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ORTHOMODULAR LATTICES CAN BE CONVERTED INTO LEFT RESIDUATED L-GROUPOIDS

IVAN CHAJDA AND HELMUT LÄNGER

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Abstract. We show that every orthomodular lattice can be considered as a left residuated l-groupoid satisfying divisibility, antitony, the double negation law and three more additional conditions expressed in the language of residuated structures. Also conversely, every left residuated l-groupoid satisfying the mentioned conditions can be organized into an orthomodular lattice.

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It is well-known that residuated structures form an algebraic axiomatization of fuzzy logics, see e. g. [1] for an overview. The reader can find necessary concepts and definitions concerning residuated structures in [5], however this paper is self-contained. Orthomodular lattices were introduced by G. Birkhoff and J. von Neumann as an algebraic axiomatization of the logic of quantum mechanics, see e. g. [4], [6] or [2] for details. Hence it is a natural question if these two concepts have a common base, i. e. if orthomodular lattices can be considered as residuated structures and hence as an axiomatization of certain fuzzy logic and, conversely, if certain residuated structures can be converted into orthomodular lattices, i. e. if the logic of quantum mechanics can be considered as a kind of fuzzy logic. For the theory of orthomodular lattices cf. the monographs [6] and [2] as well as the paper [3].

We start with the definition of an orthomodular lattice.

Definition 1. An *orthomodular lattice* is an algebra $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ satisfying (i) – (v) for all $x, y \in L$:

(i) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.
(ii) $x \vee x' = 1$
(iii) $x \leq y$ implies $y' \leq x'$.

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(iv) $(x')' = x$
(v) $x \leq y$ implies $y = x \vee (y \wedge x')$.

Remark 1. In every lattice $(L, \vee, \wedge)$ with a unary operation $'$ satisfying (iii) and (iv) the so-called de Morgan laws

$$(x \vee y)' = x' \wedge y' \text{ and } (x \wedge y)' = x' \vee y'$$

hold.

Remark 2. According to the de Morgan laws condition (v) can be replaced by

(vi) $x \leq y$ implies $x = y \wedge (x \vee y')$.

Now we introduce left residuated l-groupoids.

Definition 2. A *left residuated l-groupoid* is an algebra $\mathcal{A} = (A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying (i) – (iii) for all $x, y, z \in A$:

(i) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice.
(ii) $x \odot 1 = 1 \odot x = x$.
(iii) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$.

Condition (iii) is called *left adjointness*. $\mathcal{A}$ is said to satisfy *divisibility* if

$$(x \rightarrow y) \odot x = x \wedge y$$

for all $x, y \in A$. We define a unary operation $'$ on A by

$$x' := x \rightarrow 0$$

for all $x \in A$. $\mathcal{A}$ is said to satisfy *antitony* if

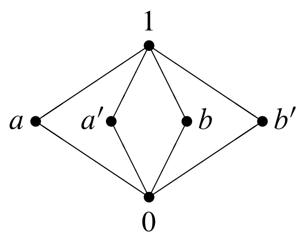
$$x \leq y \text{ implies } y' \leq x'$$

for all $x, y \in A$ and $\mathcal{A}$ is said to satisfy the *double negation law* if

$$(x')' = x$$

for all $x \in A$.

Example 1. If $A := \{0, a, a', b, b', 1\}$, $(A, \vee, \wedge, 0, 1)$ denotes the bounded lattice with the Hasse diagram



and the binary operations $\odot$ and $\to$ are defined by the tables

$\odot$	0	a	a'	b	b'	1
0	0	0	0	0	0	0
a	0	a	0	b	b'	a
a'	0	0	a'	b	b'	a'
b	0	a	a'	b	0	b
b'	0	a	a'	0	b'	b'
1	0	a	a'	b	b'	1

$\to$	0	a	a'	b	b'	1
0	1	1	1	1	1	1
a	a'	1	a'	a'	a'	1
a'	a	a	1	a	a	1
b	b'	b'	b'	1	b'	1
b'	b	b	b	b	1	1
1	0	a	a'	b	b'	1

then $(A,\vee,\wedge,\odot,\to,0,1)$ is a left residuated l-groupoid satisfying divisibility, antitony and the double negation law. The mentioned lattice is the smallest orthomodular lattice which is not a Boolean algebra and it is usually denoted by MO2.

The following theorem says that to every orthomodular lattice there can be assigned a left residuated l-groupoid in a natural way.

Theorem 1. *Let $\mathcal{L} = (L,\vee,\wedge,',0,1)$ be an orthomodular lattice and define binary operations $\odot$ and $\to$ on L by the following formulas:*

$$x \odot y = (x \vee y') \wedge y, \tag{0.1}$$

$$x \to y = (y \wedge x) \vee x'. \tag{0.2}$$

Then $\mathbf{A}(\mathcal{L}) = (L,\vee,\wedge,\odot,\to,0,1)$ is a left residuated l-groupoid satisfying divisibility, antitony, the double negation law as well as the following identity:

$$x \odot (x \vee y) = x. \tag{0.3}$$

Moreover, $x' = x \to 0$ for all $x \in L$.

