_2 + \int_0^t (t-s)(\mathcal{Q}x)(s)\,\mathrm{d}s, \quad t \in J, \tag{3.4}$$

$$\int_0^T (\mathcal{Q}x)(s)\,\mathrm{d}s = 0, \tag{3.5}$$

$$\Lambda(x) = 0. \tag{3.6}$$

It follows from (3.4) and Lemma 5 (for $h(t) = f^*(t, x(t), {}^cD^\mu x(t), x'(t))$) that $x(0) = c_2$, $x'(0) = c_1$ and x is a solution of (3.3).

Since (cf. (3.4))

$$x'(t) = c_1 + \int_0^t (\mathcal{Q}x)(s)\,\mathrm{d}s, \quad t \in J,$$

we conclude from (3.5) that $x'(T) = c_1$. Hence $x'(0) = x'(T)$. The last equality together with (3.6) give that x satisfies the boundary conditions (1.2). Consequently, x is a solution of problem (3.3), (1.2) and $x'(0) = c_1$, $x(0) = c_2$. □

In order to prove that the operator $\mathcal{S}$ admits a fixed point, for $\lambda \in [0,1]$, we first introduce an operator $\mathcal{K}_\lambda : C^1(J) \times \mathbb{R}^2 \to C^1(J) \times \mathbb{R}^2$,

$$\mathcal{K}_\lambda(x, c_1, c_2) = \left(c_1 t + c_2, c_1 + (1-\lambda)x'(0) + \lambda \int_0^T (\mathcal{Q}x)(s)\,\mathrm{d}s, c_2 + \Lambda(x) + (\lambda - 1)\Lambda(-x)\right).$$

Let

$$\Omega = \Big\{(x, c_1, c_2) \in C^1(J) \times \mathbb{R}^2 : \|x\| < \Delta T + 1, \|x'\| < \Delta + 1, |c_1| < \Delta + 1, |c_2| < \Delta T + 1\Big\}, \tag{3.7}$$

where Δ is from (H).

Lemma 7. *Let (H) hold and let $A > 0$. Then*

$$\deg\Big(\mathcal{J} - \mathcal{K}_1, \Omega, 0\Big) \neq 0, \tag{3.8}$$

where "deg" stands for the Leray-Schauder degree and $\mathcal{J}$ is the identity operator on $C^1(J) \times \mathbb{R}^2$.

Proof. Let $M : [0,1] \times C^1(J) \times \mathbb{R} \to C^1(J) \times \mathbb{R}$, $M(\lambda, x, c_1, c_2) = \mathcal{K}_\lambda(x, c_1, c_2)$. Since $f^* \in C(J \times \mathbb{R}^3)$, we conclude from Lemma 4 that $\mathcal{Q}$ is a continuous operator. As Λ is continuous and takes bounded sets into bounded sets, it is easy to prove that M is a completely continuous operator.

Due to

$$\mathcal{K}_0(-x, -c_1, -c_2) = -\mathcal{K}_0(x, c_1, c_2) \quad \text{for } (x, c_1, c_2) \in C^1(J) \times \mathbb{R}^2,$$

$\mathcal{K}_0$ is an odd operator.

Assume that $M(\lambda_0, x, c_1, c_2) = (x, c_1, c_2)$ for some $(x, c_1, c_2) \in C^1(J) \times \mathbb{R}^2$ and $\lambda_0 \in [0,1]$. Then

$$x(t) = c_1 t + c_2, \quad t \in J, \tag{3.9}$$

$$(1-\lambda_0)x'(0) + \lambda_0 \int_0^T (\mathcal{Q}x)(s)\,\mathrm{d}s = 0, \tag{3.10}$$

$$\Lambda(x) + (\lambda_0 - 1)\Lambda(-x) = 0. \tag{3.11}$$

Lemma 1 together with (3.11) give $x(\xi) = 0$ for some $\xi \in J$. Hence (cf. (3.9)) $c_1\xi + c_2 = 0$, and therefore $x(t) = c_1(t-\xi)$ on J.

