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INEQUALITIES FOR log−CONVEX FUNCTIONS AND P−FUNCTIONS

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Abstract. In this paper we obtain some new integral inequalities like Hermite-Hadamard type for log−convex functions and $P-$functions.

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1. INTRODUCTION

We shall recall the definitions of log−convex functions and $P-$functions.

A function is called log−convex or multiplicatively convex on a real interval $I = [a,b]$, if $\log f$ is convex , or, equivalently if for all $x, y \in I$ and all $\alpha \in [0,1]$, one has the inequality

$$f(\alpha x + (1-\alpha)y) \leq [f(x)]^{\alpha}[f(y)]^{(1-\alpha)}. \tag{1.1}$$

It is said to be log−concave if the inequality in (1.1) is reversed.

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is $P-$function or f belongs to the class of $P(I)$, if it is nonnegative and for all $x, y \in I$ and $\alpha \in [0,1]$, satisfies the following inequality

$$f(\alpha x + (1-\alpha)y) \leq f(x) + f(y).$$

For further information about log−convex functions and $P-$functions see [1, 2, 4–7, 9–11].

The following inequality is called Hermite-Hadamard inequality:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then the double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

holds.

The aim of this paper is to obtain some new integral inequalities like Hermite-Hadamard type for twice differentiable log−convex functions and $P-$functions.

In order to prove our main results we need the following Lemma from [8]:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $f'' \in L([a,b])$, then*

$$
\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx \\
&=\frac{(b-a)^2}{54}\left[\int_0^1 \left(2-t-t^2\right) f''\left(ta+(1-t)\frac{2a+b}{3}\right)dt\right. \\
&+\int_0^1 \left(2+t-t^2\right) f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)dt \\
&\left.+\int_0^1 \left(3t-t^2\right) f''\left(t\frac{a+2b}{3}+(1-t)b\right)dt\right].
\end{aligned}
$$

2. INEQUALITIES VIA log−CONVEXITY AND $P-$ FUNCTIONS

We shall start with the following result:

Theorem 1. *Let $f : I \to [0,\infty)$ be a twice differentiable mapping on I° such that $f'' \in L[a,b]$ where $a, b \in I^{\circ}$ with $a < b$. If $|f''|$ is log−convex on $[a,b]$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{54}\left\{\left|f''\left(\frac{2a+b}{3}\right)\right|\mu_K+\left|f''\left(\frac{a+2b}{3}\right)\right|\mu_M+\left|f''(b)\right|\mu_N\right\}
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_K &= -\frac{2}{\ln K}+\frac{3K-1}{(\ln K)^2}+\frac{2-2K}{(\ln K)^3}, \\
\mu_M &= \frac{2M-2}{\ln M}+\frac{M+1}{(\ln M)^2}+\frac{2-2M}{(\ln M)^3} \\
\mu_N &= \frac{2N}{\ln N}+\frac{3-N}{(\ln N)^2}+\frac{2-2N}{(\ln N)^3}
\end{aligned}
$$

and

$$
K=\frac{|f''(a)|}{\left|f''\left(\frac{2a+b}{3}\right)\right|}, \quad M=\frac{\left|f''\left(\frac{2a+b}{3}\right)\right|}{\left|f''\left(\frac{a+2b}{3}\right)\right|}, N=\frac{\left|f''\left(\frac{a+2b}{3}\right)\right|}{|f''(b)|}
$$

In the sequel of the paper, we set $K, M, N \neq 1$

Proof. From Lemma 1, property of the modulus and log−convexity of $|f''|$ we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\le \frac{(b-a)^2}{54}\left\{\int_0^1 \left(2-t-t^2\right)\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|dt\right. \\
&\quad+\int_0^1 \left(2+t-t^2\right)\left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right|dt \\
&\quad\left.+\int_0^1 \left(3t-t^2\right)\left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right|dt\right\} \\
&\le \frac{(b-a)^2}{54}\left\{\left|f''\left(\frac{2a+b}{3}\right)\right|\int_0^1 \left(2-t-t^2\right)\left[\frac{|f''(a)|}{\left|f''\left(\frac{2a+b}{3}\right)\right|}\right]^t dt\right. \\
&\quad+\left|f''\left(\frac{a+2b}{3}\right)\right|\int_0^1 \left(2+t-t^2\right)\left[\frac{\left|f''\left(\frac{2a+b}{3}\right)\right|}{\left|f''\left(\frac{a+2b}{3}\right)\right|}\right]^t dt \\
&\quad\left.+\left|f''(b)\right|\int_0^1 \left(3t-t^2\right)\left[\frac{\left|f''\left(\frac{a+2b}{3}\right)\right|}{|f''(b)|}\right]^t dt\right\}.
\end{aligned}$$

