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# PI-PROPERTIES OF SOME MATRIX ALGEBRAS WITH INVOLUTION

TSETSKA RASHKOVA

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*Abstract.* We define the nilpotency index of the $\flat$-variables in second order matrix algebras with Grassmann entries and involution $\flat$. Identities of minimal degree are found for a concrete subalgebra of the matrix algebra $M_4(K)$. When it has an involution $\phi$ as well some of its $\phi$-identities are given. For an analogue of this subalgebra over finite dimensional Grassmann algebras a new involution $(\flat)$ is introduced and its $(\flat)$-identities are discussed.

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## 1. INTRODUCTION

The classical PI-theory (the theory of the polynomial identities) has its development for algebras with involution as well. The contributions of Amitsur [1], Levchenko [9], Rowen [14], Wenxin and Racine [17], Giambruno and Valenti [6], Drensky and Giambruno [5], Rashkova [11], La Mattina and Misso [8] are only a part of it.

In 1973 Krasovski and Regev [7] described completely the $T$-ideal of the identities of the Grassmann algebra $E$ and it was a natural step to investigate the PI-structure of algebras not only over fields (with any characteristic) but over algebras as well, especially Grassmann algebras [4, 12, 16].

In the paper we consider mainly finite dimensional Grassmann algebras and special matrix algebras over them.

We recall the definition of the Grassmann algebra $E$ as:

$$E = K\langle e_1, e_2, ...| e_i e_j + e_j e_i = 0,\ i, j = 1, 2, ...\rangle,$$

where $K$ is a field of characteristic zero.

We cite basic propositions from [3, 7]. The notation $[x, y, z] = [[x, y], z] = [x, y]z - z[x, y]$ will be used.

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**Proposition 1** ([7, Corollary, p. 437])**.** *The $T$ - ideal of the Grassmann algebra $E$ is generated by the identity $[x,y,z]=0$.*

**Proposition 2** ([3, Lemma 6.1])**.** *For any $n,k \geq 2$ in the algebra $E$ the identity $S_n^k(x_1,...,x_n)=0$ holds, where*

$$S_n(x_1,...,x_n)=\sum_{\sigma\in Sym(n)}(-1)^{\sigma}x_{\sigma(1)}...x_{\sigma(n)}$$

*is the $n$-th standard polynomial.*

**Proposition 3** ([3, Lemma 6.6])**.** *The matrix algebra $M_n(E)$ does not satisfy the identity*

$$S_m^n(x_1,...,x_m)=0$$

*for any $m$.*

There are subalgebras of $M_n(E)$ however being counter examples of Proposition 3 for concrete $m$.

We use the notation $E_n^{'}$ for a non unitary Grassmann algebra with generators $e_1,...,e_n$.

The existence of nilpotent elements of minimal nilpotency index both in finite dimensional Grassmann algebras and in matrix algebras over them was investigated in [12, 13]. We state some of the results needed:

**Proposition 4** ([13, Proposition 13])**.** *The identity $x^3=0$ holds for the algebra $E_4^{'}$.*

**Proposition 5** ([13, Proposition 16])**.** *The algebra $M_2(E_4^{'})$ satisfies the identity $X^4=0$.*

In [13] examples were given as well of subalgebras $\mathfrak{A}_{\mathrm{i}}, i=1,2$ of $M_n(\mathfrak{R})$ such that the identities $x^4=0$ and $[x,y,z]=0$ in $\mathfrak{R}$ imply the identity $X^4=0$ in $\mathfrak{A}_{\mathrm{i}}, i=1,2$.

An involution $\psi$ on the Grassmann algebras $E_2^{'}$ and $E_3^{'}$ defines an involution $\phi$ on the corresponding $2\times 2$ matrix algebra over any of them. In that case the classes of symmetric and of skew-symmetric to the involution $\phi$ matrices of nilpotency indices 2 and 3 were described in [12].

In the present paper we continue the investigations started in [12]:

We define the nilpotency index of the $\flat$-variables in the considered algebras with involution $\phi=\flat$.

For a concrete subalgebra of the matrix algebra $M_4(K)$ identities of minimal degree are found. When additionally the algebra has an involution $\phi$ some of its $\phi$-identities are given.

For an analogue of this subalgebra over finite dimensional Grassmann algebras a new involution $\phi=(\flat)$ is introduced and some $(\flat)$-identities are discussed.

## 2. Results

### 2.1. *PI-properties of involution second order matrix algebras with Grassmann entries*

We recall the definition of an involution on an algebra $R$: it is a second order antiautomorphism $\psi$ such that $\psi(ab) = \psi(b)\psi(a)$ for all $a, b \in R$.

