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BOUNDS FOR THE ARITHMETIC MEAN IN TERMS OF THE TOADER MEAN AND OTHER BIVARIATE MEANS

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Abstract. We find the greatest values α_1 and α_2, and the least values β_1 and β_2, such that the double inequalities $\alpha_1 T(a,b)+(1-\alpha_1)H(a,b) < A(a,b) < \beta_1 T(a,b)+(1-\beta_1)H(a,b)$ and $\alpha_2 T(a,b)+(1-\alpha_2)\overline{H}(a,b) < A(a,b) < \beta_2 T(a,b)+(1-\beta_2)\overline{H}(a,b)$ hold for all $a,b > 0$ with $a \neq b$. Here, $\overline{H}(a,b) = \sqrt{2}ab/\sqrt{a^2+b^2}$, $H(a,b) = 2ab/(a+b)$, $A(a,b) = (a+b)/2$, and $T(a,b) = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\, d\theta$ denote the harmonic root-square, harmonic, arithmetic and Toader means of a and b, respectively.

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Keywords: harmonic root-square mean, harmonic, arithmetic, Toader mean, complete elliptic integrals

1. INTRODUCTION

In [16], Toader introduced the Toader mean $T(a,b)$ of two positive numbers a and b as follows:

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\, d\theta = \begin{cases} 2a\mathcal{E}(\sqrt{1-(b/a)^2})/\pi, & a > b, \\ 2b\mathcal{E}(\sqrt{1-(a/b)^2})/\pi, & a < b, \\ a, & a = b. \end{cases} \tag{1.1}$$

where $\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}d\theta (r \in [0,1])$ is the complete elliptic integral of the second kind.

Recently, the Toader mean has been the object of intensive research. In particular, many remarkable inequalities for the Toader mean $T(a,b)$ can be found in the literatures [9–12, 14–16]

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Let $\overline{H}(a,b) = \sqrt{2}ab/\sqrt{a^2+b^2}, H(a,b) = 2ab/(a+b), G(a,b) = \sqrt{ab}$, $A(a,b) = (a+b)/2, S(a,b) = \sqrt{(a^2+b^2)/2}, C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic root-squre, harmonic, geometric, arithmetic, quadratic, and contraharmonic means of a and b, respectively. Then it is known that the inequalities

$$\overline{H}(a,b) < H(a,b) < G(a,b) < A(a,b) < S(a,b) < C(a,b) \tag{1.2}$$

hold for all $a,b > 0$ with $a \neq b$.

For $p \in \mathbb{R}, a,b > 0$ with $a \neq b$ the power mean $M_p(a,b)$ is defined by

$$M_p(a,b) = \begin{cases} \left(\frac{a^p+a^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \tag{1.3}$$

It is well known that the power mean $M_p(a,b)$ is continuous and strictly increasing with respect to p. Many means are special case of $M_p(a,b)$, for example, $M_{-1}(a,b) = H(a,b) = 2ab/(a+b), M_0(a,b) = G(a,b) = \sqrt{ab}, M_1(a,b) = A(a,b) = (a+b)/2, M_2(a,b) = S(a,b) = \sqrt{(a^2+b^2)/2}$.

Vuorinen [17] conjectured that

$$M_{3/2}(a,b) < T(a,b) \tag{1.4}$$

for all $a,b > 0$ with $a \neq b$. This conjecture was solved by Qiu and Shen[14], and Barnard, Pearce and Richards in [6], respectively.

In [1], Alzer and Qiu presented that

$$T(a,b) < M_{\log 2/\log(\pi/2)}(a,b) \tag{1.5}$$

for all $a,b > 0$ with $a \neq b$, which gives a best possible upper bound for Toader mean in terms of the power mean.

From (1.4) and (1.5) one concludes that

$$A(a,b) < T(a,b) < S(a,b) \tag{1.6}$$

for all $a,b > 0$ with $a \neq b$.

In [11], the authors demonstrated that the double inequality

$$\alpha S(a,b) + (1-\alpha)A(a,b) < T(a,b) < \beta S(a,b) + (1-\beta)A(a,b) \tag{1.7}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq \frac{1}{2}$ and $\beta \geq \frac{4-\pi}{(\sqrt{2}-1)\pi}$.

