Miskolc Mathematical Notes
Vol. 18 (2017), No. 1, pp. 417–425

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2017.1729

SOME COEFFICIENT PROPERTIES RELATING TO A CERTAIN CLASS OF STARLIKE FUNCTIONS

RAVINDER KRISHNA RAINA AND JANUSZ SOKÓŁ

Received 18 September, 2015

Abstract. This paper considers the problem of determining coefficients in a class Δ^* of normalized starlike functions f analytic in the open unit disk $|z|<1$ satisfying the inequality that

$$\left| \left\{ \frac{zf'(z)}{f(z)} \right\}^2 - 1 \right| < 2 \left| \frac{zf'(z)}{f(z)} \right|.$$

2010 *Mathematics Subject Classification:* 30C45

Keywords: analytic functions, convex functions, starlike functions, differential subordination

1. INTRODUCTION

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Also, let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ comprising of functions f normalized by $f(0) = 0$, $f'(0) = 1$, and let $\mathcal{S} \subset \mathcal{A}$ denote the class of functions which are univalent in $\mathbb{U}$. Let a function f be analytic univalent in the unit disc $\mathbb{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$ with the normalization $f(0) = 0$, then f maps $\mathbb{U}$ onto a starlike domain with respect to $w_0 = 0$ if and only if

$$\mathfrak{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in \mathbb{U}). \tag{1.1}$$

It is well known that if an analytic function f satisfies (1.1) and $f(0) = 0$, $f'(0) \neq 0$, then f is univalent and starlike in $\mathbb{U}$. The set of all functions $f \in \mathcal{A}$ that are starlike univalent in $\mathbb{U}$ will be denoted by $\mathcal{S}^*$.

For the purpose of this paper, we represent by Δ^* a class which is defined by

$$\Delta^* = \left\{ f \in \mathcal{S}^* : \left| \left\{ \frac{zf'(z)}{f(z)} \right\}^2 - 1 \right| < 2 \left| \frac{zf'(z)}{f(z)} \right|, z \in \mathbb{U} \right\} \tag{1.2}$$

and a related class studied by Rønning [8] was defined by

$$\mathcal{S}_p = \left\{ f \in \mathcal{S} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < \mathfrak{Re}\, \frac{zf'(z)}{f(z)},\ z \in \mathbb{U} \right\}. \tag{1.3}$$

Interpreting geometrically the condition in (1.3), we note that $zf'(z)/f(z)$ lies inside the parabola

$$(\mathfrak{Im}w)^2 < 2\mathfrak{Re}w - 1,$$

and in this way the class $\mathcal{S}_p$ was observed to be connected with certain conic domains. In recent papers [1–4, 6, 10], certain function classes were considered and were defined under the condition that $zf'(z)/f(z)$ lies in a domain which possesses some geometric properties. If we interpret the condition in (1.2) geometrically, then we observe that the product of the distances of $zf'(z)/f(z)$ from the foci -1 and 1 is less than twice the distance of $zf'(z)/f(z)$ from the origin. The shape of the domain for $Q(z) = zf'(z)/f(z)$ is described in Theorem 1 below and the shape of $Q(\mathbb{U})$ is depicted in Figure 1.

Theorem 1 ([9]). *If $f(z) \in \Delta^*$, then*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{U} \tag{1.4}$$

and

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \sqrt{2} \quad and \quad \left|\frac{zf'(z)}{f(z)} + 1\right| > \sqrt{2}, \quad z \in \mathbb{U}. \tag{1.5}$$

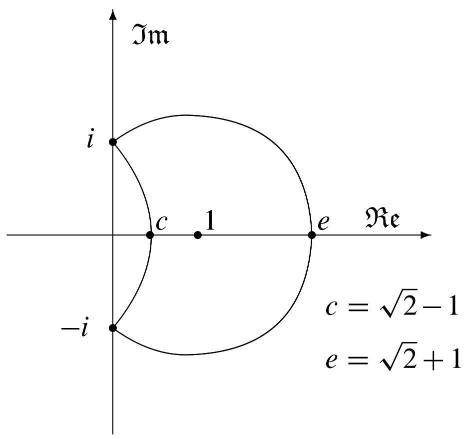

FIGURE 1. The domain for $zf'(z)/f(z)$, $f \in \Delta^*$.

