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THIRD HANKEL DETERMINANT FOR CERTAIN SUBCLASS OF p-VALENT ANALYTIC FUNCTIONS

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This paper is dedicated to Professor T. RAMREDDY on his 72^{nd} birthday.

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Abstract. The objective of this paper is to obtain an upper bound to the third Hankel determinant for certain subclass of p-valent functions, using Toeplitz determinants.

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1. INTRODUCTION

Let A_p denote the class of functions f of the form

$$f(z) = z^p + a_{p+1}z^{p+1} + \cdots, \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$ with $p \in \mathbb{N} = \{1,2,3,\ldots\}$. Let S be the subclass of $A_1 = A$, consisting of univalent functions. In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture, i.e., for a univalent function its n^{th}- coefficient is bounded by n (see [3]). The bounds for the coefficients of these functions give information about their geometric properties. In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ (when $p = 1$) was defined by Pommerenke [10] as follows and has been extensively studied.

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.2}$$

One can easily observe that the Fekete-Szegő functional is $H_2(1) = a_3 - a_2^2$. Fekete and Szegő then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in S$. Further, sharp upper bounds for the functional $H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2$, the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant (functional), were obtained for various subclasses of univalent and multivalent analytic functions. Janteng et al. [6] have considered the functional $|a_2a_4 - a_3^2|$ and found a sharp upper bound for the function f in the subclass $\mathcal{R}$ of S, consisting of functions whose derivative has a positive real part (also called bounded turning functions) studied by MacGregor [9] and have showed that $|H_2(2)| \leq \frac{4}{9}$. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 3$ and $n = p$, denoted by $H_3(p)$, given by

$$H_3(p) = \begin{vmatrix} a_p & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \end{vmatrix}. \tag{1.3}$$

For $f \in A_p$, $a_p = 1$, so that, we have

$$H_3(p) = a_{p+2}(a_{p+1}a_{p+3} - a_{p+2}^2) - a_{p+3}(a_{p+3} - a_{p+1}a_{p+2}) + a_{p+4}(a_{p+2} - a_{p+1}^2)$$

and by applying the triangle inequality, we obtain

$$|H_3(p)| \leq |a_{p+2}||a_{p+1}a_{p+3} - a_{p+2}^2| + |a_{p+3}||a_{p+1}a_{p+2} - a_{p+3}| + |a_{p+4}||a_{p+2} - a_{p+1}^2|. \tag{1.4}$$

Incidentally, all of the functionals on the right hand side of the inequality (1.4) have known (and sharp) upper bounds except $|a_{p+1}a_{p+2} - a_{p+3}|$. It was known that if $f \in \mathcal{R}_p$, the class of p-valent bounded turning functions, then $|a_k| \leq \frac{2p}{k}$, where $k \in \{p+1,\ p+2,\ ...\}$ and $|a_{p+2} - a_{p+1}^2| \leq \frac{2p}{p+2}$, with $p \in \mathbb{N}$.

Motivated by the result obtained by Babalola [1] in finding the sharp upper bound to the Hankel determinant $|H_3(1)|$ for the class $\mathcal{R}$, in this paper we obtain an upper bound to the functional $|a_{p+1}a_{p+2} - a_{p+3}|$ and hence for $|H_3(p)|$, for the function f given in (1.1), belonging to certain subclass of p-valent analytic functions, as follows.

Definition 1 ([13])**.** A function $f \in A_p$ is said to be in the class $I_p(\beta)$(β is real), if it satisfies the condition

$$\operatorname{Re}\left\{(1-\beta)\frac{f(z)}{z^p} + \beta\frac{f'(z)}{pz^{p-1}}\right\} > 0, \quad z \in E. \tag{1.5}$$

(1) Choosing $\beta = 1$ and $p = 1$, we obtain $I_1(1) = \mathcal{R}$.
(2) Selecting $\beta = 1$, we get $I_p(1) = \mathcal{R}_p$.

2. Preliminary Results

In this section some preliminary lemmas are stated which are required for proving our results.

Let $\mathcal{P}$ denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + ... = 1 + \sum_{n=1}^{\infty} c_nz^n, \tag{2.1}$$

which are analytic in the open unit disc E and satisfy $\operatorname{Re}p(z) > 0$ for any $z \in E$. Here $p(z)$ is called Carathéodory function [4].

Lemma 1 ([11, 12]). *If $p \in \mathcal{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$.*

Lemma 2 ([5]). *The power series for $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ given in (2.1) converges in the open unit disc E to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \; n = 1,2,3....$$

and $c_{-k} = \bar{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z)$, with $\sum_{k=1}^{m} \rho_k = 1$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

We may assume without restriction that $c_1 \geq 0$. Using Lemma 2, for $n = 2$ and $n = 3$, for some complex values x and z with $|x| \leq 1$ and $|z| \leq 1$ respectively, we have

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.2}$$

$$\text{and } \; 4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z. \tag{2.3}$$

To obtain our results, we refer to the classical method devised by Libera and Zlotkiewicz [8], which is used by many authors in the literature.

