PI-PROPERTIES OF SOME MATRIX ALGEBRAS WITH INVOLUTION

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Abstract. We define the nilpotency index of the \(b\)-variables in second order matrix algebras with Grassmann entries and involution \(b\). Identities of minimal degree are found for a concrete subalgebra of the matrix algebra \(M_4(K)\). When it has an involution \(\phi\) as well some of its \(\phi\)-identities are given. For an analogue of this subalgebra over finite dimensional Grassmann algebras a new involution \(\langle b\rangle\) is introduced and its \(\langle b\rangle\)-identities are discussed.

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1. INTRODUCTION

The classical PI-theory (the theory of the polynomial identities) has its development for algebras with involution as well. The contributions of Amitsur [1], Levchenko [9], Rowen [14], Wenxin and Racine [17], Giambruno and Valenti [6], Drensky and Giambruno [5], Rashkova [11], La Mattina and Misso [8] are only a part of it.

In 1973 Krasovski and Regev [7] described completely the \(T\)-ideal of the identities of the Grassmann algebra \(E\) and it was a natural step to investigate the PI-structure of algebras not only over fields (with any characteristic) but over algebras as well, especially Grassmann algebras [4, 12, 16].

In the paper we consider mainly finite dimensional Grassmann algebras and special matrix algebras over them.

We recall the definition of the Grassmann algebra \(E\) as:

\[ E = K\langle e_1, e_2, \ldots | e_i e_j + e_j e_i = 0, i, j = 1, 2, \ldots \rangle, \]

where \(K\) is a field of characteristic zero.

We cite basic propositions from [3, 7]. The notation \([x, y, z] = [[x, y], z] = [x, y]z - z[x, y]\) will be used.

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Proposition 1 ([7, Corollary, p. 437]). The $T$-ideal of the Grassmann algebra $E$ is generated by the identity $[x, y, z] = 0$.

Proposition 2 ([3, Lemma 6.1]). For any $n, k \geq 2$ in the algebra $E$ the identity $S_n^k(x_1, \ldots, x_n) = 0$ holds, where

$$S_n(x_1, \ldots, x_n) = \sum_{\sigma \in \text{Sym}(n)} (-1)^{\sigma} x_{\sigma(1)} \cdots x_{\sigma(n)}$$

is the $n$-th standard polynomial.

Proposition 3 ([3, Lemma 6.6]). The matrix algebra $M_n(E)$ does not satisfy the identity

$$S_m^n(x_1, \ldots, x_m) = 0$$

for any $m$.

There are subalgebras of $M_n(E)$ however being counter examples of Proposition 3 for concrete $m$.

We use the notation $E'_n$ for a non unitary Grassmann algebra with generators $e_1, \ldots, e_n$.

The existence of nilpotent elements of minimal nilpotency index both in finite dimensional Grassmann algebras and in matrix algebras over them was investigated in [12, 13]. We state some of the results needed:

Proposition 4 ([13, Proposition 13]). The identity $x^3 = 0$ holds for the algebra $E_4'$.

Proposition 5 ([13, Proposition 16]). The algebra $M_2(E_4')$ satisfies the identity $X^4 = 0$.

In [13] examples were given as well of subalgebras $\mathfrak{A}_i, i = 1, 2$ of $M_n(\mathfrak{R})$ such that the identities $x^4 = 0$ and $[x, y, z] = 0$ in $\mathfrak{R}$ imply the identity $X^4 = 0$ in $\mathfrak{A}_i, i = 1, 2$.

An involution $\psi$ on the Grassmann algebras $E'_2$ and $E'_3$ defines an involution $\phi$ on the corresponding $2 \times 2$ matrix algebra over any of them. In that case the classes of symmetric and of skew-symmetric to the involution $\phi$ matrices of nilpotency indices 2 and 3 were described in [12].

In the present paper we continue the investigations started in [12];

We define the nilpotency index of the $b$-variables in the considered algebras with involution $\phi = b$.

For a concrete subalgebra of the matrix algebra $M_4(K)$ identities of minimal degree are found. When additionally the algebra has an involution $\phi$ some of its $\phi$-identities are given.

For an analogue of this subalgebra over finite dimensional Grassmann algebras a new involution $\phi = (b)$ is introduced and some $(b)$-identities are discussed.
2. Results

2.1. PI-properties of involution second order matrix algebras with Grassmann entries

We recall the definition of an involution on an algebra $R$: it is a second order antiautomorphism $\psi$ such that $\psi(ab) = \psi(b)\psi(a)$ for all $a, b \in R$.