By $R^-$ we denote the skew-symmetric due to the involution elements of $R$, namely $z_1, .., z_i, ...$ and by $R^+$ we denote the symmetric due to the involution elements $y_1, .., y_j, ....$ It is important to consider $\psi$-variables (symmetric and skew-symmetric) as the elements of $R^+$ form a Jordan algebra due to the multiplication $y_1 \circ y_2 = y_1 y_2 + y_2 y_1$ and the elements of $R^-$ form a Lie algebra due to the operation $[z_1, z_2]$.

**Definition 1.** Let $f = f(x_1, ..., x_m) \in K\langle x_1, ..., x_n\rangle$, the free associative algebra on $n$ generators over $K$. We say that $f$ is a $\psi$-identity in skew variables for the algebra $R$ over $K$ if $f(z_1, ..., z_m) = 0$ for all $z_1, ..., z_m \in R^-$. Accordingly $f$ is a $\psi$-identity in symmetric variables for the algebra $R$ over $K$ if $f(y_1, ..., y_m) = 0$ for all $y_1, ..., y_m \in R^+$.

We say that $f$ is a $\psi$-identity if $f(z_1, ..., z_i, y_{i+1}, ..., y_m) = 0$ for any $z_1, ..., z_i \in R^-$ and any $y_{i+1}, ..., y_m \in R^+$.

We denote an involution on the basic field or algebra as $\psi$ while $\phi$ will mean an involution on the corresponding matrix algebra.

If a ring $R$ has an involution $\psi = *$ two involutions $\phi_1 = \sharp$ and $\phi_2 = \flat$ on $M_2(R)$ are defined as follows [15]:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\sharp} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\flat} = \begin{pmatrix} d^* & b^* \\ c^* & a^* \end{pmatrix}.$$

It is known [2] that two involutions play an important role in the Grassmann algebra: the involution $\psi_1$ acting on the generators $e_i$ of $E$ as $\psi_1(e_{2k}) = e_{2k-1}$, $\psi_1(e_{2k-1}) = e_{2k}$ and the trivial on the generators involution $\psi_2$ for which $\psi_2(e_i) = e_i$ for all $e_i$.

Here we consider the algebra $(M_2(E_4^{'}, \psi_2), \flat)$ and continue some of the investigations made in [12] by finding the nilpotency index of the $\flat$-variables of $(M_2(E_4^{'}, \psi_2), \flat)$.

**Theorem 1.** *The algebra* $(M_2(E_4^{'}, \psi_2), \flat)$ *satisfies the* $\flat$*-identity* $Y^4 = 0$ *in* $\flat$*-symmetric variables and the* $\flat$*-identity* $Z^3 = 0$ *in* $\flat$*-skew symmetric variables.*

*Proof of Theorem 1.* As Proposition 5 holds we have to prove only that $Z^3 = 0$ in $\flat$-skew symmetric variables.

Let $Z = \begin{pmatrix} y_1 & z_1 \\ z_2 & y_2 \end{pmatrix}$. The condition $\phi_2(Z) = -Z$ means that $\psi_2(z_1) = -z_1$, $\psi_2(z_2) = -z_2$, $\psi_2(y_1) = -y_2$ and $\psi_2(y_2) = -y_1$. Thus we get that

$$z_1 = \alpha_5 e_1 e_2 + \alpha_6 e_1 e_3 + \alpha_7 e_1 e_4 + \alpha_8 e_2 e_3 + \alpha_9 e_2 e_4 + \alpha_{10} e_3 e_4$$

$$\begin{aligned}
&+\alpha_{11}e_1e_2e_3+\alpha_{12}e_1e_2e_4+\alpha_{13}e_1e_3e_4+\alpha_{14}e_2e_3e_4;\\
z_2&=\beta_5e_1e_2+\beta_6e_1e_3+\beta_7e_1e_4+\beta_8e_2e_3+\beta_9e_2e_4+\beta_{10}e_3e_4\\
&+\beta_{11}e_1e_2e_3+\beta_{12}e_1e_2e_4+\beta_{13}e_1e_3e_4+\beta_{14}e_2e_3e_4;\\
y_1&=\gamma_1e_1+\gamma_2e_2+\gamma_3e_3+\gamma_4e_4\\
&+\gamma_5e_1e_2+\gamma_6e_1e_3+\gamma_7e_1e_4+\gamma_8e_2e_3+\gamma_9e_2e_4+\gamma_{10}e_3e_4\\
&+\gamma_{11}e_1e_2e_3+\gamma_{12}e_1e_2e_4+\gamma_{13}e_1e_3e_4+\gamma_{14}e_2e_3e_4+\gamma_{15}e_1e_2e_3e_4;\\
y_2&=-\gamma_1e_1-\gamma_2e_2-\gamma_3e_3-\gamma_4e_4\\
&+\gamma_5e_1e_2+\gamma_6e_1e_3+\gamma_7e_1e_4+\gamma_8e_2e_3+\gamma_9e_2e_4+\gamma_{10}e_3e_4\\
&+\gamma_{11}e_1e_2e_3+\gamma_{12}e_1e_2e_4+\gamma_{13}e_1e_3e_4+\gamma_{14}e_2e_3e_4-\gamma_{15}e_1e_2e_3e_4.
\end{aligned}$$

