ON THE MONOTONICITY OF $q$-SCHURER-STANCU TYPE POLYNOMIALS

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Abstract. Some properties of monotonicity and convexity of the q-Schurer-Stancu operators are considered. The paper contains also numerical examples based on Matlab algorithms, which verify these properties.

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1. PRELIMINARIES

In the last decades, the application of q-calculus represents one of the most interesting areas of research in approximation theory. Lupas [12] introduced in 1987 a q-type of the Bernstein operators and in 1997 another generalization of these operators based on q-integers was introduced by Phillips [16]. Their approximation properties were studied by Videnskii [18], N. Mahmudov [13], T. Acar and A. Aral [1] and O. Dalmanoglu [9, 10]. In time, many authors have been studied new classes of q-generalized operators ([2–4, 6, 7, 17]).

Before proceeding further, we mention some basic definitions and notations from q-calculus. For any fixed real number $q > 0$, the q-integer $[k]_q$, for $k \in \mathbb{N}$ is defined as

$$[k]_q = \begin{cases} 
\frac{1-q^k}{1-q}, & q \neq 1, \\
1, & q = 1.
\end{cases}$$

The q-factorial integer and the q-binomial coefficients are:

$$[k]_q! = \begin{cases} 
[k]_q[k-1]_q \ldots [1]_q, & k = 1,2, \ldots \\
1, & k = 0,
\end{cases}$$

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, (n \geq k \geq 0).$$
The q-analogue of \((x-a)^n_q\) is the polynomial
\[
(x-a)^n_q = \begin{cases} 
1, & \text{if } n = 0, \\
(x-a)(x-qa)\ldots(x-q^{n-1}a), & \text{if } n \geq 1.
\end{cases}
\]

Let \(p\) be a non-negative integer and let \(\alpha, \beta\) be some real parameters satisfying the conditions \(0 \leq \alpha \leq \beta\). In 2003, D. Bărboșu [8] introduced for any \(f \in C[0,1+p]\) and \(x \in [0,1]\) the Schurer-Stancu operators as follows
\[
S_{m,p}^{(\alpha,\beta)}(f,q,x) = \sum_{k=0}^{m+p} p_{m,k}(x) f \left( \frac{k + \alpha}{m + \beta} \right),
\]
where \(p_{m,k}(x) = \begin{pmatrix} m + p \\ k \end{pmatrix}_q x^k (1-x)^{m+p-k}.
\]

Recently, P.N. Agrawal, V. Gupta and A.S. Kumar [5] introduced the class of q-Schurer-Stancu operators. For any \(m \in \mathbb{N}\), \(p\) a fixed non negative integer number and \(\alpha, \beta\) some real parameters satisfying the conditions \(0 \leq \alpha \leq \beta\), they constructed the class of generalized q-Schurer-Stancu operators \(\mathcal{S}_{m,p}^{(\alpha,\beta)}\) as follows
\[
\mathcal{S}_{m,p}^{(\alpha,\beta)}(f,q,x) = \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) f \left( \frac{[k]_q + \alpha}{[m]_q + \beta} \right), \quad x \in [0,1],
\]
where \(\tilde{p}_{m,k}(x) = \begin{pmatrix} m + p \\ k \end{pmatrix}_q x^k (1-x)^{m+p-k}.
\]

If \(\alpha = \beta = 0\) the above operators reduce to the Bernstein-Schurer operators introduced by Muraru in [14].

**Lemma 1 ([5]).** For the operators defined in (1.1) the following properties hold

1. \(\mathcal{S}_{m,p}^{(\alpha,\beta)}(e_0,q,x) = 1\),
2. \(\mathcal{S}_{m,p}^{(\alpha,\beta)}(e_1,q,x) = \frac{\alpha}{[m]_q + \beta} + \frac{[m+p]_q}{[m]_q + \beta} x\),
3. \(\mathcal{S}_{m,p}^{(\alpha,\beta)}(e_2,q,x) = \frac{\alpha^2}{([m]_q + \beta)^2} + \frac{[m+p]_q^2}{([m]_q + \beta)^2} x^2 + \frac{2\alpha [m+p]_q x}{([m]_q + \beta)^2} + \frac{[m+p]_q x(1-x)}{([m]_q + \beta)^2}\).
ON THE MONOTONICITY OF \( q \)-SCHURER-STANCU TYPE POLYNOMIALS