By $R^-$ we denote the skew-symmetric due to the involution elements of $R$, namely $z_1, ..., z_i, ...$ and by $R^+$ we denote the symmetric due to the involution elements $y_1, ..., y_j, ...$. It is important to consider $\psi$-variables (symmetric and skew-symmetric) as the elements of $R^+$ form a Jordan algebra due to the multiplication $y_1 \circ y_2 = y_1y_2 + y_2y_1$ and the elements of $R^-$ form a Lie algebra due to the operation $[z_1, z_2]$.

**Definition 1.** Let $f = f(x_1, ..., x_m) \in K(x_1, ..., x_n)$, the free associative algebra on $n$ generators over $K$. We say that $f$ is a $\psi$-identity in skew variables for the algebra $R$ over $K$ if $f(z_1, ..., z_m) = 0$ for all $z_1, ..., z_m \in R^-$. Accordingly $f$ is a $\psi$-identity in symmetric variables for the algebra $R$ over $K$ if $f(y_1, ..., y_m) = 0$ for all $y_1, ..., y_m \in R^+$.

We say that $f$ is a $\psi$-identity if $f(z_1, ..., z_i, y_{i+1}, ..., y_m) = 0$ for any $z_1, ..., z_i \in R^-$ and any $y_{i+1}, ..., y_m \in R^+$.

We denote an involution on the basic field or algebra as $\psi$ while $\phi$ will mean an involution on the corresponding matrix algebra.

If a ring $R$ has an involution $\psi = \ast$ two involutions $\phi_1 = \ast$ and $\phi_2 = b$ on $M_2(R)$ are defined as follows [15]:

$$
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}^\ast = \begin{pmatrix}
  a^* & c^* \\
  b^* & d^*
\end{pmatrix},
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}^b = \begin{pmatrix}
  d^* & b^* \\
  c^* & a^*
\end{pmatrix}.
$$

It is known [2] that two involutions play an important role in the Grassmann algebra: the involution $\psi_1$ acting on the generators $e_i$ of $E$ as $\psi_1(e_{2k}) = e_{2k-1}$, $\psi_1(e_{2k-1}) = e_{2k}$ and the trivial on the generators involution $\psi_2$ for which $\psi_2(e_i) = e_i$ for all $e_i$.

Here we consider the algebra $(M_2(E_4', \psi_2), b)$ and continue some of the investigations made in [12] by finding the nilpotency index of the $b$-variables of $(M_2(E_4', \psi_2), b)$.

**Theorem 1.** The algebra $(M_2(E_4', \psi_2), b)$ satisfies the $b$-identity $Y^4 = 0$ in $b$-symmetric variables and the $b$-identity $Z^3 = 0$ in $b$-skew symmetric variables.

**Proof of Theorem 1.** As Proposition 5 holds we have to prove only that $Z^3 = 0$ in $b$-skew symmetric variables.

Let $Z = \begin{pmatrix}
  y_1 & z_1 \\
  z_2 & y_2
\end{pmatrix}$. The condition $\phi_2(Z) = -Z$ means that $\psi_2(z_1) = -z_1$, $\psi_2(z_2) = -z_2$, $\psi_2(y_1) = -y_2$ and $\psi_2(y_2) = -y_1$. Thus we get that