In [15], the authors proved that the double inequalities

$$\begin{gathered} \alpha_1 C(a,b) + (1-\alpha_1)A(a,b) < T(a,b) < \beta_1 C(a,b) + (1-\beta_1)A(a,b), \\ \frac{\alpha_2}{A(a,b)} + \frac{1-\alpha_2}{C(a,b)} < \frac{1}{T(a,b)} < \frac{\beta_2}{A(a,b)} + \frac{1-\beta_2}{C(a,b)} \end{gathered} \tag{1.8}$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/4, \beta_1 \geq 4/\pi - 1$ and $\alpha_2 \leq \pi/2 - 1, \beta_2 \geq 3/4$.

The main purpose of this paper is to find the greatest values α_1, α_2 and the least values β_1, β_2 in $(0,1)$ such that the double inequalities

$$\alpha_1 T(a,b) + (1-\alpha_1)H(a,b) < A(a,b) < \beta_1 T(a,b) + (1-\beta_1)H(a,b),$$
$$\alpha_2 T(a,b) + (1-\alpha_2)\overline{H}(a,b) < A(a,b) < \beta_2 T(a,b) + (1-\beta_2)\overline{H}(a,b),$$

hold for all $a,b > 0$ with $a \neq b$.

As an application, we get a new lower bound for the complete elliptic integral of the second kind in terms of elementary functions, which improves some well-known results.

2. Preliminaries and lemmas

In order to establish our main result we need several lemmas, which we present in this section.

For $0 < r < 1$ and $r' = \sqrt{1-r^2}$ Legendre's complete elliptic integrals of the first and second kinds [7, 8] are defined by

$$\begin{cases} \mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} (1-r^2\sin^2\theta)^{-1/2} d\theta, \\ \mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{K}(0) = \pi/2, \qquad \mathcal{K}(1) = \infty \end{cases}$$

and

$$\begin{cases} \mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} (1-r^2\sin^2\theta)^{1/2} d\theta, \\ \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r'), \\ \mathcal{E}(0) = \pi/2, \qquad \mathcal{E}(1) = 1, \end{cases}$$

respectively.

For $0 < r < 1$, the following formulas were presented in [4, Appendix E, pp. 474-475]:

$$\frac{d\mathcal{K}}{dr} = \frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2}, \qquad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E} - \mathcal{K}}{r},$$
$$\frac{d(\mathcal{E} - r'^2\mathcal{K})}{dr} = r\mathcal{K}, \qquad \frac{d(\mathcal{K} - \mathcal{E})}{dr} = \frac{r\mathcal{E}}{r'^2},$$
$$\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\mathcal{E} - r'^2\mathcal{K}}{1+r}. \tag{2.1}$$

The following lemma can be found in [4, Theorem 3.21 (1)].

Lemma 1. $(\mathcal{E} - r'^2\mathcal{K})/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4, 1)$.

Lemma 2 (Theorem 1.25 in [4]). *For $-\infty < a < b < \infty$, let $f,g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b), let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \text{ and } \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

3. MAIN RESULTS

Now we are in a position to state and prove our main results.

Theorem 1. *The double inequality*

$$\alpha_1 T(a,b) + (1-\alpha_1)H(a,b) < A(a,b) < \beta_1 T(a,b) + (1-\beta_1)H(a,b) \tag{3.1}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq \frac{\pi}{4}$ and $\beta_1 \geq \frac{4}{5}$.

Proof. Without loss of generality, we assume that $a > b$. Let $t = b/a \in (0,1)$ and $r = \frac{1-t}{1+t}$, then

$$\begin{aligned}
\frac{A(a,b)-H(a,b)}{T(a,b)-H(a,b)} &= \frac{\frac{1+t}{2} - \frac{2t}{1+t}}{\frac{2}{\pi}\mathcal{E}'(t) - \frac{2t}{1+t}} \\
&= \frac{\frac{1}{1+r} - (1-r)}{\frac{2}{\pi}\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) - (1-r)} \\
&= \frac{r^2}{\frac{2}{\pi}(2\mathcal{E} - r'^2\mathcal{K}) - 1 + r^2}.
\end{aligned} \tag{3.2}$$

Let $f_1(r) = r^2$, $f_2(r) = \frac{2}{\pi}(2\mathcal{E} - r'^2\mathcal{K}) - 1 + r^2$, and

$$f(r) = \frac{f_1(r)}{f_2(r)} = \frac{r^2}{\frac{2}{\pi}(2\mathcal{E} - r'^2\mathcal{K}) - 1 + r^2}. \tag{3.3}$$