The proof is completed by making use of the necessary computation. □

Theorem 2. *Let $f : I \to [0,\infty)$ be a twice differentiable mapping on I° such that $f'' \in L[a,b]$ where $a,b \in I^\circ$ with $a < b$. If $|f''|^q$ is* log−*convex on $[a,b]$, then the following inequality holds for some fixed $q > 1$*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\le \frac{(b-a)^2}{54}\left\{\left(\frac{3^{p+2}-2^{p+1}(p+4)}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left|f''\left(\frac{2a+b}{3}\right)\right|\left(\frac{K^q-1-q\ln K}{(q\ln K)^2}\right)^{\frac{1}{q}}\right. \\
&\quad+\left(\frac{2^{p+1}(p+4)-2p-5}{(p+1)(p+2)}\right)\left|f''\left(\frac{a+2b}{3}\right)\right|\left(\frac{(2M^q-1)q\ln M+1-M^q}{(q\ln M)^2}\right)^{\frac{1}{q}} \\
&\quad\left.+\left(\frac{3^{p+2}-2^{p+1}(p+4)}{(p+1)(p+2)}\right)^{\frac{1}{p}}|f''(b)|\left(\frac{N^q q\ln N-N^q+1}{(q\ln N)^2}\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* K,M,N *are as in the aforementioned Theorem.*

Proof. From Lemma 1 and using the weighted version of Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{54} \\
&\quad\times\left\{\left(\int_0^1 (1-t)(2+t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad\left(\int_0^1 (1+t)(2-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1+t)\left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad\left.+\left(\int_0^1 t(3-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

If we use the log-convexity of $|f''|^q$ above and calculate the integrals we get the desired result. □

Theorem 3. *Let* $f: I\to[0,\infty)$ *be a twice differentiable mapping on* I° *such that* $f''\in L[a,b]$ *where* $a,b\in I^\circ$ *with* $a<b$. *If* $|f''|$ *is* $P-$*function on* $[a,b]$, *then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{54}\left[\frac{7|f''(a)|+20\left[\left|f''\left(\frac{2a+b}{3}\right)\right|+\left|f''\left(\frac{a+2b}{3}\right)\right|\right]+7|f''(b)|}{6}\right].
\end{aligned}$$

Proof. Since $|f''|^q$ is $P-$function on $[a,b]$, from Lemma 1 and properties of the modulus we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{54}\left\{\int_0^1 (2-t-t^2)\left[|f''(a)|+\left|f''\left(\frac{2a+b}{3}\right)\right|\right]\right. \\
&\quad+\int_0^1 (2+t-t^2)\left[\left|f''\left(\frac{2a+b}{3}\right)\right|+\left|f''\left(\frac{a+2b}{3}\right)\right|\right]
\end{aligned}$$

$$+\int_0^1 \left(3t-t^2\right)\left[\left|f''\left(\frac{a+2b}{3}\right)\right|+\left|f''(b)\right|\right]\right\}.$$

The proof is completed by making use of the necessary computation. □

Theorem 4. *Let $f : I \to [0,\infty)$ be a twice differentiable mapping on I° such that $f'' \in L[a,b]$ where $a,b \in I^\circ$ with $a < b$. If $|f''|^q$ is P−function on $[a,b]$, then the following inequality holds for some fixed $q > 1$*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)^2}{54}\left\{\left(\frac{3^{p+2}-2^{p+1}(p+4)}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left[\left(\frac{|f''(a)|^q+\left|f''\left(\frac{2a+b}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\frac{\left|f''\left(\frac{a+2b}{3}\right)\right|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}\right]$$

$$+\left(\frac{2^{p+1}(p+4)-2p-5}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{3\left[\left|f''\left(\frac{a+2b}{3}\right)\right|^q+\left|f''\left(\frac{2a+b}{3}\right)\right|^q\right]}{2}\right)^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $|f''|^q$ is P−function on $[a,b]$, from Lemma 1 and using the weighted version of Hölder's integral inequality, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)^2}{54}\left\{\left(\int_0^1 (1-t)(2+t)^p\,dt\right)^{\frac{1}{p}}\right.$$

$$\times\left(\int_0^1 (1-t)\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 (1+t)(2-t)^p\,dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 (1+t)\left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 t\,(3-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg\}$$

$$\leq \frac{(b-a)^2}{54}\Bigg\{\left(\int_0^1 (1-t)\,(2+t)^p\,dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 (1-t)\left[\left|f''(a)\right|^q+\left|f''\left(\frac{2a+b}{3}\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 (1+t)\,(2-t)^p\,dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 (1+t)\left[\left|f''\left(\frac{a+2b}{3}\right)\right|^q+\left|f''\left(\frac{2a+b}{3}\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 t\,(3-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left[\left|f''\left(\frac{a+2b}{3}\right)\right|^q+\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\Bigg\}.$$