$$
z_1 = \alpha_5 e_1 e_2 + \alpha_6 e_1 e_3 + \alpha_7 e_1 e_4 + \alpha_8 e_2 e_3 + \alpha_9 e_2 e_4 + \alpha_{10} e_3 e_4
$$
The conditions
It is equal to
\[ y = \gamma \]
In this case
\[ \text{Proof of Theorem 2.} \]
We consider the four summands of degree 3
\[ y_2 = -\gamma y_1 - y_2 y_2 - y_3 y_3 - y_4 y_4 \]
As in \( z \) the least degree of the summands is 4 we have \( x z_j z_k = 0, z_j x z_k = 0, \) \( z_j z_k x = 0 \) for any entry \( x \) of the matrix \( Z \). As the least degree of the summands in \( y_1 z_1 \) is 3 we get that \( y_1 z_1 z_k = 0 \). The least degree in \( y_2^2 \) is 3 and we have \( y_2^2 z_1 = 0 \) and \( z_1 y_2^2 = 0 \) as well. Thus for the matrix \( Z^2 = (a_{ij}) \) we get \( a_{11} = a_{22} = 0, a_{12} = y_1 z_1 y_2 \) and \( a_{21} = y_2 z_2 y_1 \).
We consider the four summands of degree 3 (the minimal one) in \( y_1 z_1 \):
\[
\begin{align*}
\alpha e_1 e_2 e_3 & \rightarrow \alpha = \gamma \alpha_8 - \gamma_2 \alpha_6 + \gamma_3 \alpha_5 \\
\beta e_1 e_2 e_3 & \rightarrow \beta = \gamma \alpha_9 - \gamma_2 \alpha_7 + \gamma_4 \alpha_5 \\
\gamma e_1 e_2 e_3 & \rightarrow \gamma = \gamma \alpha_{10} - \gamma \alpha_7 + \gamma_4 \alpha_6 \\
\delta e_1 e_2 e_3 & \rightarrow \delta = \gamma \alpha_{10} - \gamma_3 \alpha_9 + \gamma_4 \alpha_8.
\end{align*}
\]
Now we define the coefficient of the only summand (of degree 4) in \( a_{12} = y_1 z_1 y_2 \).
It is equal to
\[
-\gamma_4 (\gamma \alpha_8 - \gamma_2 \alpha_6 + \gamma_3 \alpha_5) + \gamma_3 (\gamma_1 \alpha_9 - \gamma_2 \alpha_7 + \gamma_4 \alpha_5)
\]
\[
-\gamma_2 (\gamma_1 \alpha_{10} - \gamma_3 \alpha_7 + \gamma_4 \alpha_6) + \gamma_1 (\gamma_2 \alpha_{10} - \gamma_3 \alpha_9 + \gamma_4 \alpha_8)
\]
\[
\equiv 0.
\]
The same is valid for \( a_{21} = y_2 z_2 y_1 \) as well. Thus \( Z^3 \) is the zero matrix.

If we change the involution \( \psi_2 \), considered in \( E_4' \), with the involution \( \psi_1 \), the \( b \)-variables of \( (M_2(E_4', \psi_1), b) \) do not have a lower nilpotency index, namely

**Theorem 2.** The algebra \( (M_2(E_4', \psi_1), b) \) satisfies the \( b \)-identity \( A^4 = 0 \) for \( A \) being any \( b \)-variable.

**Proof of Theorem 2.** We mach only the crucial steps of the proof.
In this case \( \psi_1(e_1) = e_2 (\psi_1(x)) = e_1 \) and \( \psi_1(e_3) = e_4 (\psi_1(e_4)) = e_3 \).
We have to consider only the case when \( A = Z \) is a \( b \)-skew symmetric variable.
The conditions \( \psi_1(z_1) = -z_1 \) and \( \psi_1(y_1) = -y_2 \) give that
\[
\begin{align*}
z_1 &= \alpha_1 (e_1 - e_2) + \alpha_3 (e_3 - e_4) + \alpha_6 (e_1 e_3 + e_2 e_4) + \alpha_7 (e_1 e_4 + e_2 e_3) \\
&+ \alpha_{11} (e_1 e_2 e_3 - e_1 e_2 e_4) + \alpha_{13} (e_1 e_3 e_4 - e_2 e_3 e_4);
\end{align*}
\[
\begin{align*}
z_2 &= \beta_1 (e_1 - e_2) + \beta_3 (e_3 - e_4) + \beta_6 (e_1 e_3 + e_2 e_4) + \beta_7 (e_1 e_4 + e_2 e_3)
\end{align*}
\]

\[ + \beta_{11}(e_1e_2e_3 - e_1e_2e_4) + \beta_{13}(e_1e_3e_4 - e_2e_3e_4): \]
\[ y_1 = \gamma_1e_1 + \gamma_2e_2 + \gamma_3e_3 + \gamma_4e_4 \]
\[ + \gamma_5e_1e_2 + \gamma_6e_1e_3 + \gamma_7e_1e_4 + \gamma_8e_2e_3 + \gamma_9e_2e_4 + \gamma_{10}e_3e_4 \]
\[ + \gamma_{11}e_1e_2e_3 + \gamma_{12}e_1e_2e_4 + \gamma_{13}e_1e_3e_4 + \gamma_{14}e_2e_3e_4: \]
\[ y_2 = -\gamma_2e_1 - \gamma_1e_2 - \gamma_4e_3 - \gamma_3e_4 \]
\[ - \gamma_5e_1e_2 + \gamma_9e_1e_3 + \gamma_7e_1e_4 + \gamma_6e_2e_3 - \gamma_10e_3e_4 \]
\[ - \gamma_{12}e_1e_2e_3 - \gamma_{11}e_1e_2e_4 - \gamma_{13}e_1e_3e_4 - \gamma_{14}e_2e_3e_4. \]