We now prove that

$$|c_1| \le \Delta. \tag{3.12}$$

Let $c_1 > \Delta$. Then $x' = c_1 > \Delta$ on J, and therefore $f^*(t, x(t), {}^cD^\mu x(t), x'(t)) > 0$ for $t \in J$ by (3.1). This fact together with $A > 0$ and Lemma 4 imply $(\mathcal{Q}x)(t) > 0$ on J. Hence $(1-\lambda_0)c_1 + \lambda_0 \int_0^T (\mathcal{Q}x)(s)\,ds > 0$, which contradicts (3.10). Therefore $c_1 \le \Delta$. Similarly, if $c_1 < -\Delta$, then we have $f^*(t, x(t), {}^cD^\mu x(t), x'(t)) < 0$ and $(\mathcal{Q}x)(t) < 0$ for $t \in J$, which again contradicts (3.10). Hence (3.12) is true.

Consequently, $|x(t)| = |c_1(t-\xi)| \le \Delta T$, $|x'(t)| = |c_1| \le \Delta$, $|{}^cD^\mu x(t)| = |I^{1-\mu}x'(t)| \le \Delta K$ on J and $|c_2| = |x(0)| \le \Delta T$. As a result,

$$M(\lambda, x, c_1, c_2) \ne (x, c_1, c_2) \quad \text{for } (x, c_1, c_2) \in \partial\Omega \text{ and } \lambda \in [0,1].$$

Hence, by the Borsuk antipodal theorem and the homotopy property, the relations

$$\deg\left(\mathcal{I} - \mathcal{K}_0, \Omega, 0\right) \ne 0,$$

$$\deg\left(\mathcal{I} - \mathcal{K}_0, \Omega, 0\right) = \deg\left(\mathcal{I} - \mathcal{K}_1, \Omega, 0\right)$$

hold. Combining these relations we obtain (3.8). ☐

Finally, let for $\lambda \in [0,1]$ an operator $\mathcal{H}_\lambda : C^1(J) \times \mathbb{R}^2 \to C^1(J) \times \mathbb{R}^2$ be defined as

$$\mathcal{H}_\lambda(x, c_1, c_2) = \left(c_1 t + c_2 + \lambda \int_0^t (t-s)(\mathcal{Q}x)(s)\,ds,\ c_1 + \int_0^T (\mathcal{Q}x)(s)\,ds,\ c_2 + \Lambda(x)\right).$$

Then, for $(x, c_1, c_2) \in C^1(J) \times \mathbb{R}^2$,

$$\mathcal{H}_0(x, c_1, c_2) = \mathcal{K}_1(x, c_1, c_2), \tag{3.13}$$

$$\mathcal{H}_1(x, c_1, c_2) = \mathcal{S}(x, c_1, c_2). \tag{3.14}$$

Lemma 8. *Let (H) hold. Let $V : [0,1] \times C^1(J) \times \mathbb{R} \to C^1(J) \times \mathbb{R}$ and $V(\lambda, x, c_1, c_2) = \mathcal{H}_\lambda(x, c_1, c_2)$. Then V is a completely continuous operator.*

Proof. We first prove that V is continuous. To this end let $\{x_n\} \subset C^1(J)$, $\{c_{n,i}\} \subset \mathbb{R}$, $i = 1,2$, $\{\lambda_n\} \subset [0,1]$ be convergent sequences and let $\lim_{n\to\infty} x_n = x$ in $C^1(J)$, $\lim_{n\to\infty} c_{n,i} = c_i$, $\lim_{n\to\infty} \lambda_n = \lambda$ in $\mathbb{R}$, where $x \in C^1(J)$, $c_i, \lambda \in \mathbb{R}$, $i = 1,2$. Then $\lim_{n\to\infty} f^*(t, x_n(t), {}^cD^\mu x_n(t), x_n'(t)) = f^*(t, x(t), {}^cD^\mu x(t), x'(t))$ uniformly on J. This together with Lemma 4 imply that $\lim_{n\to\infty} (\mathcal{Q}x_n)(t) = (\mathcal{Q}x)(t)$ uniformly on J. Hence