2. Coefficient estimates

Theorem 2. *If* $f(z) \in \Delta^*$ *and*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{U}, \tag{2.1}$$

then

$$|a_2| \le 1, \quad |a_3| \le 3/4, \quad |a_4| \le 1/2. \tag{2.2}$$

Proof. In view of (1.2), we have

$$\left\{ \frac{zf'(z)}{f(z)} \right\}^2 - 1 = 2w(z)\frac{zf'(z)}{f(z)},$$

where

$$|w(z)| < 1 \quad z \in \mathbb{U}, \quad w(z) = \sum_{k=1}^{\infty} c_k z^k. \tag{2.3}$$

Thus, we obtain

$$(zf'(z) - f(z))(zf'(z) + f(z)) = 2w(z)zf'(z)f(z). \tag{2.4}$$

If we assume that $a_1 = 1$, then from (2.1) and (2.3), we at once have

$$\begin{aligned} &\left(\sum_{k=1}^{\infty}(k-1)a_k z^k\right)\left(\sum_{k=1}^{\infty}(k+1)a_k z^k\right) \\ &= 2\left(\sum_{k=1}^{\infty} c_k z^k\right)\left(\sum_{k=1}^{\infty} k a_k z^k\right)\left(\sum_{k=1}^{\infty} a_k z^k\right). \end{aligned} \tag{2.5}$$

Hence, we obtain

$$\begin{aligned} &\left(\sum_{k=1}^{\infty}(k+1)a_k z^k\right)\left(\sum_{k=1}^{\infty}(k-1)a_k z^k\right) \\ &= (2z + 3a_2z^2 + 4a_3z^3 + \cdots)(a_2z^2 + 2a_3z^3 + 3a_4z^4 + \cdots) \\ &= 2a_2z^3 + (4a_3 + 3a_2^2)z^4 + (6a_4 + 10a_2a_3)z^5 + \cdots \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &2\left(\sum_{k=1}^{\infty} c_k z^k\right)\left(\sum_{k=1}^{\infty} k a_k z^k\right)\left(\sum_{k=1}^{\infty} a_k z^k\right) \\ &=2(c_1z + c_2z^2 + c_3z^3 + \cdots)(z + 2a_2z^2 + 3a_3z^3 + \cdots)(z + a_2z^2 + a_3z^3 + \cdots) \\ &=2c_1z^3 + 2(3a_2c_1 + c_2)z^4 + 2(4a_3c_1 + 2a_2^2c_1 + 3a_2c_2 + c_3)z^5 + \cdots. \end{aligned} \tag{2.7}$$

Equating now the coefficients of like powers of z in (2.6) and (2.7), we have

(*i*) $a_2 = c_1$,

(ii) $4a_3 + 3a_2^2 = 6a_2c_1 + 2c_2$,
(iii) $6a_4 + 10a_2a_3 = 8a_3c_1 + 4a_2^2c_1 + 6a_2c_2 + 2c_3$.

It is well known that the coefficients of the bounded function $w(z)$ satisfies the inequality that $|c_k| \le 1$, $(k = 1,2,3,\ldots)$, and hence from (i), we have the first inequality of (2.2) that $|a_2| \le 1$. Now, from (i) and (ii), we have

$$|4a_3| = 2\left|c_2 + \frac{3}{2}c_1^2\right|. \tag{2.8}$$

Using the estimate (see [7]) that if $w(z)$ has the form (2.3), then

$$|c_2 - \mu c_1^2| \le \max\{1, |\mu|\}, \quad \text{for all } \ \mu \in \mathbb{C}, \tag{2.9}$$

and we obtain from (2.8) and (2.9) that

$$|a_3| \le \frac{3}{4},$$

which gives the second inequality of of (2.2). From (iii), we find that

$$|6a_4| = \left|-10a_2a_3 + 8a_3c_1 + 4a_2^2c_1 + 6a_2c_2 + 2c_3\right|. \tag{2.10}$$