We follow the coefficient of \( e_1e_2e_3 \) in the entry \( a_{11} = z_1z_2y_1 + y_1z_1z_2 + z_1y_2z_2 \) of the matrix \( Z^3 = (a_{ij}) \). Forming \( z_1z_2 \) we find the coefficient of \( e_1e_2e_3 \) in the product \( y_1(z_1z_2) \), namely \(- (\gamma_1 + \gamma_2)(\alpha_1\beta_3 - \alpha_3\beta_1)\).

The same holds for the coefficient of \( e_1e_2e_3 \) in the products \( z_1z_2y_1 \) and in \( z_1y_2z_2 \). Thus \( Z^3 \) is not a zero matrix.

Taking into account the conditions on the entries of a \( b \)-symmetric matrix \( Y \) we see that the coefficient of \( e_1e_2e_3 \) in the entry \( b_{11} \) of the matrix \( Y^3 = (b_{ij}) \) is \( 3(\gamma_1 - \gamma_2)(\alpha_1\beta_3 - \alpha_3\beta_1) \).

\[ \Box \]

2.2. PI-properties of some fourth order matrix algebras

We define the 8-th dimensional matrix algebra \( AM_4(K) \) as the algebra of the matrices of type
\[
\begin{pmatrix}
  a_{11} & 0 & a_{13} & 0 \\
  0 & a_{22} & 0 & a_{24} \\
  a_{31} & 0 & a_{33} & 0 \\
  0 & a_{42} & 0 & a_{44}
\end{pmatrix}, \quad a_{ij} \in K.
\]

The following theorem holds:

**Theorem 3.** The algebra \( AM_4(K) \) satisfies the Hall identity \([ [X_1, X_2]^2, X_3] = 0 \).

**Proof of Theorem 3.** For \( X_1, X_2 \in AM_4(K) \) in \( [X_1, X_2] = (c_{ij}) \) we have \( c_{33} = -c_{11} \) and \( c_{44} = -c_{22} \). The matrix \([X_1, X_2]^2 = (d_{ij}) \) is a diagonal matrix with \( d_{33} = d_{11} \) and \( d_{44} = d_{22} \). Thus \([ [X_1, X_2]^2, X_3] = 0 \). \[ \Box \]

By the system for computer algebra Mathematica we see that \( AM_4(K) \) satisfies the identity \( S_4(X_1, X_2, X_3, X_4) = 0 \) as well.

The \( n \)-th analogue of \( AM_4(K) \) is the algebra \( AM_{2n}(K) \). Its elements are of type \((a_{ij})\) with non-zero entries only among \( a_{ij} \) for \( i = 1, \ldots, 2n \), \( a_{j,n+j} \) and \( a_{n+j,j} \) for \( j = 1, \ldots, n \). The two identities in \( AM_4(K) \) hold in \( AM_{2n}(K) \) as well.

It is known that in a matrix algebra over a field \( K \) of characteristic zero up to isomorphism there are two types of involutions - the transpose one \( t \) and the symplectic involution \( \ast \), the latter defined on an even \( 2k \) order matrix algebra as
\[
\begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix}^\ast = \begin{pmatrix}
  D & -B^t \\
  -C^t & A
\end{pmatrix}.
\]
where $A, B, C, D$ are $k \times k$ matrices.

We recall that the Hall identity $[[Y_1, Y_2]^2, Y_3] = 0$ is a $*$-identity of minimal degree in $*$-symmetric variables for the algebra $(M_4(K), *)$ [5].

Next we consider the matrix algebra $AM_4(K)$ with the symplectic involution $*$.

**Theorem 4.** The algebra $(AM_4(K), *)$ satisfies the $*$-identity $[Y_1, Y_2] = 0$ in $*$-symmetric variables.

**Proof of Theorem 4.** From

$$
\begin{pmatrix}
  a_{11} & 0 & a_{13} & 0 \\
  0 & a_{22} & 0 & a_{24} \\
  a_{31} & 0 & a_{33} & 0 \\
  0 & a_{42} & 0 & a_{44}
\end{pmatrix}^*
= \begin{pmatrix}
  a_{33} & 0 & -a_{13} & 0 \\
  0 & a_{44} & 0 & -a_{24} \\
  -a_{31} & 0 & a_{11} & 0 \\
  0 & -a_{42} & 0 & a_{22}
\end{pmatrix}
= \begin{pmatrix}
  a_{11} & 0 & a_{13} & 0 \\
  0 & a_{22} & 0 & a_{24} \\
  a_{31} & 0 & a_{33} & 0 \\
  0 & a_{42} & 0 & a_{44}
\end{pmatrix}
$$

we see that the $*$-symmetric elements of $(AM_4(K), *)$ are diagonal matrices. □