$$\lim_{n\to\infty}\left(c_{n,1}t + c_{n,2} + \lambda_n \int_0^t (t-s)(\mathcal{Q}x_n)(s)\,ds\right) = c_1 t + c_2 + \lambda \int_0^t (t-s)(\mathcal{Q}x)(s)\,ds,$$

$$\lim_{n\to\infty}\left(c_{n,1} + \lambda_n \int_0^t (\mathcal{Q}x_n)(s)\,ds\right) = c_1 + \lambda \int_0^t (\mathcal{Q}x)(s)\,ds$$

uniformly on J. Besides,

$$\lim_{n\to\infty}\left(c_{n,1}+\int_0^T(\mathcal{Q}x_n)(s)\,\mathrm{d}s\right)=c_1+\int_0^T(\mathcal{Q}x)(s)\,\mathrm{d}s,$$

$$\lim_{n\to\infty}\big(c_{n,2}+\Lambda(x_n)\big)=c_2+\Lambda(x).$$

Consequently, V is a continuous operator.

Let $\Phi\subset C^1(J)\times\mathbb{R}^2$ be bounded and let $\|x\|\le L$, $\|x'\|\le L$, $|c_1|\le L$, $|c_2|\le L$ for $(x,c_1,c_2)\in\Phi$, where L is a positive constant. Let $W=E_{2-\alpha,2-\alpha}\left(|A|T^{2-\alpha}\right)$. Then, by (3.2) and Lemma 4, the relation

$$\begin{aligned}|(\mathcal{Q}x)(t)|&\le E+|A|E\int_0^t(t-s)^{1-\alpha}E_{2-\alpha,2-\alpha}\left(A(t-s)^{2-\alpha}\right)\mathrm{d}s\\&\le E+|A|EW\int_0^t(t-s)^{1-\alpha}\,\mathrm{d}s\le E+|A|EW\frac{T^{2-\alpha}}{2-\alpha}=H\end{aligned}$$

holds for $t\in J$ and $(x,c_1,c_2)\in\Phi$. Hence

$$\left|c_1t+c_2+\lambda\int_0^t(t-s)(\mathcal{Q}x)(s)\,\mathrm{d}s\right|\le L(T+1)+\frac{HT^2}{2},$$

$$\left|c_1+\lambda\int_0^T(\mathcal{Q}x)(s)\,\mathrm{d}s\right|\le L+HT,$$

$$|c_2+\Lambda(x)|\le L+\max\{|\Lambda(-L)|,\Lambda(L)\}$$

for $t\in J$, $(x,c_1,c_2)\in\Phi$ and $\lambda\in[0,1]$, and therefore the set $V([0,1]\times\Phi)=\{V(\lambda,x,c_1,c_2):\lambda\in[0,1],\ (x,c_1,c_2)\in\Phi\}$ is bounded in $C^1(J)\times\mathbb{R}^2$. In view of $\|\mathcal{Q}x\|\le H$ we see that the set $\left\{c_1+\lambda\int_0^t(\mathcal{Q}x)(s)\,\mathrm{d}s:(x,c_1,c_2)\in\Phi,\ \lambda\in[0,1]\right\}$ is equicontinuous on J.

Hence the Arzelà-Ascoli theorem and the Bolzano–Weierstrass compactness theorem in $\mathbb{R}$ guarantee that the set $V([0,1]\times\Phi)$ is relatively compact in $C^1(J)\times\mathbb{R}^2$. Consequently, V is completely continuous. □