Because $a_2 = c_1$, (2.10) becomes

$$|6a_4| = \left|-2a_3c_1 + 4c_1^3 + 6c_1c_2 + 2c_3\right|. \tag{2.11}$$

Moreover, from $(i) - (ii)$, we have

$$a_3 = \frac{1}{2}c_2 + \frac{3}{4}c_1^2, \tag{2.12}$$

and from (2.11) and (2.12), we obtain that

$$\begin{aligned} |6a_4| &= \left|-2\left(\frac{1}{2}c_2 + \frac{3}{4}c_1^2\right)c_1 + 4c_1^3 + 6c_1c_2 + 2c_3\right| \\ &= \left|\frac{5}{2}c_1^3 + 5c_1c_2 + 2c_3\right| \\ &= \left|\frac{5}{2}\left(c_1^3 + 2c_1c_2 + c_3\right) - \frac{1}{2}c_3\right|. \end{aligned} \tag{2.13}$$

To find the bound for the coefficient a_4, we next derive some properties of the coefficients c_k involved in (2.13). It is known that the function $p(z)$ given by

$$\frac{1 + w(z)}{1 - w(z)} = 1 + p_1z + p_2z^2 + \cdots =: p(z) \tag{2.14}$$

defines a Caratheodory function with the property that $\mathfrak{Re}\{p(z)\} > 0$ in $\mathbb{U}$ and that $|p_k| \le 2$ $(k = 1,2,3,\ldots)$.

Using (2.3) and equating the coefficients of like powers of z in (2.14), we get

$$p_2 = 2(c_1^2 + c_2) \ \text{ and } \ p_3 = 2(c_1^3 + 2c_1c_2 + c_3).$$

Hence $|c_1^2+c_2| \leq 1$ and

$$|c_1^3+2c_1c_2+c_3| \leq 1, \tag{2.15}$$

and upon using (2.13) and (2.15), we finally find that

$$\begin{aligned}|6a_4| &\leq \left|\frac{5}{2}\left(c_1^3+2c_1c_2+c_3\right)\right| + \left|\frac{1}{2}c_3\right| \\ &\leq \frac{5}{2}+\frac{1}{2}=3,\end{aligned}$$

which gives the third inequality of (2.2) that $|a_4| \leq 1/2$. □

Remark. We deem it worthwhile to point out here the sharpness of the estimates of the coefficients given by (2.2) of Theorem 2. Therefore, let us consider the function $f_1(z)$ by

$$\begin{aligned}f_1(z) &= z\exp\int_0^z \frac{\sqrt{1+t^2}+t-1}{t}\mathrm{d}t \\ &= \frac{2\sqrt{1+z^2}-2}{z}\exp\left\{z-1+\sqrt{1+z^2}\right\} \\ &= z+z^2+\frac{3}{4}z^3+\frac{5}{12}z^4+\frac{1}{6}z^5+\frac{1}{20}z^6+\frac{49}{2880}z^7\cdots. \quad (z \in \mathbb{U}).\end{aligned} \tag{2.16}$$

To show that $f_1(z) \in \varDelta^*$, we need to show that

$$f_1(z) \in \mathcal{S}^* \quad \text{and} \quad \left|\left\{\frac{zf_1'(z)}{f_1(z)}\right\}^2-1\right| < 2\left|\frac{zf_1'(z)}{f_1(z)}\right|. \tag{2.17}$$

We note that

$$0 \leq \mathfrak{Re}\left\{e^{it}+\sqrt{e^{2it}+1}\right\} = \begin{cases} \cos t+\sqrt{|2\cos t|}\cos t/2 & \text{for} \quad t \in [0,\pi/2], \\ \cos t+\sqrt{|2\cos t|}\sin t/2 & \text{for} \quad t \in (\pi/2,3\pi/2], \\ \cos t-\sqrt{|2\cos t|}\cos t/2 & \text{for} \quad t \in (3\pi/2,2\pi), \end{cases}$$

and therefore, for the above function f_1, we have

$$\mathfrak{Re}\left\{\frac{zf_1'(z)}{f_1(z)}\right\} = \mathfrak{Re}\left\{z+\sqrt{1+z^2}\right\} > 0 \quad (z \in \mathbb{U}).$$