As $z^2$ is $*$-symmetric we come to

**Corollary 1.** The algebra $(AM_4(K), *)$ satisfies the $*$-identity $[Z_1^2, Z_2^2] = 0$ in $*$-skew symmetric variables.

Now the matrix algebras considered will have entries that are elements of a Grassmann algebra. In the statements below we use Proposition 4. As it was proved in [13] using the system for computer algebra Mathematica we give here its analytic proof.

**Proof of Proposition 4.** Without loss of generality we consider $x \in E'_4$ with summands of length 1 and 2 only (the other ones will give zeros either in $x^2$ or in $x^3$). Thus

$$x = \alpha_1 e_1 e_2 + \alpha_2 e_2 e_4 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_1 e_2 + \alpha_6 e_1 e_3 + \alpha_7 e_1 e_4 + \alpha_8 e_2 e_3 + \alpha_9 e_2 e_4 + \alpha_{10} e_3 e_4.$$

We define the coefficients of the four summands of length 3 in $x^2$. They are:

$$
\begin{align*}
\alpha e_1 e_2 e_3 & \mapsto \alpha = 2(\alpha_1 \alpha_8 - \alpha_2 \alpha_6 + \alpha_3 \alpha_5) \\
\beta e_1 e_2 e_4 & \mapsto \beta = 2(\alpha_1 \alpha_9 - \alpha_2 \alpha_7 + \alpha_4 \alpha_5) \\
\gamma e_1 e_3 e_4 & \mapsto \gamma = 2(\alpha_1 \alpha_{10} - \alpha_3 \alpha_7 + \alpha_4 \alpha_6) \\
\delta e_2 e_3 e_4 & \mapsto \delta = 2(\alpha_2 \alpha_{10} - \alpha_3 \alpha_9 + \alpha_4 \alpha_8).
\end{align*}
$$

The coefficient of the only summand (which is of length 4) of $x^3$ is proportional to

$$-\alpha_1 (\alpha_2 \alpha_{10} - \alpha_3 \alpha_9 + \alpha_4 \alpha_8) + \alpha_2 (\alpha_1 \alpha_{10} - \alpha_3 \alpha_7 + \alpha_4 \alpha_6) + \alpha_3 (\alpha_1 \alpha_9 - \alpha_2 \alpha_7 + \alpha_4 \alpha_5) + \alpha_4 (\alpha_1 \alpha_8 - \alpha_2 \alpha_6 + \alpha_3 \alpha_5) = 0.$$
The identity \([y, x, x] = 0\) and the linearization of \(x^3 = 0\) lead to

**Corollary 2.** In \(E_4'\) the following identities hold:
\[
x^2 y + yx^2 = 0,\ xyx = 0,\ xyz + zyx = 0,\xy^2 z = -zyxy = 0,\ y^2 xz = -zyxy = 0,\ zxy^2 = -yxyz = 0.
\]

**Theorem 5.** The algebra \(AM_4(E_4')\) is a nil algebra with nil index 4.

**Proof of Theorem 5.** For a matrix \(A \in AM_4(E_4')\), where
\[
A = \begin{pmatrix}
y_1 & 0 & z_1 & 0 \\
0 & y_2 & 0 & z_2 \\
z_3 & 0 & y_3 & 0 \\
0 & z_4 & 0 & y_4
\end{pmatrix}
\]
and \(A^3 = (a_{ij})\) we get
\[
\begin{align*}
a_{11} &= z_1 z_3 y_1 + y_1 z_1 z_3 + z_1 y_3 z_3, \\
a_{13} &= y_1^2 z_1 + z_1 z_3 z_1 + y_1 z_1 y_3 + z_1 y_3^2, \\
a_{22} &= z_2 z_4 y_2 + y_2 z_2 z_4 + z_2 y_4 z_4, \\
a_{24} &= y_2^2 z_2 + z_2 z_4 z_2 + y_2 z_4 z_4 + z_4 y_4^2, \\
a_{31} &= z_3 y_1^2 + y_3 z_3 y_1 + z_3 z_1 z_3 + y_3^2 z_3, \\
a_{33} &= z_3 y_1 z_1 + y_3 z_3 z_1 + z_3 z_1 y_3, \\
a_{42} &= z_4 y_2^2 + y_4 z_4 y_2 + z_4 z_2 z_4 + y_4^2 z_4, \\
a_{44} &= z_2 y_2 z_2 + y_4 z_4 z_4 + z_4 z_4 y_4.
\end{align*}
\]