4. THE PROOF OF THEOREM 1 AND AN EXAMPLE

Proof. Suppose that $(x,c_1,c_2)\in C^1(J)\times\mathbb{R}^2$ is a fixed point of $\mathcal{H}_\lambda$ for some $\lambda\in[0,1]$, that is, $\mathcal{H}_\lambda(x,c_1,c_2)=(x,c_1,c_2)$. If $\lambda=0$, then it follows from the proof of Lemma 7 (cf. (3.13)) that $(x,c_1,c_2)\in\Omega$, where Ω is given in (3.7). Let $\lambda\in(0,1]$. Then

$$x(t)=c_1t+c_2+\lambda\int_0^t(t-s)(\mathcal{Q}x)(s)\,\mathrm{d}s,\quad t\in J,\tag{4.1}$$

$$\int_0^T(\mathcal{Q}x)(s)\,\mathrm{d}s=0,\tag{4.2}$$

$$\Lambda(x)=0.\tag{4.3}$$

Hence

$$x'(t) = c_1 + \lambda \int_0^t (\mathcal{Q}x)(s)\,\mathrm{d}s, \quad t \in J, \tag{4.4}$$

so $x'(0) = c_1$, and, by (4.2), $x'(T) = c_1 + \lambda \int_0^T (\mathcal{Q}x)(s)\,\mathrm{d}s = c_1$. Consequently,

$$x'(0) = x'(T). \tag{4.5}$$

Suppose that $c_1 > \Delta$, where Δ is from (H). Then $f^*(0,x(0),0,c_1) > 0$ by (3.1), and therefore $f^*(t,x(t),{}^cD^\mu x(t),x'(t)) > 0$ on a right neighbourhood of $t = 0$. If there is some $\xi \in (0,T]$ such that $f^*(t,x(t),{}^cD^\mu x(t),x'(t)) > 0$ on $[0,\xi)$ and $f^*(\xi,x(\xi),{}^cD^\mu x(t)|_{t=\xi},x'(\xi)) = 0$, then $(\mathcal{Q}x)(t) > 0$ on $[0,\xi]$ because $A > 0$, which gives $x'(t) > c_1$ for $t \in (0,\xi]$. Hence $f^*(t,x(t),{}^cD^\mu x(t),x'(t)) > 0$ on $[0,\xi]$, contrary to $f^*(\xi,x(\xi),{}^cD^\mu x(t)|_{t=\xi},x'(\xi)) = 0$. Consequently,

$$f^*(t,x(t),{}^cD^\mu x(t),x'(t)) > 0, \ \ (Qx)(t) > 0, \quad t \in J.$$

Thus $x'(T) > c_1 = x'(0)$, which contradicts (4.5). Hence $c_1 \le \Delta$. Similarly, we can prove that $c_1 \ge -\Delta$. To summarize, $|c_1| \le \Delta$.

Suppose that $\max\{x'(t) : t \in J\} = x'(\xi) > \Delta$. Then $\xi \in (0,T)$ and $x'(\xi) - x'(0) > 0$. By (4.4), $x \in C^2(J)$ and $x'' = \lambda\mathcal{Q}x$. Hence $x''(\xi) = 0$ and by Lemma 5 and (2.3) (for $h(t) = \lambda f^*(t,x(t),{}^cD^\mu x(t),x'(t))$) the equality

$$x''(t) = A\,{}^cD^{\alpha-1}x'(t) + \lambda f^*(t,x(t),{}^cD^\mu x(t),x'(t)), \ \ t \in J,$$

holds. Lemma 2 (for $t_0 = \xi$, $\gamma = \alpha - 1$ and x replaced by x') shows that ${}^cD^{\alpha-1}x'(t)|_{t=\xi} > 0$. Hence

$$x''(\xi) = A\,{}^cD^{\alpha-1}x'(t)|_{t=\xi} + \lambda f^*(\xi,x(\xi),{}^cD^\mu x(t)|_{t=\xi},x'(\xi)) > 0,$$

which is impossible. Hence $x'(t) \le \Delta$ for $t \in J$. Similarly, by Corollary 1, we can prove that $x' \ge -\Delta$ on J. Consequently,

$$|x'(t)| \le \Delta, \ \ t \in J.$$

Next, it follows from (4.3) and Lemma 1 that $x(\tau) = 0$ for some $\tau \in J$. Therefore $|x(t)| = \left|\int_\tau^t x'(s)\,\mathrm{d}s\right| \le \Delta|t-\tau| \le \Delta T$, $|{}^cD^\mu x(t)| = |I^{1-\mu}x'(t)| \le \Delta K$. As $c_1 = x'(0)$ and $c_2 = x(0)$, we have proved