Then simple computations lead to

$$f_1(0) = f_2(0) = 0, \tag{3.4}$$

$$f_1'(r) = 2r, \tag{3.5}$$

$$f_2'(r) = \frac{2}{\pi}\frac{\mathcal{E} - r'^2\mathcal{K}}{r} + 2r, \tag{3.6}$$

$$\frac{f_1'(r)}{f_2'(r)} = \frac{1}{\frac{1}{\pi}\frac{\mathcal{E}-r'^2\mathcal{K}}{r^2} + 1}. \tag{3.7}$$

It follows from (3.7) with Lemma 1 and Lemma 2 that $f(r)$ is strictly decreasing in $(0,1)$. Moreover, using L'Hôpital's rule we get

$$\lim_{x\to 0^+} f(x) = \frac{4}{5}, \tag{3.8}$$

and

$$\lim_{x\to 1^-} f(x) = \frac{\pi}{4}. \tag{3.9}$$

Therefore, Theorem 1 follows from (3.8) and (3.9) together with the monotonicity of $f(r)$. □

From Theorem1 we get a new bound for the complete elliptic integral $\mathcal{E}(r)$ of the second kind in terms of elementary functions as follows.

Corollary 1. *For $r\in(0,1)$ and $r'=\sqrt{1-r^2}$, we have*

$$\frac{\pi}{2}\left[\frac{5}{8}(1+r')-\frac{r'}{2(1+r')}\right] < \mathcal{E}(r) < \frac{\pi}{2}\left[\frac{2(1+r')}{\pi}-(1-\frac{4}{\pi})\frac{2r'}{1+r'}\right]. \tag{3.10}$$

Proof. Without loss of generality, assume that $a>b$. Substituting $r'=\frac{b}{a}, \alpha_1=\frac{\pi}{4}, \beta_1=\frac{4}{5}$ into Theorem 1 produces Corollary 1. □

Theorem 2. *The double inequality*

$$\alpha_2 T(a,b)+(1-\alpha_2)\overline{H}(a,b) < A(a,b) < \beta_2 T(a,b)+(1-\beta_2)\overline{H}(a,b), \tag{3.11}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_2\le\frac{\pi}{4}$ and $\beta_2\ge\frac{6}{7}$.

Proof. Without loss of generality, we assume that $a>b$. Let $t=b/a\in(0,1)$ and $r=\frac{1-t}{1+t}$, then

$$\begin{aligned}\frac{A(a,b)-\overline{H}(a,b)}{T(a,b)-\overline{H}(a,b)} &= \frac{\frac{1+t}{2}-\frac{\sqrt{2}t}{\sqrt{1+t^2}}}{\frac{\pi}{2}\mathcal{E}'(t)-\frac{\sqrt{2}t}{\sqrt{1+t^2}}}\\ &= \frac{\frac{1}{1+r}-\frac{1-r}{\sqrt{1+r^2}}}{\frac{2}{\pi}\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right)-\frac{1-r}{\sqrt{1+r^2}}}\\ &= \frac{1-\frac{1-r^2}{\sqrt{1+r^2}}}{\frac{2}{\pi}(2\mathcal{E}-r'^2\mathcal{K})-\frac{1-r^2}{\sqrt{1+r^2}}}\end{aligned} \tag{3.12}$$

Let $g_1(r)=1-\frac{1-r^2}{\sqrt{1+r^2}}$, $g_2(r)=\frac{2}{\pi}(2\mathcal{E}-r'^2\mathcal{K})-\frac{1-r^2}{\sqrt{1+r^2}}$ and

$$g(r)=\frac{g_1(r)}{g_2(r)}=\frac{1-\frac{1-r^2}{\sqrt{1+r^2}}}{\frac{2}{\pi}(2\mathcal{E}-r'^2\mathcal{K})-\frac{1-r^2}{\sqrt{1+r^2}}}. \tag{3.13}$$