Now we investigate the entries of \(A^4 = (b_{ij})\):
\[
\begin{align*}
b_{11} &= z_1 z_3 y_1^2 + y_1 z_1 z_3 y_1 + z_1 y_3 z_3 y_1 + y_1^2 z_1 z_3 \\
&\quad + z_1 z_3 z_1 z_3 + y_1 z_1 y_1 z_3 + z_1 y_3^2 z_3 \\
b_{11} &= z_1 y_3 z_3 y_1 = -zyxyz \text{ gives}\n\end{align*}
\]
\[
\begin{align*}
z_1 y_3 z_3 y_1 &= -zyxyz = y_3 z_1 y_1 z_3 = -y_1 y_1 y_3 z_3.
\end{align*}
\]

Thus \(b_{11} = 0\).

In an analogous way we investigate the other entries of \(A^4\):
\[
\begin{align*}
b_{13} &= z_1 z_3 y_1 z_1 + y_1 z_1 z_3 z_1 + z_1 y_3 z_3 z_1 + y_1^2 z_1 y_3 \\
&\quad + z_1 z_3 z_1 y_3 + y_1 z_1 y_3^2 + z_1 y_3^3
\end{align*}
\]

According to Corollary 2 we have \(b_{13} = 0\).
Now we consider
\[ b_{22} = z_2z_4y_2^2 + y_2z_2z_4y_2 + z_2y_4z_4y_2 + y_2^2z_2z_4 + z_2z_2z_2z_4 + y_2z_2y_4z_4 + z_4y_4^2z_4. \]

The same Corollary leads to \( b_{22} = z_2y_4z_4y_2 + y_2z_2y_4z_4. \) As
\[ z_2y_4z_4y_2 = -z_4y_2y_4z_2 = y_4z_2y_2z_4 = -y_2z_2y_4z_4 \]
we get \( b_{22} = 0. \)

Applying Corollary 2 we get \( b_{24} = b_{31} = 0. \) In \( b_{33} \) we have to consider only the part \( y_3z_3y_3z_1 + z_3y_3z_1y_3. \) As
\[ y_3z_3y_3z_1 = -y_1z_1z_3y_3 = z_3z_1y_1y_3 = -y_3y_1z_3z_1 = z_1y_1z_3y_3 = -z_3y_1z_1y_3 \]
we get \( b_{33} = 0. \)

The identities in Corollary 2 immediately lead to \( b_{42} = 0, b_{44} = 0. \) Thus \( A^4 = 0. \)

Now we consider the subalgebra \( ASM_4(E) \) of the matrices of type
\[
\begin{pmatrix}
a & 0 & a & 0 \\
0 & b & 0 & b \\
c & 0 & c & 0 \\
0 & d & 0 & d
\end{pmatrix}
\]
and prove that it is a PI-algebra.

**Theorem 6.** The algebra \( ASM_4(E) \) satisfies the identity \( U[X,Y,Z] = 0. \)

**Proof of Theorem 6.** Let \( X, Y, Z \) be matrices from \( ASM_4(E) \) denoting its entries by \( a_i, b_i, c_i, d_i \) for \( i = 1, 2, 3 \) respectively. We form the diagonal entries of \([X,Y] = (a_{ij}), \) namely
\[
\begin{align*}
a_{11} &= [a_1, a_2] + a_1c_2 - a_2c_1, \\
a_{22} &= [b_1, b_2] + b_1d_2 - b_2d_1, \\
a_{33} &= [c_1, c_2] + c_1a_2 - c_2a_1, \\
a_{44} &= [d_1, d_2] + d_1b_2 - d_2b_1.
\end{align*}
\]