As in $z_iz_j$ the least degree of the summands is 4 we have $xz_jz_k=0$, $z_jxz_k=0$, $z_jz_kx=0$ for any entry $x$ of the matrix $Z$. As the least degree of the summands in $y_iz_j$ is 3 we get that $y_iz_jz_k=0$. The least degree in $y_i^2$ is 3 and we have $y_i^2z_j=0$ and $z_iy_j^2=0$ as well. Thus for the matrix $Z^3=(a_{ij})$ we get $a_{11}=a_{22}=0$, $a_{12}=y_1z_1y_2$ and $a_{21}=y_2z_2y_1$.

We consider the four summands of degree 3 (the minimal one) in $y_1z_1$:

$$\begin{array}{lcl}
\alpha e_1e_2e_3 & \to & \alpha=\gamma_1\alpha_8-\gamma_2\alpha_6+\gamma_3\alpha_5\\
\beta e_1e_2e_3 & \to & \beta=\gamma_1\alpha_9-\gamma_2\alpha_7+\gamma_4\alpha_5\\
\gamma e_1e_2e_3 & \to & \gamma=\gamma_1\alpha_{10}-\gamma_3\alpha_7+\gamma_4\alpha_6\\
\delta e_1e_2e_3 & \to & \delta=\gamma_2\alpha_{10}-\gamma_3\alpha_9+\gamma_4\alpha_8.
\end{array}$$

Now we define the coefficient of the only summand (of degree 4) in $a_{12}=y_1z_1y_2$. It is equal to

$$\begin{aligned}
&-\gamma_4(\gamma_1\alpha_8-\gamma_2\alpha_6+\gamma_3\alpha_5)+\gamma_3(\gamma_1\alpha_9-\gamma_2\alpha_7+\gamma_4\alpha_5)\\
&-\gamma_2(\gamma_1\alpha_{10}-\gamma_3\alpha_7+\gamma_4\alpha_6)+\gamma_1(\gamma_2\alpha_{10}-\gamma_3\alpha_9+\gamma_4\alpha_8)\equiv 0.
\end{aligned}$$

The same is valid for $a_{21}=y_2z_2y_1$ as well. Thus $Z^3$ is the zero matrix. □

If we change the involution $\psi_2$, considered in $E_4'$, with the involution $\psi_1$, the $\flat$-variables of $(M_2(E_4',\psi_1),\flat)$ do not have a lower nilpotency index, namely

**Theorem 2.** *The algebra* $(M_2(E_4',\psi_1),\flat)$ *satisfies the* $\flat$*-identity* $A^4=0$ *for* $A$ *being any* $\flat$*-variable.*

*Proof of Theorem 2.* We mach only the crucial steps of the proof.

In this case $\psi_1(e_1)=e_2$ $(\psi_1(e_2)=e_1)$ and $\psi_1(e_3)=e_4$ $(\psi_1(e_4)=e_3)$.

We have to consider only the case when $A=Z$ is a $\flat$-skew symmetric variable. The conditions $\psi_1(z_i)=-z_i$ and $\psi_1(y_1)=-y_2$ give that

$$\begin{aligned}
z_1&=\alpha_1(e_1-e_2)+\alpha_3(e_3-e_4)+\alpha_6(e_1e_3+e_2e_4)+\alpha_7(e_1e_4+e_2e_3)\\
&+\alpha_{11}(e_1e_2e_3-e_1e_2e_4)+\alpha_{13}(e_1e_3e_4-e_2e_3e_4);\\
z_2&=\beta_1(e_1-e_2)+\beta_3(e_3-e_4)+\beta_6(e_1e_3+e_2e_4)+\beta_7(e_1e_4+e_2e_3)
\end{aligned}$$