3. Main Results

Theorem 1. *If $f \in I_p(\beta)$ $(0 < \beta \leq 1)$ with $p \in \mathbb{N}$, then*

$$|a_{p+1}a_{p+2} - a_{p+3}| \leq \frac{2p}{p + 3\beta}.$$

Proof. For $f = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in I_p(\beta)$, by virtue of Definition 1, there exists an analytic function $p \in \mathcal{P}$ in the open unit disc E with $p(0) = 1$ and $\mathrm{Re}\, p(z) > 0$ such that

$$(1-\beta)\frac{f(z)}{z^p} + \beta\frac{f'(z)}{pz^{p-1}} = p(z) \Leftrightarrow (1-\beta)pf(z) + \beta f'(z) = pz^p p(z). \tag{3.1}$$

$$(1-\beta)p\left\{z^p + \sum_{n=p+1}^{\infty} a_n z^n\right\} + \beta\left\{pz^{p-1} + \sum_{n=p+1}^{\infty} n a_n z^{n-1}\right\} = pz^p\left\{1 + \sum_{n=1}^{\infty} c_n z^n\right\}.$$

Upon simplification, we obtain

$$\begin{aligned}(p+\beta)a_{p+1}z^{p+1} + (p+2\beta)a_{p+2}z^{p+2} + (p+3\beta)a_{p+3}z^{p+3} + (p+4\beta)a_{p+4}z^{p+4} + \ldots \\ = pc_1 z^{p+1} + pc_2 z^{p+2} + pc_3 z^{p+3} + pc_4 z^{p+4} + \ldots. \end{aligned} \tag{3.2}$$

Equating the coefficients of z^{p+1}, z^{p+2}, z^{p+3} and z^{p+4} respectively in 3.2, we have

$$a_{p+1}=\frac{pc_1}{p+\beta};\ a_{p+2}=\frac{pc_2}{p+2\beta};\ a_{p+3}=\frac{pc_3}{p+3\beta}\ \text{ and }\ a_{p+4}=\frac{pc_4}{p+4\beta}. \tag{3.3}$$

Substituting the values of a_{p+1}, a_{p+2} and a_{p+3} from (3.3) in the functional $|a_{p+1}a_{p+2}-a_{p+3}|$, after simplifying, we get

$$|a_{p+1}a_{p+2}-a_{p+3}|=\frac{p}{(p+\beta)(p+2\beta)(p+3\beta)}\left|p(p+3\beta)c_1c_2-(p+\beta)(p+2\beta)c_3\right|.$$

The above expression is equivalent to

$$|a_{p+1}a_{p+2}-a_{p+3}|=\frac{p}{(p+\beta)(p+2\beta)(p+3\beta)}\left|d_1c_1c_2+d_2c_3\right|, \tag{3.4}$$

$$\text{where}\quad d_1=p(p+3\beta);\ \ d_2=-(p+\beta)(p+2\beta). \tag{3.5}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.3) respectively from Lemma 2 on the right-hand side of (3.4), we have

$$\begin{aligned}|d_1c_1c_2+d_2c_3|=&\,|d_1c_1\times\frac{1}{2}\{c_1^2+x(4-c_1^2)\}+d_2\\ &\times\frac{1}{4}\{c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2)z\}|.\end{aligned}$$

Using the facts $|z|\le 1$ and $|xa+yb|\le|x||a|+|y||b|$, where x,y,a and b are real numbers, which simplifies to

$$\begin{aligned}4\left|d_1c_1c_2+d_2c_3\right|\le&\,[|(2d_1+d_2)||c_1|^3+2|d_2||(4-c_1^2)|+2|(d_1+d_2)||c_1||(4-c_1^2)||x|\\ &+|d_2||(c_1+2)||(4-c_1^2)||x|^2].\end{aligned} \tag{3.6}$$

From (3.5), we can write

$$2d_1+d_2=p^2+3p\beta-2\beta^2;\ d_1+d_2=-2\beta^2. \tag{3.7}$$

Substituting the calculated values from (3.7) along with (3.5) on the right-hand side of (3.6), we have

$$\begin{aligned}4|d_1c_1c_2+d_2c_3|\le&\,[(p^2+3p\beta-2\beta^2)c_1^3+2(p+\beta)(p+2\beta)(4-c_1^2)\\ &+4\beta^2c_1(4-c_1^2)|x|+(c_1+2)(p+\beta)(p+2\beta)(4-c_1^2)|x|^2].\end{aligned}$$