Theorem 1 ([5]). If \( f \in C[0, 1 + p] \) and \( q \in (0, 1) \) then
\[
\left\| \tilde{S}_{m,p}^{(\alpha, \beta)}(f, q, x) - f(x) \right\| \leq \frac{5}{4} \omega_f(\delta_m)
\]
holds, where
\[
\delta_m = \frac{1}{[m]_q + \beta} \sqrt{[m + p]_q + 4(q^{m}[p]_q + \alpha - \beta)^2}.
\]

2. MONOTONICITY OF THE \( q \)-SCHURER-STANCU OPERATORS

Oruc and Philips [15] showed that for a convex function \( f \) on \([0,1]\), the \( q \)-Bernstein polynomials are monotonic decreasing. In this section we will prove a similar result for \( q \)-Schurer-Stancu operators.

Theorem 2. Let \( f \) be a convex and increasing function on \([0, p + 1]\). Then, for \( 0 < q < 1 \) and \( \beta \leq \frac{[p]_q}{q^p} \),
\[
\tilde{S}_{m-1,p}^{(\alpha, \beta)}(f, q, x) \geq \tilde{S}_{m,p}^{(\alpha, \beta)}(f, q, x),
\]
for \( 0 \leq x \leq 1 \) and \( m \geq 2 \).

Proof. For \( 0 < q \leq 1 \) we have
\[
\sum_{k=0}^{m+p-1} (1 - q^s x)^{-1} \tilde{S}_{m-1,p}^{(\alpha, \beta)}(f, q, x) - \tilde{S}_{m,p}^{(\alpha, \beta)}(f, q, x)
\]
\[
= \sum_{k=0}^{m+p-1} \left[ \sum_{k=0}^{m+p-1} (1 - q^s x)^{-1} f \left( \frac{[k]_q + \alpha}{[m-1]_q + \beta} \right) \right]
\]
\[
- \sum_{k=0}^{m+p} \left[ \sum_{k=0}^{m+p} (1 - q^s x)^{-1} f \left( \frac{[k]_q + \alpha}{[m]_q + \beta} \right) \right]
\]
Denote
\[
\psi_k(x) = x^k \prod_{s=m+p-k}^{m+p-1} (1 - q^s x)^{-1}
\]
and using the following relation
\[
x^k \prod_{s=m+p-k-1}^{m+p-1} (1 - q^s x)^{-1} = \psi_k(x) + q^{m+p-k-1} \psi_{k+1}(x)
\]
we find
\[
\tilde{S}_{m-1,p}^{(\alpha, \beta)}(f, q, x) - \tilde{S}_{m,p}^{(\alpha, \beta)}(f, q, x)
\]
Let $a$ be such that $0 < a < 1$. Then

$$\alpha_k \psi_k(x) + \left\{ f \left( \frac{[m+p-k]_q}{[m]_q} \right) - f \left( \frac{[m+p]_q}{[m]_q} \right) \right\} \psi_{m+p}(x)$$

$$+ \left\{ f \left( \frac{[m+p]_q}{[m]_q} \right) - f \left( \frac{\alpha}{[m]_q} \right) \right\} \psi_0(x),$$

where

$$a_k = f \left( \frac{[k]_q + \alpha}{[m-1]_q + \beta} \right) \left[ m + p - k \right]_q q^{m+p-k} f \left( \frac{[k]_q + \alpha}{[m-1]_q + \beta} \right) \frac{[k]_q}{[m+p]_q}$$

$$- f \left( \frac{[k]_q + \alpha}{[m]_q + \beta} \right).$$

From (2.2) it is clear that each $\psi_k(x)$ is non-negative on $[0, 1]$ for $0 \leq q \leq 1$ and thus, it suffices to show that each $a_k$ is non-negative.

Since $f$ is convex on $[0, p + 1]$, for any $t_0, t_1$ such that $0 \leq t_0 < t_1 \leq p + 1$ and any $\lambda, 0 < \lambda < 1$, we have

$$f(\lambda t_0 + (1-\lambda) t_1) \leq \lambda f(t_0) + (1-\lambda) f(t_1).$$

(2.3)