$$+\beta_{11}(e_1e_2e_3-e_1e_2e_4)+\beta_{13}(e_1e_3e_4-e_2e_3e_4);$$
$$y_1=\gamma_1e_1+\gamma_2e_2+\gamma_3e_3+\gamma_4e_4$$
$$+\gamma_5e_1e_2+\gamma_6e_1e_3+\gamma_7e_1e_4+\gamma_8e_2e_3+\gamma_9e_2e_4+\gamma_{10}e_3e_4$$
$$+\gamma_{11}e_1e_2e_3+\gamma_{12}e_1e_2e_4+\gamma_{13}e_1e_3e_4+\gamma_{14}e_2e_3e_4;$$
$$y_2=-\gamma_2e_1-\gamma_1e_2-\gamma_4e_3-\gamma_3e_4$$
$$-\gamma_5e_1e_2+\gamma_9e_1e_3+\gamma_8e_1e_4+\gamma_7e_2e_3+\gamma_6e_2e_4-\gamma_{10}e_3e_4$$
$$-\gamma_{12}e_1e_2e_3-\gamma_{11}e_1e_2e_4-\gamma_{14}e_1e_3e_4-\gamma_{13}e_2e_3e_4.$$

We follow the coefficient of $e_1e_2e_3$ in the entry $a_{11}=z_1z_2y_1+y_1z_1z_2+z_1y_2z_2$ of the matrix $Z^3=(a_{ij})$. Forming $z_1z_2$ we find the coefficient of $e_1e_2e_3$ in the product $y_1(z_1z_2)$, namely $-(\gamma_1+\gamma_2)(\alpha_1\beta_3-\alpha_3\beta_1)$.

The same holds for the coefficient of $e_1e_2e_3$ in the products $z_1z_2y_1$ and in $z_1y_2z_2$. Thus $Z^3$ is not a zero matrix.

Taking into account the conditions on the entries of a $\flat$-symmetric matrix $Y$ we see that the coefficient of $e_1e_2e_3$ in the entry $b_{11}$ of the matrix $Y^3=(b_{ij})$ is $3(\gamma_1-\gamma_2)(\alpha_1\beta_3-\alpha_3\beta_1)$. □

### 2.2. *PI-properties of some fourth order matrix algebras*

We define the 8-th dimensional matrix algebra $AM_4(K)$ as the algebra of the matrices of type
$\begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & 0 & a_{24} \\ a_{31} & 0 & a_{33} & 0 \\ 0 & a_{42} & 0 & a_{44} \end{pmatrix}$, $a_{ij}\in K$. The following theorem holds:

**Theorem 3.** *The algebra $AM_4(K)$ satisfies the Hall identity $[[X_1,X_2]^2,X_3]=0$.*

*Proof of Theorem 3.* For $X_1,X_2\in AM_4(K)$ in $[X_1,X_2]=(c_{ij})$ we have $c_{33}=-c_{11}$ and $c_{44}=-c_{22}$. The matrix $[X_1,X_2]^2=(d_{ij})$ is a diagonal matrix with $d_{33}=d_{11}$ and $d_{44}=d_{22}$. Thus $[[X_1,X_2]^2,X_3]=0$. □

By the system for computer algebra *Mathematica* we see that $AM_4(K)$ satisfies the identity $S_4(X_1,X_2,X_3,X_4)=0$ as well.

The n-th analogue of $AM_4(K)$ is the algebra $AM_{2n}(K)$. Its elements are of type $(a_{ij})$ with non-zero entries only among $a_{ii}$ for $i=1,...,2n$, $a_{j,n+j}$ and $a_{n+j,j}$ for $j=1,...,n$. The two identities in $AM_4(K)$ hold in $AM_{2n}(K)$ as well.

It is known that in a matrix algebra over a field $K$ of characteristic zero up to isomorphism there are two types of involutions - the transpose one $t$ and the symplectic involution $*$, the latter defined on an even $2k$ order matrix algebra as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} D & -B^t \\ -C^t & A \end{pmatrix},$$

where $A, B, C, D$ are $k \times k$ matrices.

We recall that the Hall identity $[[Y_1, Y_2]^2, Y_3] = 0$ is a $*$-identity of minimal degree in $*$-symmetric variables for the algebra $(M_4(K), *)$ [5].

Next we consider the matrix algebra $AM_4(K)$ with the symplectic involution $*$.