$$\|x\| \le \Delta T, \ \|{}^cD^\mu x\| \le \Delta K, \ \|x'\| \le \Delta, \ |c_1| \le \Delta T, \ |c_2| \le \Delta, \tag{4.6}$$

which implies $V(\lambda,x,c_1,c_2) \ne (x,c_1,c_2)$ for $(x,c_1,c_2) \in \partial\Omega$ and $\lambda \in [0,1]$, where V is from Lemma 8. Combinig Lemma 8 with the homotopy property we have

$$\deg\left(\mathcal{I} - \mathcal{H}_0, \Omega, 0\right) = \deg\left(\mathcal{I} - \mathcal{H}_1, \Omega, 0\right).$$

This equality together with (3.8) and (3.13) give

$$\deg\left(\mathcal{I} - \mathcal{H}_1, \Omega, 0\right) \ne 0.$$

Hence there exists a fixed point (x, c_1, c_2) of $\mathcal{H}_1$. Lemma 6 and (3.14) guarantee that x is a fixed point of problem (3.3), (1.2) and $c_1 = x'(0)$, $c_2 = x(0)$. Due to (4.6), $f^*(t, x(t), {}^cD^\mu x(t), x'(t)) = f(t, x(t), {}^cD^\mu x(t), x'(t))$ for $t \in J$, and therefore x is a solution of problem (1.1), (1.2). □

Example 2. Let $p \in C(J \times \mathbb{R}^3)$ be bounded, $a, b, c \in C(J)$, $c > 0$ on J, and $n \in \mathbb{N}$, $\beta, \gamma \in (0, 2n-1)$. Then the function

$$f(t,x,y,z) = p(t,x,y,z) + a(t)|x|^{\beta-1}x + b(t)|y|^\gamma + c(t)z^{2n-1}$$

satisfies condition (H). Really, let $|p(t,x,y,z)| \le L$ for $(t,x,y,z) \in J \times \mathbb{R}^3$ and $c_* = \min\{c(t) : t \in J\}$. Since

$$\lim_{v\to\infty}\left(Lv^{1-2n} + \|a\|T^\beta v^{\beta+1-2n} + \|b\|K^\gamma v^{\gamma+1-2n}\right) = 0, \quad K = \frac{T^{1-\mu}}{\Gamma(2-\mu)},$$

there exists $\Delta > 0$ such that

$$L\Delta^{1-2n} + \|a\|T^\beta \Delta^{\beta+1-2n} + \|b\|K^\gamma \Delta^{\gamma+1-2n} \le c_*.$$

Hence $L + \|a\|(\Delta T)^\beta + \|b\|(\Delta K)^\gamma \le c_*\Delta^{2n-1}$, and therefore for $(t,x,y) \in \mathcal{D}$, where $\mathcal{D}$ is from (H), the inequalities

$$f(t,x,y,\Delta) \ge -L - \|a\|(\Delta T)^\beta - \|b\|(\Delta K)^\gamma + c_*\Delta^{2n-1} \ge 0,$$
$$f(t,x,y,-\Delta) \le L + \|a\|(\Delta T)^\beta + \|b\|(\Delta K)^\gamma - c_*\Delta^{2n-1} \le 0$$

hold. Theorem 1 gives that the equation

$$\begin{aligned} u'' = {} & A\,{}^cD^\alpha u + p(t,u,{}^cD^\mu u, u') \\ & + a(t)|u|^{\beta-1}u + b(t)|{}^cD^\mu u|^\gamma + c(t)(u')^{2n-1}, \quad A > 0, \end{aligned} \tag{4.7}$$

has at least one solution u satisfying the boundary conditions (1.2) and $\|u\| \le \Delta T$, $\|{}^cD^\mu u\| \le \Delta K$, $\|u'\| \le \Delta$.

In particular, there exists a solution of (4.7) satisfying the boundary conditions

$$\min\{u(t) : t \in J\} = 0, \quad u'(0) = u'(T),$$

that is, u is a nonnegative solution of the problem.

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