Since $c_1=c\in[0,2]$, noting that $c_1-a\le c_1+a$, where $a\ge 0$ and replacing $|x|$ by μ on the right-hand side of the above inequality, we get

$$\begin{aligned}4|d_1c_1c_2+d_2c_3|\le&\,[(p^2+3p\beta-2\beta^2)c^3+2(p+\beta)(p+2\beta)(4-c^2)+4\beta^2c(4-c^2)\mu\\ &+(c-2)(p+\beta)(p+2\beta)(4-c^2)\mu^2]=F(c,\mu),\end{aligned} \tag{3.8}$$

for $0\le\mu=|x|\le 1$ and $0\le c\le 2$, where

$$\begin{aligned}F(c,\mu)=&\,(p^2+3p\beta-2\beta^2)c^3+2(p+\beta)(p+2\beta)(4-c^2)+4\beta^2c(4-c^2)\mu\\ &+(c-2)(p+\beta)(p+2\beta)(4-c^2)\mu^2.\end{aligned} \tag{3.9}$$

Next, we need to find the maximum value of the function $F(c,\mu)$ on the closed region $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ given in (3.9) partially with respect to μ and c respectively, we obtain

$$\frac{\partial F}{\partial \mu}=4\beta^2c(4-c^2)+2(p+\beta)(p+2\beta)(4c-c^3-8+2c^2)\mu. \tag{3.10}$$

and

$$\frac{\partial F}{\partial c}=3(p^2+3p\beta-2\beta^2)c^2-4c(p+\beta)(p+2\beta)+16\beta^2\mu-12\beta^2c^2\mu \\ +(p+\beta)(p+2\beta)(4-3c^2+4c)\mu^2. \tag{3.11}$$

For the extreme values of $F(c,\mu)$, consider

$$\frac{\partial F}{\partial \mu}=0 \quad \text{and} \quad \frac{\partial F}{\partial c}=0. \tag{3.12}$$

In view of (3.12), on solving the equations in (3.10) and (3.11), we obtain the only critical point for the function $F(c,\mu)$ which lies in the closed region $[0,2]\times[0,1]$ is $(0,0)$. At the critical point $(0,0)$, we observe that

$$\begin{aligned} \frac{\partial^2 F}{\partial \mu^2}&=-4(p+\beta)(p+2\beta)<0;\\ \frac{\partial^2 F}{\partial c^2}&=-16(p+\beta)(p+2\beta)<0;\\ \frac{\partial^2 F}{\partial c\partial \mu}&=16\beta^2;\\ \left[\left(\frac{\partial^2 F}{\partial \mu^2}\right)\left(\frac{\partial^2 F}{\partial c^2}\right)-\left(\frac{\partial^2 F}{\partial c\partial \mu}\right)^2\right]&=64[(p+\beta)^2(p+2\beta)^2-4\beta^4]>0, \end{aligned}$$

with $p\in\mathbb{N}$ and $0<\beta\le 1$.

Therefore, the function $F(c,\mu)$ has maximum value at the point $(0,0)$, from (3.9), it is given by

$$G_{max}=F(0,0)=8(p+\beta)(p+2\beta). \tag{3.13}$$

Simplifying the expressions (3.4) and (3.8) together with (3.13), we obtain

$$|a_{p+1}a_{p+2}-a_{p+3}|\le\frac{2p}{p+3\beta}. \tag{3.14}$$

This completes the proof of our theorem. □

Remark 1. Choosing $p=1$ and $\beta=1$ in (3.14), we obtain $|a_2a_3-a_4|\le\frac{1}{2}$, this inequality is sharp and coincides with the result of Bansal et al. [2].

Theorem 2. *If $f \in I_p(\beta)$ $(0 < \beta \leq 1)$ with $p \in \mathbb{N}$ then*

$$|a_{p+2} - a_{p+1}^2| \leq \frac{2p}{p+2\beta}$$

and the inequality is sharp for the values $c_1 = c = 0,\ c_2 = 2$ and $x = 1$.

Proof. On substituting the values of a_{p+1} and a_{p+2} from (3.3) in the functional $|a_{p+2} - a_{p+1}^2|$, which simplifies to

$$|a_{p+2} - a_{p+1}^2| = \frac{p}{(p+\beta)^2(p+2\beta)} \left|(p+\beta)^2 c_2 - p(p+2\beta)c_1^2\right|. \tag{3.15}$$

The above expression is equivalent to

$$|a_{p+2} - a_{p+1}^2| = \frac{p}{(p+\beta)^2(p+2\beta)} \left|d_1 c_2 + d_2 c_1^2\right|, \tag{3.16}$$

$$\text{where} \quad d_1 = (p+\beta)^2 \quad \text{and} \quad d_2 = -p(p+2\beta). \tag{3.17}$$