**Theorem 4.** *The algebra $(AM_4(K), *)$ satisfies the $*$-identity $[Y_1, Y_2] = 0$ in $*$-symmetric variables.*

*Proof of Theorem 4.* From

$$\begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & 0 & a_{24} \\ a_{31} & 0 & a_{33} & 0 \\ 0 & a_{42} & 0 & a_{44} \end{pmatrix}^*$$

$$= \begin{pmatrix} a_{33} & 0 & -a_{13} & 0 \\ 0 & a_{44} & 0 & -a_{24} \\ -a_{31} & 0 & a_{11} & 0 \\ 0 & -a_{42} & 0 & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & 0 & a_{24} \\ a_{31} & 0 & a_{33} & 0 \\ 0 & a_{42} & 0 & a_{44} \end{pmatrix}$$

we see that the $*$-symmetric elements of $(AM_4(K), *)$ are diagonal matrices. □

As $z^2$ is $*$-symmetric we come to

**Corollary 1.** *The algebra $(AM_4(K), *)$ satisfies the $*$-identity $[Z_1^2, Z_2^2] = 0$ in $*$-skew symmetric variables.*

Now the matrix algebras considered will have entries that are elements of a Grassmann algebra. In the statements below we use Proposition 4. As it was proved in [13] using the system for computer algebra *Mathematica* we give here its analytic proof.

*Proof of Proposition 4.* Without loss of generality we consider $x \in E_4'$ with summands of length 1 and 2 only (the other ones will give zeros either in $x^2$ or in $x^3$). Thus

$$\begin{aligned} x &= \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_1 e_2 + \alpha_6 e_1 e_3 \\ &+ \alpha_7 e_1 e_4 + \alpha_8 e_2 e_3 + \alpha_9 e_2 e_4 + \alpha_{10} e_3 e_4. \end{aligned}$$

We define the coefficients of the four summands of length 3 in $x^2$. They are:

$$\begin{aligned} \alpha e_1 e_2 e_3 &\mapsto \alpha = 2(\alpha_1\alpha_8 - \alpha_2\alpha_6 + \alpha_3\alpha_5) \\ \beta e_1 e_2 e_4 &\mapsto \beta = 2(\alpha_1\alpha_9 - \alpha_2\alpha_7 + \alpha_4\alpha_5) \\ \gamma e_1 e_3 e_4 &\mapsto \gamma = 2(\alpha_1\alpha_{10} - \alpha_3\alpha_7 + \alpha_4\alpha_6) \\ \delta e_2 e_3 e_4 &\mapsto \delta = 2(\alpha_2\alpha_{10} - \alpha_3\alpha_9 + \alpha_4\alpha_8). \end{aligned}$$

The coefficient of the only summand (which is of length 4) of $x^3$ is proportional to

$$\begin{aligned} &-\alpha_1(\alpha_2\alpha_{10} - \alpha_3\alpha_9 + \alpha_4\alpha_8) + \alpha_2(\alpha_1\alpha_{10} - \alpha_3\alpha_7 + \alpha_4\alpha_6) \\ &-\alpha_3(\alpha_1\alpha_9 - \alpha_2\alpha_7 + \alpha_4\alpha_5) + \alpha_4(\alpha_1\alpha_8 - \alpha_2\alpha_6 + \alpha_3\alpha_5) \equiv 0. \end{aligned}$$

□

The identity $[y,x,x] = 0$ and the linearization of $x^3 = 0$ lead to

**Corollary 2.** *In $E_4^{'}$ the following identities hold:*

$$x^2y + yx^2 = 0,\ xyx = 0,\ xyz + zyx = 0,$$
$$xy^2z = -zyxy = 0,\ y^2xz = -zyxy = 0,\ zxy^2 = -yxyz = 0.$$

**Theorem 5.** *The algebra $AM_4(E_4^{'})$ is a nil algebra with nil index* 4.

*Proof of Theorem 5.* For a matrix $A \in AM_4(E_4^{'})$, where

$$A = \begin{pmatrix} y_1 & 0 & z_1 & 0 \\ 0 & y_2 & 0 & z_2 \\ z_3 & 0 & y_3 & 0 \\ 0 & z_4 & 0 & y_4 \end{pmatrix}$$

and $A^3 = (a_{ij})$ we get

$$a_{11} = z_1z_3y_1 + y_1z_1z_3 + z_1y_3z_3,$$
$$a_{13} = y_1^2z_1 + z_1z_3z_1 + y_1z_1y_3 + z_1y_3^2,$$
$$a_{22} = z_2z_4y_2 + y_2z_2z_4 + z_2y_4z_4,$$
$$a_{24} = y_2^2z_2 + z_2z_4z_2 + y_2z_2y_4 + z_4y_4^2,$$
$$a_{31} = z_3y_1^2 + y_3z_3y_1 + z_3z_1z_3 + y_3^2z_3,$$
$$a_{33} = z_3y_1z_1 + y_3z_3z_1 + z_3z_1y_3,$$
$$a_{42} = z_4y_2^2 + y_4z_4y_2 + z_4z_2z_4 + y_4^2z_4,$$
$$a_{44} = z_2y_2z_2 + y_4z_4z_1 + z_4z_2y_4.$$