For the matrix \([X,Y,Z] = (b_{ij}) \) we have modulo \([x,y,z] = 0 \) for \( x, y, z \in E \) that
\[
b_{11} + b_{33} = [a_1c_2 - a_2c_1, a_3] + ([a_1, a_2] + a_1c_2 - a_2c_1)c_3 - a_3(c_1, c_2 + c_1a_2 - c_2a_1) \\
+ [c_1a_2 - c_2a_1, c_3] + ([c_1, c_2] + c_1a_2 - c_2a_1)a_3 - c_3([a_1, a_2] + a_1c_2 - a_2c_1) \\
= [a_1c_2 - a_2c_1, a_3] + (a_1c_2 - a_2c_1)c_3 - a_3(c_1, c_2 + c_1a_2 - c_2a_1) \\
+ [c_1a_2 - c_2a_1, c_3] + (c_1a_2 - c_2a_1)a_3 - c_3(a_1c_2 - a_2c_1) \\
= [a_1c_2 - a_2c_1, a_3] + [c_1a_2 - c_2a_1, c_3] + [a_1c_2 - a_2c_1, c_3] + [c_1a_2 - c_2a_1, a_3] \\
= [[a_1, c_2] + [c_1, a_2], a_3] + [[c_1, a_2] + [a_1, c_2], c_3] \equiv 0.
\]
Analogously we get that $b_{22} + b_{44} = 0$. Thus $U[X, Y, Z] = 0$ for any matrix $U \in ASM_4(E)$. □

The analogue of $ASM_4(E)$ in the general case is the matrix algebra $ASM_{2n}(E)$. Its elements are of type $(a_{ij})$, where $a_{ii} = a_{i,n+i}$ for $i = 1, \ldots, n$ and $a_{jj} = a_{j,n-j}$ for $j = n+1, \ldots, 2n$. The algebra $ASM_{2n}(E)$ satisfies the same identity $U[X, Y, Z] = 0$.

For now we are able to find involutions in $M_n(E)$ for $n > 2$ only considering an involution in $E$. We generalize the case $n = 2$, namely

**Proposition 6.** The mapping $(b)$, defined as

$$
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34} \\
  a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix}^{(b)} = \begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix}^{(b)} = \begin{pmatrix}
  (D)^b & (B)^b \\
  (C)^b & (A)^b
\end{pmatrix}
$$

is an involution on $M_4(E, \psi = *)$.

**Proof of Proposition 6.** Considering in details the entries of the two matrices $(AB)^{(b)}$ and $(B)^{(b)}(A)^{(b)}$ we see that their corresponding entries are equal i.e. the mapping $(b)$ is an involution. □

We cover the following special case: Let $E_3'$ be the non-unitary finite dimensional Grassmann algebra with generators $e_1, e_2, e_3$ and $AM(2)(E_3')$ be the subalgebra of $AM_4(E_3')$ defined by the matrices of type

$$
\begin{pmatrix}
  y_1 & 0 & z_1 & 0 \\
  0 & y_2 & 0 & z_2 \\
  z_3 & 0 & y_3 & 0 \\
  0 & z_4 & 0 & y_4
\end{pmatrix},
$$

where $y_i$ are even elements (of even length) of $E_3'$, while $z_i$ are odd elements (of odd length) of $E_3', i = 1, \ldots, 4$. We equip the algebra $AM(2)(E_3', \psi_2)$ with the involution $(b)$ as defined in Proposition 6.

We characterize the $(b)$-symmetric elements $Y_i$ and the $(b)$-skew symmetric elements $Z_j$ of the algebra $(AM(2)(E_3', \psi_2), (b))$.

**Theorem 7.** The algebra $(AM(2)(E_3', \psi_2), (b))$ satisfies the $(b)$-identity $Y^3 = 0$ in $(b)$-symmetric variables.
Proof of Theorem 7. Let consider a \((\nu)\)-symmetric element \(Y\). Denoting for short
\[\psi_2\] as \(*\) in the equality
\[
\begin{pmatrix}
  y_4^* & 0 & z_2^* & 0 \\
  0 & y_3^* & 0 & z_1^* \\
  z_4^* & 0 & y_2^* & 0 \\
  0 & z_3^* & 0 & y_1^*
\end{pmatrix}
= 
\begin{pmatrix}
  y_1 & 0 & z_1 & 0 \\
  0 & y_2 & 0 & z_2 \\
  z_3 & 0 & y_3 & 0 \\
  0 & z_4 & 0 & y_4
\end{pmatrix}
\]
we get the following conditions on the entries of \(Y\): \(\psi_2(y_4) = y_1, \psi_2(y_3) = y_2, \psi_2(z_2) = z_1\) and \(\psi_2(z_4) = z_3\).