Simple calculations show that

$$g_1(0) = g_2(0) = 0, \tag{3.14}$$

$$g_1'(r) = \frac{r(r^2+3)}{(1+r^2)^{3/2}}, \tag{3.15}$$

$$g_2'(r) = \frac{2}{\pi}\frac{\mathcal{E} - r'^2\mathcal{K}}{r} + \frac{r(r^2+3)}{(1+r^2)^{3/2}}, \tag{3.16}$$

$$\frac{g_1'(r)}{g_2'(r)} = \frac{\frac{(r^2+3)}{(1+r^2)^{3/2}}}{\frac{2}{\pi}\frac{\mathcal{E}-r'^2\mathcal{K}}{r^2} + \frac{(r^2+3)}{(1+r^2)^{3/2}}} = \frac{1}{\frac{\frac{2}{\pi}\frac{\mathcal{E}-r'^2\mathcal{K}}{r^2}}{\frac{(r^2+3)}{(1+r^2)^{3/2}}} + 1}. \tag{3.17}$$

It is easy to verify that the function $\frac{(r^2+3)}{(1+r^2)^{3/2}}$ is positive and strictly decreasing in $(0,1)$, then (3.17) and Lemma 1 lead to the conclusion that $\frac{g_1'(r)}{g_2'(r)}$ is strictly decreasing in $(0,1)$. Hence, $g(r)$ is strictly decreasing directly from Lemma 2. Moreover, the usage of L'Hôpital's rule and standard argument show that

$$\lim_{x\to 0^+} g(x) = \frac{6}{7}, \tag{3.18}$$

and

$$\lim_{x\to 1^-} g(x) = \frac{\pi}{4}. \tag{3.19}$$

Thus, Theorem 2 follows from (3.18) and (3.19) together with the monotonicity of $g(r)$.

□

4. COMPARISON WITH SOME WELL-KNOWN RESULTS

Recently, the complete elliptic integrals have attracted the attention of numerous mathematicians. In particular, many remarkable properties and inequalities for the complete elliptic integrals can be found in the literature [1–3, 5, 11, 13, 18–21].

In [11], the authors obtained that

$$\frac{\pi}{2}\left[\frac{1}{2}\sqrt{\frac{1+r'^2}{2}} + \frac{1+r'}{4}\right] < \mathcal{E}(r) < \frac{\pi}{2}\left[\left(\frac{4-\pi}{(\sqrt{2}-1)\pi}\sqrt{\frac{1+r'^2}{2}} + \frac{(\sqrt{2}\pi-4)(1+r')}{2(\sqrt{2}-1)\pi}\right],\right. \tag{4.1}$$

for all $r \in (0,1)$ and $r' = \sqrt{1-r^2}$.

Guo and Qi [13] proved that

$$\frac{\pi}{2} - \frac{1}{2}\log\frac{(1+r)^{1-r}}{(1-r)^{1+r}} < \mathcal{E}(r) < \frac{\pi-1}{2} + \frac{1-r^2}{4r}\log\frac{1+r}{1-r}, \tag{4.2}$$

for all $r \in (0,1)$.

It was pointed out in [11] that the bounds in (4.1) for $\mathcal{E}(r)$ are better than that in (4.2) for some $r \in (0,1)$.

Let $g(x) = \frac{5}{8}(1+x) - \frac{x}{2(1+x)} - \left[\frac{1}{2}\sqrt{\frac{1+x^2}{2}} + \frac{1+x}{4}\right], x \in (0,1)$. Then simple computations lead to

$$g(x) = \frac{3x^2+2x+3-2\sqrt{2(1+x^2)}(1+x)}{8(1+x)}. \tag{4.3}$$

Since

$$\left(3x^2+2x+3\right)^2 - \left(2\sqrt{2(1+x^2)}(1+x)\right)^2 = (1-x)^4 > 0,$$

we clearly see that $g(x) > 0$, which shows that the lower bounds in (3.10) for $\mathcal{E}(r)$ is better than the one in (4.1).

Very recently, Yin and Qi obtained in [20] that

$$\frac{\pi}{2}\frac{\sqrt{6+2\sqrt{1-r^2}-3r^2}}{2\sqrt{2}} \le \mathcal{E}(r) \le \frac{\pi}{2}\frac{\sqrt{10-2\sqrt{1-r^2}-5r^2}}{2\sqrt{2}}. \tag{4.4}$$

Since

$$\frac{5}{8}(1+x) - \frac{x}{2(1+x)} > \frac{\sqrt{6+2x-3(1-x^2)}}{2\sqrt{2}}$$

is equivalent to

$$\left(5x^2+6x+5\right)^2 > 8(x+1)^2\left(3x^2+2x+3\right)$$

and $(x-1)^4 > 0$, the lower bound in (3.10) for $\mathcal{E}(r)$ is better than the one in (4.4).

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