Now we investigate the entries of $A^4 = (b_{ij})$:

$$b_{11} = z_1z_3y_1^2 + y_1z_1z_3y_1 + z_1y_3z_3y_1 + y_1^2z_1z_3$$
$$+ z_1z_3z_1z_3 + y_1z_1y_1z_3 + z_1y_3^2z_3$$

Applying Corollary 2 we simplify $b_{11}$ and get $b_{11} = z_1y_3z_3y_1 + y_1z_1y_3z_3$. The identity $xyz = -zyx$ gives

$$z_1y_3z_3y_1 = -z_3y_1y_3z_1 = y_3z_1y_1z_3 = -y_1z_1y_3z_3.$$

Thus $b_{11} = 0$.

In an analogous way we investigate the other entries of $A^4$:

$$b_{13} = z_1z_3y_1z_1 + y_1z_1z_3z_1 + z_1y_3z_3z_1 + y_1^2z_1y_3$$
$$+ z_1z_3z_1y_3 + y_1z_1y_3^2 + z_1y_3^3$$

According to Corollary 2 we have $b_{13} = 0$.

Now we consider

$$b_{22} = z_2z_4y_2^2 + y_2z_2z_4y_2 + z_2y_4z_4y_2 + y_2^2z_2z_4 + z_2z_2z_2z_4 + y_2z_2y_4z_4 + z_4y_4^2z_4.$$

The same Corollary leads to $b_{22} = z_2y_4z_4y_2 + y_2z_2y_4z_4$. As

$$z_2y_4z_4y_2 = -z_4y_2y_4z_2 = y_4z_2y_2z_4 = -y_2z_2y_4z_4$$

we get $b_{22} = 0$.

Applying Corollary 2 we get $b_{24} = b_{31} = 0$. In $b_{33}$ we have to consider only the part $y_3z_3y_1z_1 + z_3y_1z_1y_3$. As

$$y_3z_3y_1z_1 = -y_1z_1z_3y_3 = z_3z_1y_1y_3 = -y_3y_1z_3z_1 = z_1y_1z_3y_3 = -z_3y_1z_1y_3$$

we get $b_{33} = 0$.

The identities in Corollary 2 immediately lead to $b_{42} = 0$, $b_{44} = 0$. Thus $A^4 = 0$. □

Now we consider the subalgebra $ASM_4(E)$ of the matrices of type $\begin{pmatrix} a & 0 & a & 0 \\ 0 & b & 0 & b \\ c & 0 & c & 0 \\ 0 & d & 0 & d \end{pmatrix}$ and prove that it is a PI-algebra.

**Theorem 6.** *The algebra $ASM_4(E)$ satisfies the identity $U[X,Y,Z] = 0$.*

*Proof of Theorem 6.* Let $X, Y, Z$ be matrices from $ASM_4(E)$ denoting its entries by $a_i, b_i, c_i, d_i$ for $i = 1,2,3$ respectively. We form the diagonal entries of $[X,Y] = (a_{ij})$, namely

$$\begin{aligned} a_{11} &= [a_1,a_2] + a_1c_2 - a_2c_1, \\ a_{22} &= [b_1,b_2] + b_1d_2 - b_2d_1, \\ a_{33} &= [c_1,c_2] + c_1a_2 - c_2a_1, \\ a_{44} &= [d_1,d_2] + d_1b_2 - d_2b_1. \end{aligned}$$

For the matrix $[X,Y,Z] = (b_{ij})$ we have modulo $[x,y,z] = 0$ for $x,y,z \in E$ that

$$\begin{aligned} & b_{11} + b_{33} \\ &= [a_1c_2 - a_2c_1, a_3] + ([a_1,a_2] + a_1c_2 - a_2c_1)c_3 - a_3([c_1,c_2] + c_1a_2 - c_2a_1) \\ &+ [c_1a_2 - c_2a_1, c_3] + ([c_1,c_2] + c_1a_2 - c_2a_1)a_3 - c_3([a_1,a_2] + a_1c_2 - a_2c_1) \\ &= [a_1c_2 - a_2c_1, a_3] + (a_1c_2 - a_2c_1)c_3 - a_3(c_1a_2 - c_2a_1) \\ &+ [c_1a_2 - c_2a_1, c_3] + (c_1a_2 - c_2a_1)a_3 - c_3(a_1c_2 - a_2c_1) \\ &= [a_1c_2 - a_2c_1, a_3] + [c_1a_2 - c_2a_1, c_3] + [a_1c_2 - a_2c_1, c_3] + [c_1a_2 - c_2a_1, a_3] \\ &= [[a_1,c_2] + [c_1,a_2], a_3] + [[c_1,a_2] + [a_1,c_2], c_3] \equiv 0. \end{aligned}$$