Let \(y_1 = s_1 e_1 e_2 + s_2 e_1 e_3 + s_3 e_2 e_3\). Then \(y_4 = \psi_2(y_1) = -y_1\). For \(y_2 = t_1 e_1 e_2 + t_2 e_1 e_3 + t_3 e_2 e_3\) we get \(y_3 = \psi_2(y_2) = -y_2\). Obviously \(y_1^2 = y_2^2 = 0\).

As the entries are from \(E_3\) we could work with odd entries having summands of degree 1 only. Let \(z_1 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3\) and \(z_3 = m_1 e_1 + m_2 e_2 + m_3 e_3\). Then \(z_2 = \psi_2(z_1) = z_1, z_4 = \psi_2(z_3) = z_3\). Considering \(Y^3 = Y^2 Y = (a_{ij})\) as
\[
\begin{pmatrix}
  z_1 z_3 & 0 & y_1 z_1 - z_1 y_2 & 0 \\
  0 & z_1 z_3 & 0 & y_2 z_1 - z_1 y_1 \\
  z_3 y_1 - y_2 z_3 & 0 & z_3 z_1 & 0 \\
  0 & z_3 y_2 - y_1 z_3 & 0 & z_3 z_1
\end{pmatrix}
\]
we see that manipulating with the generators \(e_1, e_2, e_3\), probably nontrivial entries could be only
\[a_{13} = a_{24} = z_1 z_3 z_1 = \beta_1 e_1 e_2 e_3, \ a_{31} = a_{42} = z_3 z_1 z_3 = \beta_2 e_1 e_2 e_3.\]

Applying Corollary 2 we get that both of them are zero. \(\square\)

Theorem 8. The algebra \((AM(2)(E_3, \psi_2), (\nu))\) satisfies the \((\nu)\)-identity \(Z^3 = 0\) in \((\nu)\)-skew symmetric variables.

Proof of Theorem 8. Using the same notations for the matrix entries of \(Z\) as in the previous theorem, in this case we have
\[y_4 = -\psi_2(y_1) = y_1, \ y_3 = -\psi_2(y_2) = y_2, \ y_1^2 = y_2^2 = 0, \ z_2 = -\psi_2(z_1) = -z_1, \ z_4 = -\psi_2(z_3) = -z_3.\]

In \(Z^3 = Z^2 Z = (b_{ij})\) nonzero could be only the entries \(b_{13} = -b_{24} = z_1 z_3 z_1\) and \(b_{31} = -b_{42} = z_3 z_1 z_3\). Corollary 2 proves they both are zero. \(\square\)

We consider the subalgebra \((AM(2)(E_3, \psi_2), (\nu))\) instead of the algebra \((AM(4)(E_3, \psi_2), (\nu))\) itself as if \(A^n = 0\) for a \(b\)-variable \(A\) of \((AM(4)(E_3, \psi_2), (\nu))\) we
have \( n > 3 \). Thus the algebras \( (AM_4(E_4, \psi_2), (b)) \) and \( AM_4(E_4') \) have equal nil indices.

We give an example of another matrix algebra with involution \((b)\) having lower nilpotency index of its \((b)\)-skew symmetric variables:

Let \( BM(2)(E_3') \) be the algebra defined by the matrices of type

\[
\begin{pmatrix}
  y_1 & 0 & 0 & z_1 \\
  0 & y_2 & z_2 & 0 \\
  0 & z_3 & y_3 & 0 \\
  z_4 & 0 & 0 & y_4
\end{pmatrix}
\]

, where \( y_i \) are even elements of \( E_3' \), while \( z_i \) are odd elements of \( E_3' \), \( i = 1, \ldots, 4 \). We equip the algebra \( BM(2)(E_3', \psi_2) \) with the involution \((b)\) as defined in Proposition 6.

**Theorem 9.** The algebra \( (BM(2)(E_3', \psi_2), (b)) \) satisfies the \((b)\)-identity \( Y^3 = 0 \) in \((b)\)-symmetric variables and the \((b)\)-identity \( Z^2 = 0 \) in \((b)\)-skew symmetric variables.

**Proof of Theorem 9.** In the algebra \( (BM(2)(E_3', \psi_2), (b)) \) any \((b)\)-skew symmetric variable \( Z \) is a diagonal matrix and \( Z^2 = 0 \) as \( y_i^2 = 0 \) for \( i = 1, \ldots, 4 \). \( \square \)

There is a package written in the system for computer algebra Mathematica [10] for manipulating in finite dimensional Grassmann algebras. Using it a programme was written by the author giving an alternative way of confirming the validity of the corresponding theorems in the paper.

**References**


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