Proof. Let $a,b \in L$. We have

$$a \to 0 = (0 \wedge a) \vee a' = 0 \vee a' = a'.$$

Of course, $(L,\vee,\wedge,0,1)$ is a bounded lattice. Moreover,

$$a \odot 1 = (a \vee 1') \wedge 1 = (a \vee 0) \wedge 1 = a \wedge 1 = a \text{ and}$$
$$1 \odot a = (1 \vee a') \wedge a = 1 \wedge a = a.$$

If $a \odot b \leq c$ then $(a \vee b') \wedge b \leq c$ and hence

$$a \leq a \vee b' = ((a \vee b') \wedge b) \vee b' = (((a \vee b') \wedge b) \wedge b) \vee b' \leq (c \wedge b) \vee b' = b \to c.$$

If, conversely, $a \leq b \to c$ then $a \leq (c \wedge b) \vee b'$ and hence

$$a \odot b = (a \vee b') \wedge b \leq (((c \wedge b) \vee b') \vee b') \wedge b = ((c \wedge b) \vee b') \wedge b = c \wedge b \leq c.$$

Now, using orthomodularity (i. e. (v) of Definition 1), we have

$$(a \to b) \odot a = (((b \wedge a) \vee a') \vee a') \wedge a = ((b \wedge a) \vee a') \wedge a = a \wedge b.$$

In view of Definition 1, $a \leq b$ implies $b' \leq a'$ and we have $(a')' = a$. Finally, by applying (0.1) and (vi) of Remark 2 we obtain

$$a \odot (a \vee b) = (a \vee (a \vee b)') \wedge (a \vee b) = a.$$

□

Remark 3. The operation $x \odot y := (x \vee y') \wedge y$ is called the *Sasaki projection* of x onto y (cf. [6] and [2]).

Conversely, certain left residuated l-groupoids give rise to an orthomodular lattice.

Theorem 2. *Let $\mathcal{A} = (A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a left residuated l-groupoid satisfying antitony, the double negation law as well as identities (0.1) and (0.3) of Theorem 1. Moreover, define $x' := x \rightarrow 0$ for all $x \in A$. Then $\mathbf{L}(\mathcal{A}) = (A, \vee, \wedge, ', 0, 1)$ is an orthomodular lattice.*

Proof. Let $a, b \in A$. Clearly, $(A, \vee, \wedge, 0, 1)$ is a bounded lattice and $a' = a \rightarrow 0$. Using antitony we see that $a \leq b$ implies $b' \leq a'$. Moreover, we have $(a')' = a$ according to the double negation law. Finally, if $a \leq b$ then, using (0.3) and (0.1), we have

$$b = (b')' = (b' \odot (b' \vee a'))' = (b' \odot a')' = ((b' \vee a) \wedge a')' = a \vee (b \wedge a')$$

and hence $a \vee a' = a \vee (1 \wedge a') = 1$. □

Finally, we prove that the correspondence described in the last two theorems is one-to-one.

Theorem 3. *We have $\mathbf{L}(\mathbf{A}(\mathcal{L})) = \mathcal{L}$ for every orthomodular lattice $\mathcal{L}$ and $\mathbf{A}(\mathbf{L}(\mathcal{A})) = \mathcal{A}$ for every left residuated l-groupoid satisfying antitony, the double negation law as well as identities (0.1) – (0.3) of Theorem 1.*

Proof. If $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ is an orthomodular lattice, $\mathbf{A}(\mathcal{L}) = (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ and $\mathbf{L}(\mathbf{A}(\mathcal{L})) = (L, \vee, \wedge, ^*, 0, 1)$ then

$$x^* = x \rightarrow 0 = (0 \wedge x) \vee x' = 0 \vee x' = x'$$

for all $x \in L$, therefore we obtain $\mathbf{L}(\mathbf{A}(\mathcal{L})) = \mathcal{L}$. Conversely, if $\mathcal{A} = (A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a left residuated l-groupoid satisfying divisibility, antitony, the double negation law as well as identities (0.1) – (0.3) of Theorem 1, $\mathbf{L}(\mathcal{A}) = (A, \vee, \wedge, ', 0, 1)$ and $\mathbf{A}(\mathbf{L}(\mathcal{A})) = (A, \vee, \wedge, \circ, \Rightarrow, 0, 1)$ then

$$x \circ y = (x \vee y') \wedge y = x \odot y \text{ and}$$
$$x \Rightarrow y = (y \wedge x) \vee x' = x \rightarrow y$$

for all $x, y \in A$, therefore we obtain $\mathbf{A}(\mathbf{L}(\mathcal{A})) = \mathcal{A}$. □

Remark 4. We have shown that orthomodular lattices can be considered as special residuated lattices and hence the logic of quantum mechanics axiomatized by them has a common base with a certain fuzzy logic axiomatized just by means of residuated lattices as pointed out in [1]. This sheds a new light on the logic of quantum mechanics and yields new tools for its investigation.

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Authors' addresses

Ivan Chajda

Palacký University Olomouc, Faculty of Science, Department of Algebra and Geometry, 17. listopadu 12, 771 46 Olomouc, Czech Republic

E-mail address: `ivan.chajda@upol.cz`

Helmut Länger

TU Wien, Faculty of Mathematics and Geoinformation, Institute of Discrete Mathematics and Geometry, Wiedner Hauptstraße 8-10, 1040 Vienna, Austria, and Palacký University Olomouc, Faculty of Science, Department of Algebra and Geometry, 17. listopadu 12, 771 46 Olomouc, Czech Republic

E-mail address: `helmut.laenger@tuwien.ac.at`