Hence, by (1.1), $f_1(z) \in \mathcal{S}^*$ and the left-hand side of the second condition in (2.17) becomes

$$\begin{aligned}\left|\left\{\frac{zf_1'(z)}{f_1(z)}\right\}^2-1\right| &= \left|\left\{z+\sqrt{1+z^2}\right\}^2-1\right| \\ &= 2|z|\left|z+\sqrt{1+z^2}\right| \\ &< 2\left|z+\sqrt{1+z^2}\right|\end{aligned}$$

$$= 2\left|\frac{zf_1'(z)}{f_1(z)}\right|,$$

which implies that $f_1(z) \in \Delta^*$. Thus, from (2.16), we see that the first and the second estimations in (2.2) are sharp. The question is whether the third estimation $|a_4| \le 1/2$ is sharp in the class. The function (2.16) suggest the following conjecture.

Conjecture 1. *If $f(z)$ defined by (2.18) belongs to Δ^*, then*

$$|a_4| \le \frac{5}{12} = 0.416\dots .$$

In the sequel, we find somewhat weaker estimation for $|a_n|$, for the next coefficients too by applying another method.

Theorem 3. *If $f(z) \in \Delta^*$ and*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{U}, \tag{2.18}$$

then for $n = 2, 3, 4, \dots$, we have

$$(n-1)^2|a_n|^2 \le \sum_{k=1}^{n-1} |a_k|^2(1 + 2k - k^2). \tag{2.19}$$

Proof. In view of (1.5), we have

$$\frac{zf'(z)}{f(z)} - 1 = \sqrt{2}w(z),$$

where

$$|w(z)| < 1 \quad z \in \mathbb{U}, \quad w(z) = \sum_{k=1}^{\infty} c_k z^k. \tag{2.20}$$

Thus, we obtain

$$\frac{1}{\sqrt{2}}(zf'(z) - f(z)) = w(z)f(z), \tag{2.21}$$

and from (2.18) and (2.20), we at once have

$$\frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} (k-1)a_k z^k = \sum_{k=1}^{\infty} c_k z^k \sum_{k=1}^{\infty} a_k z^k, \quad a_1 = 1.$$

Thus, we get

$$\frac{1}{\sqrt{2}} \sum_{k=1}^{n} (k-1)a_k z^k + \frac{1}{\sqrt{2}} \sum_{k=n+1}^{\infty} (k-1)a_k z^k = w(z)\left\{\sum_{k=1}^{n-1} a_k z^k + \sum_{k=n}^{\infty} a_k z^k\right\},$$

which gives

$$\sum_{k=1}^{n}\frac{k-1}{\sqrt{2}}a_k z^k+\sum_{k=n+1}^{\infty}\frac{k-1}{\sqrt{2}}a_k z^k-\sum_{k=1}^{\infty}c_k z^k\sum_{k=n}^{\infty}a_k z^k=w(z)\left\{\sum_{k=1}^{n-1}a_k z^k\right\}.$$

Therefore, we can write

$$\sum_{k=1}^{n}\frac{k-1}{\sqrt{2}}a_k z^k+\sum_{k=n+1}^{\infty}b_k z^k=w(z)\sum_{k=1}^{n-1}a_k z^k,$$

for some b_k, $n+1\le k<\infty$, where b_k can be expressed in terms of the following relation involving the coefficients a_k and c_k:

$$b_k=\frac{k-1}{\sqrt{2}}a_k-\sum_{j=1}^{k-n}c_j a_{k-j}.$$

This gives

$$\begin{aligned}\left|\sum_{k=1}^{n}\frac{k-1}{\sqrt{2}}a_k z^k+\sum_{k=n+1}^{\infty}b_k z^k\right|^2&=\left|w(z)\sum_{k=1}^{n-1}a_k z^k\right|^2\\&\le\left|\sum_{k=1}^{n-1}a_k z^k\right|^2,\end{aligned}\tag{2.22}$$

where

$$\sum_{k=1}^{n}\frac{k-1}{\sqrt{2}}a_k z^k+\sum_{k=n+1}^{\infty}b_k z^k:=\sum_{k=1}^{\infty}d_k z^k$$

is an analytic function in the unit disc. Making use of the known formula (see, for instance [5])