Substituting the value of c_2 from (2.2) of Lemma 2, applying the triangle inequality on the right-hand side of (3.16), after simplifying, we get

$$2\left|d_1 c_2 + d_2 c_1^2\right| \leq \left[|(d_1 + 2d_2)||c_1|^2 + |d_1|\left(4 - c_1^2\right)|x|\right]. \tag{3.18}$$

From (3.17), we can write

$$d_1 + 2d_2 = -(p^2 + 2p\beta - \beta^2);\ d_1 = (p+\beta)^2. \tag{3.19}$$

Substituting the calculated values from (3.19), taking $c_1 = c \in [0,2]$, replacing $|x|$ by μ on the right-hand side of (3.18), we obtain

$$\begin{aligned} 2\left|d_1 c_2 + d_2 c_1^2\right| &\leq \left[(p^2 + 2p\beta - \beta^2)c^2 + (p+\beta)^2\left(4 - c^2\right)\mu\right] \\ &= F(c,\mu)\ ,\ 0 \leq \mu = |x| \leq 1 \text{ and } 0 \leq c \leq 2, \end{aligned} \tag{3.20}$$

$$\text{where } F(c,\mu) = (p^2 + 2p\beta - \beta^2)c^2 + (p+\beta)^2\left(4 - c^2\right)\mu. \tag{3.21}$$

Now, we maximize the function $F(c,\mu)$ on the closed region $[0,2] \times [0,1]$. Let us suppose that there exists a maximum value for $F(c,\mu)$ at any point in the interior of the closed region $[0,2] \times [0,1]$. Differentiating $F(c,\mu)$ given in (3.21) partially with respect to μ, we obtain

$$\frac{\partial F}{\partial \mu} = (p+\beta)^2\left(4 - c^2\right) \tag{3.22}$$

For $0 < \beta \leq 1$, for fixed values of c with $0 < c < 2$ and $p \in \mathbb{N}$, from (3.22), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ which is independent of μ becomes an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed region $[0,2] \times [0,1]$. The maximum value of $F(c,\mu)$ occurs only on the boundary i.e., when $\mu = 1$. Therefore, for fixed $c \in [0,2]$, we have

$$\max_{0 \leq \mu \leq 1} F(c,\mu) = F(c,1) = G(c). \tag{3.23}$$

In view of (3.23), replacing μ by 1 in (3.21), it simplifies to

$$G(c) = -2\beta^2 c^2 + 4(p+\beta)^2, \tag{3.24}$$

$$G'(c) = -4\beta^2 c. \tag{3.25}$$

From the expression (3.25), we observe that $G'(c) \leq 0$ for each $c \in [0,2]$ and for every β with $0 < \beta \leq 1$. Therefore, $G(c)$ becomes a decreasing function of c, whose maximum value occurs at $c = 0$ only and from (3.24), it is given by

$$G_{max} = G(0) = 4(p+\beta)^2. \tag{3.26}$$

Simplifying the expressions (3.16), (3.20) along with (3.26), we obtain

$$|a_{p+2} - a_{p+1}^2| \leq \frac{2p}{p+2\beta}. \tag{3.27}$$

This completes the proof of our theorem. $\square$

Remark 2. If $p = 1$ and $\beta = 1$ in (3.27) then $|a_3 - a_2^2| \leq \frac{2}{3}$, this result coincides with that of Babalola [1].

Theorem 3. *If $f \in I_p(\beta)$ $(0 < \beta \leq 1)$ then*

$$|a_{p+k}| \leq \frac{2p}{p+k\beta}, \quad for \ \ p, \ \ k \in \mathbb{N}. \tag{3.28}$$

Proof. Using the fact that $|c_n| \leq 2$, for $n \in \mathbb{N}$, with the help of c_2 and c_3 values given in (2.2) and (2.3) respectively, together with the values obtained in (3.3), we get $|a_{p+k}| \leq \frac{2p}{p+k\beta}$, with $p\ ,k \in \mathbb{N}$. This completes the proof of our theorem. $\square$

Substituting the results of Theorems 1, 2, 3 together with the known inequality $|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \left[\frac{2p}{p+2\beta}\right]^2$ (see [7]) in the inequality given in (1.4), we obtain the following Corollary.

Corollary 1. *If $f \in I_p(\beta)$ $(0 < \beta \leq 1)$ with $p \in \mathbb{N}$ then*

$$|H_3(p)| \leq 4p^2 \left[\frac{2p}{(p+2\beta)^3} + \frac{1}{(p+3\beta)^2} + \frac{1}{(p+2\beta)(p+4\beta)}\right]. \tag{3.29}$$

Remark 3. In particular for the values $p = 1$ and $\beta = 1$ in (3.29), which simplifies to $|H_3(1)| \leq \frac{439}{540}$. This result coincides with that of Bansal et al. [2].

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