If we calculate the above integrals, we get the desired result. □

Theorem 5. *Under the assumptions of above Theorem, we can write*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)^2}{54}\left\{\left(\frac{3^{p+1}-2^{p+1}}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f''(a)|^q+\left|f''\left(\frac{2a+b}{3}\right)\right|^q}{q+1}\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\frac{\left|f''\left(\frac{a+2b}{3}\right)\right|^q+|f''(b)|^q}{q+1}\right)^{\frac{1}{q}}\right]$$

$$\left.+\left(\frac{2^{p+1}-1}{p+1}\right)^{\frac{1}{p}}\left[(2^{q+1}-1)\left(\frac{\left|f''\left(\frac{a+2b}{3}\right)\right|^q+\left|f''\left(\frac{2a+b}{3}\right)\right|^q}{q+1}\right)\right]^{\frac{1}{q}}\right\}.$$

Proof. Since $|f''|^q$ is $P-$function on $[a,b]$, from Lemma 1 and using the Hölder's integral inequality, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)^2}{54}\left\{\left(\int_0^1 (2+t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^q\left[\left|f''(a)\right|^q+\left|f''\left(\frac{2a+b}{3}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^1 (2-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1+t)^q\left[\left|f''\left(\frac{a+2b}{3}\right)\right|^q+\left|f''\left(\frac{2a+b}{3}\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^1 (3-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t^q\left[\left|f''\left(\frac{a+2b}{3}\right)\right|^q+\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}.$$

The proof is completed by making use of the necessary computation. □

Theorem 6. *Let $f : I \to [0,\infty)$, be a twice differentiable mapping on I° such that $f'' \in L[a,b]$ where $a,b \in I^\circ$ with $a < b$. If $|f''|^q$ is P−function on $[a,b]$, then the following inequality holds for some fixed $q > 1$,*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{54}\left\{\left[\left|f''(a)\right|+\left|f''\left(\frac{2a+b}{3}\right)\right|\right]\left(\frac{1}{(p+1)^{\frac{1}{p}}}\right)\left(\frac{3^{q+1}-2^{q+1}}{(q+1)}\right)^{\frac{1}{q}}\right.$$
$$+\left[\left|f''\left(\frac{a+2b}{3}\right)\right|+\left|f''\left(\frac{2a+b}{3}\right)\right|\right]\left(\frac{2^{p+1}-1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2^{q+1}-1}{q+1}\right)^{\frac{1}{q}}$$
$$\left.\left[\left|f''\left(\frac{a+2b}{3}\right)\right|+\left|f''(b)\right|\right]\left(\frac{3^{q+1}-2^{q+1}}{q+1}\right)^{\frac{1}{q}}\left(\frac{1}{(p+1)^{\frac{1}{p}}}\right)\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $|f''|^q$ is P−function on $[a,b]$, from Lemma 1 and the weighted version of Hölder's inequality (see [3, pp117]), we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{54}\left\{\int_0^1 (1-t)\,(2+t)\left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right|dt\right.$$
$$+\int_0^1 (1+t)\,(2-t)\,dt\left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right|dt$$
$$\left.+\int_0^1 t\,(3-t)\left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right|dt\right\}$$

$$\leq \frac{(b-a)^2}{54}\left\{\left(\int_0^1 (1-t)^p \left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right| dt\right)^{\frac{1}{p}}\right.$$
$$\times\left(\int_0^1 (2+t)^q \left|f''\left(ta+(1-t)\frac{2a+b}{3}\right)\right| dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 (1+t)^p \left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right| dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1 (2-t)^q \left|f''\left(t\frac{2a+b}{3}+(1-t)\frac{a+2b}{3}\right)\right| dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 t^p \left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right| dt\right)^{\frac{1}{p}}$$
$$\left.\times\left(\int_0^1 (3-t)^q \left|f''\left(t\frac{a+2b}{3}+(1-t)b\right)\right| dt\right)^{\frac{1}{q}}\right\}.$$

The proof is completed by making use of the necessary computation. □

Remark 1. One can obtain inequalities for $quasi-$convex functions. The details are omitted to the interested reader.

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