Let $t_0 = \frac{[k-1]_q + \alpha}{[m-1]_q + \beta}$, $t_1 = \frac{[k]_q + \alpha}{[m-1]_q + \beta}$ and $\lambda = q^{m+p-k} \frac{[k]_q}{[m+p]_q}$. Then $0 \leq t_0 < t_1 \leq p + 1$ and $0 < \lambda < 1$ for $1 \leq k \leq m + p - 1$. If we replace them in the relation (2.3), it follows

$$q^{m+p-k} \frac{[k]_q}{[m+p]_q} f \left( \frac{[k-1]_q + \alpha}{[m-1]_q + \beta} \right) + \frac{[m+p-k]_q}{[m]_q} f \left( \frac{[k]_q + \alpha}{[m-1]_q + \beta} \right)$$
follows from the inequality (2.4) we obtain
\[ f \left( \frac{[k]q + \alpha}{[m-1]q + \beta} \right) \geq \frac{[k]q + \alpha}{[m+1]q + \beta} + \frac{[m+p-k]q}{[m+1]q + \beta}. \]

Using the inequality \([k]q([k-1]q + \alpha) \geq ([k]q + \alpha)[k-1]q\) and \(f\) increasing function, it follows
\[ f \left( \frac{[k]q + \alpha}{[m-1]q + \beta} \right) \geq f \left( \frac{[k]q + \alpha}{[m+1]q + \beta} \right) + f \left( \frac{[m+p-k]q}{[m+1]q + \beta} \right). \]

Since \(f\) is increasing on \([0, p+1]\) and
\[ a_k = f \left( \frac{[k]q + \alpha}{[m-1]q + \beta} \right) + f \left( \frac{[m+p-k]q}{[m+1]q + \beta} \right) \]
from the inequality (2.4) we obtain \(a_k \geq 0, k = 1, m + p - 1\).

Therefore \(\tilde{S}^{(\alpha, \beta)}_{m-1,p}(f,q,x) \geq \tilde{S}^{(\alpha, \beta)}_{m,p}(f,q,x)\).

For \(q = 1\) and \(0 \leq x < 1\) in a similar way the property (2.1) is verified.

For \(q = 1\) and \(x = 1\) we have
\[ \tilde{S}^{(\alpha, \beta)}_{m-1,p}(f,1,1) - \tilde{S}^{(\alpha, \beta)}_{m,p}(f,1,1) = f \left( \frac{m+p-1+\alpha}{m+1+\beta} \right) - f \left( \frac{m+p+\alpha}{m+1+\beta} \right) \geq 0. \]

\(\square\)

**Theorem 3.** If \(f\) is convex, then for all \(m \geq 1\) and \(0 < q \leq 1\) it follows

i) \(\tilde{S}^{(\alpha, \beta)}_{m,p}(f,q,x) \geq f(x), \) for \(x \in [0,1], \) \(f\) increasing on \([0,1]\) and \(\beta = \alpha + \epsilon, \epsilon \in [0,q^m[p]q];\)

ii) \(\tilde{S}^{(\alpha, \beta)}_{m,p}(f,q,x) \geq f(x), \) for \(x \in \left[0, \frac{\alpha}{\beta - q^m[p]q}\right], \) \(f\) increasing on \([0,1]\) and \(\beta > \alpha + q^m[p]q;\)

iii) \(\tilde{S}^{(\alpha, \beta)}_{m,p}(f,q,x) \geq f(x), \) for \(x \in \left(\frac{\alpha}{\beta - q^m[p]q}, 1\right], \) \(f\) decreasing on \([0,1]\) and \(\beta > \alpha + q^m[p]q.\)

**Proof.** We consider the knots \(x_k = \frac{[k]q + \alpha}{[m]q + \beta}, \) \(0 \leq k \leq m + p.\) From Lemma 1 it follows
\[ \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) = 1, \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x)x_k = \frac{\alpha}{[m]q + \beta} + \frac{[m+p]q}{[m]q + \beta} x. \]
Using the convexity of function $f$ we have

$$
\tilde{S}^{(\alpha, \beta)}_{m, p}(f, q, x) = \sum_{k=0}^{m+p} \tilde{\beta}_{m,k}(x) f(x_k) \\
= f \left( \frac{\alpha}{[m]_q + \beta} + \frac{[m + p]_q}{[m]_q + \beta} x \right).
$$

The following inequalities hold

a) $\frac{\alpha}{[m]_q + \beta} + \frac{[m + p]_q}{[m]_q + \beta} x \geq x$ for $\beta = \alpha + \epsilon, \epsilon \in [0, q^m[p]], x \in [0, 1]$;

b) $\frac{\alpha}{[m]_q + \beta} + \frac{[m + p]_q}{[m]_q + \beta} x \geq x$ for $\beta > \alpha + q^m[p], x \in \left[0, \frac{\alpha}{\beta - q^m[p]} \right]$;

c) $\frac{\alpha}{[m]_q + \beta} + \frac{[m + p]_q}{[m]_q + \beta} x \leq x$ for $\beta > \alpha + q^m[p], x \in \left[\frac{\alpha}{\beta - q^m[p]}, 1 \right]$.