Analogously we get that $b_{22}+b_{44}=0$. Thus $U[X,Y,Z]=0$ for any matrix $U \in ASM_4(E)$. □

The analogue of $ASM_4(E)$ in the general case is the matrix algebra $ASM_{2n}(E)$. Its elements are of type $(a_{ij})$, where $a_{ii}=a_{i,n+i}$ for $i=1,...,n$ and $a_{jj}=a_{j,j-n}$ for $j=n+1,...,2n$. The algebra $ASM_{2n}(E)$ satisfies the same identity $U[X,Y,Z]=0$.

For now we are able to find involutions in $M_n(E)$ for $n>2$ only considering an involution in $E$. We generalize the case $n=2$, namely

**Proposition 6.** *The mapping* $(\flat)$*, defined as*

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}^{(\flat)}$$

$$= \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{(\flat)} = \begin{pmatrix} (D)^\flat & (B)^\flat \\ (C)^\flat & (A)^\flat \end{pmatrix} = \begin{pmatrix} a_{44}^* & a_{34}^* & a_{24}^* & a_{14}^* \\ a_{43}^* & a_{33}^* & a_{23}^* & a_{13}^* \\ a_{42}^* & a_{32}^* & a_{22}^* & a_{12}^* \\ a_{41}^* & a_{31}^* & a_{21}^* & a_{11}^* \end{pmatrix}$$

*is an involution on* $M_4(E,\psi=*)$.

*Proof of Proposition 6.* Considering in details the entries of the two matrices $(AB)^{(\flat)}$ and $(B)^{(\flat)}(A)^{(\flat)}$ we see that their corresponding entries are equal i.e. the mapping $(\flat)$ is an involution. □

We cover the following special case: Let $E_3'$ be the non-unitary finite dimensional Grassmann algebra with generators $e_1,e_2,e_3$ and $AM(2)(E_3')$ be the subalgebra of $AM_4(E_3')$ defined by the matrices of type

$$\begin{pmatrix} y_1 & 0 & z_1 & 0 \\ 0 & y_2 & 0 & z_2 \\ z_3 & 0 & y_3 & 0 \\ 0 & z_4 & 0 & y_4 \end{pmatrix},$$

where $y_i$ are even elements (of even length) of $E_3'$, while $z_i$ are odd elements (of odd length) of $E_3'$, $i=1,...,4$. We equip the algebra $AM(2)(E_3',\psi_2)$ with the involution $(\flat)$ as defined in Proposition 6.

We characterize the $(\flat)$-symmetric elements $Y_i$ and the $(\flat)$-skew symmetric elements $Z_j$ of the algebra $(AM(2)(E_3',\psi_2),(\flat))$.

**Theorem 7.** *The algebra* $(AM(2)(E_3',\psi_2),(\flat))$ *satisfies the* $(\flat)$*-identity* $Y^3=0$ *in* $(\flat)$*-symmetric variables.*

*Proof of Theorem 7.* Let consider a ($\flat$)-symmetric element $Y$. Denoting for short $\psi_2$ as $*$ in the equality $\begin{pmatrix} y_4^* & 0 & z_2^* & 0 \\ 0 & y_3* & 0 & z_1^* \\ z_4* & 0 & y_2^* & 0 \\ 0 & z_3* & 0 & y_1^* \end{pmatrix} = \begin{pmatrix} y_1 & 0 & z_1 & 0 \\ 0 & y_2 & 0 & z_2 \\ z_3 & 0 & y_3 & 0 \\ 0 & z_4 & 0 & y_4 \end{pmatrix}$ we get the following conditions on the entries of $Y$: $\psi_2(y_4) = y_1$, $\psi_2(y_3) = y_2$, $\psi_2(z_2) = z_1$ and $\psi_2(z_4) = z_3$.

Let $y_1 = s_1e_1e_2 + s_2e_1e_3 + s_3e_2e_3$. Then $y_4 = \psi_2(y_1) = -y_1$. For $y_2 = t_1e_1e_2 + t_2e_1e_3 + t_3e_2e_3$ we get $y_3 = \psi_2(y_2) = -y_2$. Obviously $y_1^2 = y_2^2 = 0$.

As the entries are from $E_3'$ we could work with odd entries having summands of degree 1 only. Let $z_1 = \alpha_1e_1 + \alpha_2e_2 + \alpha_3e_3$ and $z_3 = m_1e_1 + m_2e_2 + m_3e_3$. Then $z_2 = \psi_2(z_1) = z_1$, $z_4 = \psi_2(z_3) = z_3$. Considering $Y^3 = Y^2Y = (a_{ij})$ as

$$\begin{pmatrix} z_1z_3 & 0 & y_1z_1 - z_1y_2 & 0 \\ 0 & z_1z_3 & 0 & y_2z_1 - z_1y_1 \\ z_3y_1 - y_2z_3 & 0 & z_3z_1 & 0 \\ 0 & z_3y_2 - y_1z_3 & 0 & z_3z_1 \end{pmatrix}
\begin{pmatrix} y_1 & 0 & z_1 & 0 \\ 0 & y_2 & 0 & z_1 \\ z_3 & 0 & -y_2 & 0 \\ 0 & z_3 & 0 & -y_1 \end{pmatrix}$$

we see that manipulating with the generators $e_1, e_2, e_3$, probably nontrivial entries could be only

$$a_{13} = a_{24} = z_1z_3z_1 = \beta_1e_1e_2e_3, \ a_{31} = a_{42} = z_3z_1z_3 = \beta_2e_1e_2e_3.$$

Applying Corollary 2 we get that both of them are zero. □

**Theorem 8.** *The algebra* $(AM(2)(E_3', \psi_2), (\flat))$ *satisfies the* ($\flat$)*-identity* $Z^3 = 0$ *in* ($\flat$)*-skew symmetric variables.*

*Proof of Theorem 8.* Using the same notations for the matrix entries of $Z$ as in the previous theorem, in this case we have

$$\begin{aligned} &y_4 = -\psi_2(y_1) = y_1, \, y_3 = -\psi_2(y_2) = y_2, \\ &y_1^2 = y_2^2 = 0, \\ &z_2 = -\psi_2(z_1) = -z_1, \, z_4 = -\psi_2(z_3) = -z_3. \end{aligned}$$

In $Z^3 = Z^2Z = (b_{ij})$ nonzero could be only the entries $b_{13} = -b_{24} = z_1z_3z_1$ and $b_{31} = -b_{42} = z_3z_1z_3$. Corollary 2 proves they both are zero. □

We consider the subalgebra $(AM(2)(E_3', \psi_2), (\flat))$ instead of the algebra $(AM_4(E_3', \psi_2), (\flat))$ itself as if $A^n = 0$ for a $\flat$-variable $A$ of $(AM_4(E_4', \psi_2), (\flat))$ we have $n > 3$. Thus the algebras $(AM_4(E_4^{'}, \psi_2), (\flat))$ and $AM_4(E_4^{'})$ have equal nil indices.

We give an example of another matrix algebra with involution $(\flat)$ having lower nilpotency index of its $(\flat)$-skew symmetric variables:

Let $BM(2)(E_3^{'})$ be the algebra defined by the matrices of type

$$\begin{pmatrix} y_1 & 0 & 0 & z_1 \\ 0 & y_2 & z_2 & 0 \\ 0 & z_3 & y_3 & 0 \\ z_4 & 0 & 0 & y_4 \end{pmatrix}$$

, where $y_i$ are even elements of $E_3^{'}$, while $z_i$ are odd elements of $E_3^{'}$, $i = 1, \ldots, 4$. We equip the algebra $BM(2)(E_3^{'}, \psi_2)$ with the involution $(\flat)$ as defined in Proposition 6.

**Theorem 9.** *The algebra* $(BM(2)(E_3^{'}, \psi_2), (\flat))$ *satisfies the* $(\flat)$*-identity* $Y^3 = 0$ *in* $(\flat)$*-symmetric variables and the* $(\flat)$*-identity* $Z^2 = 0$ *in* $(\flat)$*-skew symmetric variables.*

*Proof of Theorem 9.* In the algebra $(BM(2)(E_3^{'}, \psi_2), (\flat))$ any $(\flat)$-skew symmetric variable $Z$ is a diagonal matrix and $Z^2 = 0$ as $y_i^2 = 0$ for $i = 1, \ldots, 4$. □

There is a package written in the system for computer algebra *Mathematica* [10] for manipulating in finite dimensional Grassmann algebras. Using it a programme was written by the author giving an alternative way of confirming the validity of the corresponding theorems in the paper.

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*Author's address*

**Tsetska Rashkova**
University of Ruse, Department of Mathematics, 8 Studentska Str., 7017 Ruse, Bulgaria
*E-mail address:* tsrashkova@uni-ruse.bg