The theorem is proved using the monotony of function $f$ and the inequalities a)-c).

3. NUMERICAL EXAMPLE

Davis [11] proved that for any convex function $f$, the classical Bernstein polynomial is convex and the sequence of Bernstein polynomials is monotonic decreasing. Oruc and Philips [15] extend these results for the Bernstein operators in $q$-calculus for $0 < q \leq 1$. In this section we will verify numerically these properties for the q-Schurer-Stancu operators.

**Table 1. The q-Schurer-Stancu operators**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\tilde{S}^{(3,5)}_{30,6}(f, q, x)$</th>
<th>$\tilde{S}^{(3,5)}_{90,6}(f, q, x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.029115231597413</td>
<td>0.02656252311249</td>
</tr>
<tr>
<td>0.1</td>
<td>0.089811163243826</td>
<td>0.08338394253406</td>
</tr>
<tr>
<td>0.2</td>
<td>0.191687642176340</td>
<td>0.179475155093384</td>
</tr>
<tr>
<td>0.3</td>
<td>0.347058215579779</td>
<td>0.326749960898829</td>
</tr>
<tr>
<td>0.4</td>
<td>0.570050753689721</td>
<td>0.538887425606676</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8767866566660820</td>
<td>0.831510780091276</td>
</tr>
<tr>
<td>0.6</td>
<td>1.285573442130916</td>
<td>1.222376845885256</td>
</tr>
<tr>
<td>0.7</td>
<td>1.817111513724159</td>
<td>1.731579581814080</td>
</tr>
<tr>
<td>0.8</td>
<td>2.494715950321931</td>
<td>2.381768592612226</td>
</tr>
<tr>
<td>0.9</td>
<td>3.344554197167635</td>
<td>3.198383491142964</td>
</tr>
<tr>
<td>1</td>
<td>4.395900582819147</td>
<td>4.209905050209471</td>
</tr>
</tbody>
</table>
In Table 1 are calculated the values of the q-Schurer-Stancu operators $S_{30,6}^{(3,5)}(f,q,x)$ and $S_{90,6}^{(3,5)}(f,q,x)$ for $f(x) = x^3 e^x + 1$ and $q = 0.9$. Also, in the Figure 1 are given the graphics of these operators.

**Figure 1.** The monotonicity of the q-Schurer-Stancu operators

In the next part of this section we will give some numerical examples which verify the inequalities proved in Theorem 3.

**Example 1.** If $n = 50$, $p = 5$, $\alpha = 3$, $\beta = 3.0211$, $q = 0.9$, $f(x) = x^3 e^x + 1$, it follows $S_{m,p}^{(\alpha,\beta)}(f,q,x) \geq f(x)$ for all $x \in [0,1]$.

**Figure 2.** The q-Schurer-Stancu operators for increasing function and $\beta = \alpha + \epsilon$
Example 2. If \( n = 50,\ p = 5,\ \alpha = 3,\ q = 0.9,\ \beta = 8.0211,\ f(x) = x^3 e^{x+1},\) it follows \( \tilde{S}_{m,p}^{(\alpha,\beta)}(f,q,x) \geq f(x)\) for all \( x \in [0,0.375].\)

![Figure 3](image-url)  
**Figure 3.** The \( q\)-Schurer-Stancu operators for increasing function and \( \beta > \alpha + q^m[p]_q\)

Example 3. If \( n = 50,\ p = 5,\ \alpha = 3,\ q = 0.9,\ \beta = 8.0211,\ f(x) = e^{-x^2},\) it follows \( \tilde{S}_{m,p}^{(\alpha,\beta)}(f,q,x) \geq f(x)\) for all \( x \in [0.375, 1].\)

![Figure 4](image-url)  
**Figure 4.** The \( q\)-Schurer-Stancu operators for decreasing function

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ON THE MONOTONICITY OF q-SCHURER-STANCU TYPE POLYNOMIALS 27

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