$$\int_0^{2\pi}\left|\sum_{k=1}^{\infty}d_k(re^{i\theta})^k\right|^2 d\theta=2\pi\sum_{n=1}^{\infty}|d_n|^2r^{2n},$$

and integrating on $z=re^{i\theta}$, $0<r<1$, $0\le\theta<2\pi$, both the sides of (2.22), we obtain

$$\sum_{k=1}^{n}\frac{(k-1)^2}{2}|a_k|^2r^{2k}+\sum_{k=n+1}^{\infty}|b_k|^2r^{2k}\le\sum_{k=1}^{n-1}|a_k|^2r^{2k}.$$

Therefore,

$$\frac{1}{2}\sum_{k=1}^{n}(k-1)^2|a_k|^2r^k\le\sum_{k=1}^{n-1}|a_k|^2r^{2k},$$

which upon letting $r \to 1$ gives

$$\frac{1}{2}\sum_{k=1}^{n}(k-1)^2|a_k|^2 \le \sum_{k=1}^{n-1}|a_k|^2,$$

and this leads to the desired result (2.19). □

Theorem 4. *Let $f(z)$ defined by (2.18) belong to Δ^*. Then for $n \ge 3$, we have*

$$|a_n| \le \frac{\sqrt{3}}{n-1} \quad . \tag{2.23}$$

Proof. From (2.19), we have (for $n \ge 3$)

$$\begin{aligned}(n-1)^2|a_n|^2 &\le \sum_{k=1}^{n-1}|a_k|^2(1+2k-k^2)\\ &= 2|a_1|^2+|a_2|^2-2|a_3|^2-7|a_4|^2-\dots\\ &\le 2|a_1|^2+|a_2|^2\\ &= 2+|a_2|^2.\end{aligned}$$

Furthermore, $|a_2|^2 \le 1$ by Theorem 2, and we have then

$$(n-1)^2|a_n|^2 \le 3$$

and finally we obtain (2.23). □

ACKNOWLEDGEMENT

The authors would like to express their sincerest thanks to the referees for a careful reading and various suggestions made for the improvement of the paper.

REFERENCES

[1] J. Dziok, R. K. Raina, and J. Sokół, "Certain results for a class of convex functions related to a shell-like curve connected with Fibonacci numbers," *Comput. Math. Appl.*, vol. 61, no. 9, pp. 2605–2613, 2011, doi: 10.1016/j.camwa.2011.03.006.

[2] J. Dziok, R. K. Raina, and J. Sokół, "On alpha-convex functions related to shell-like functions connected with Fibonacci numbers," *Appl. Math. Comput.*, vol. 218, pp. 966–1002, 2011, doi: 10.1016/j.amc.2011.01.059.

[3] J. Dziok, R. K. Raina, and J. Sokół, "Differential subordinations and alpha-convex functions," *Acta Math. Scientia*, vol. 33B, pp. 609–620, 2013, doi: 10.1016/S0252-9602(13)60024-7.

[4] J. Dziok, R. K. Raina, and J. Sokół, "On a class of starlike functions related to a shell-like curve connected with Fibonacci numbers," *Math. Comput. Modelling*, vol. 57, pp. 1203–1211, 2013.

[5] A. W. Goodman, *Univalent Functions, Vol. I.* Tampa, Florida: Mariner Publishing Co., 1983.

[6] R. Jurasińnska and J. Sokół, "Some problems for certain family of starlike functions," *Math. Comput. Modelling*, vol. 55, pp. 2134–2140, 2012, doi: 10.1016/j.mcm.2012.01.009.

[7] F. R. Keogh and E. P. Merkes, "A coefficient inequality for certain classes of analytic functions," *Proc. Amer. Math. Soc.*, vol. 20, pp. 8–12, 1969, doi: 10.2307/2035949.

[8] F. R. Keogh and E. P. Merkes, "Uniformly convex functions and a corresponding class of starlike functions," *Proc. Amer. Math. Soc.*, vol. 118, pp. 189–196, 1993.
[9] R. K. Raina and J. Sokół, "Some properties relating to a certain class of starlike functions," *Comptes Rendus Mathématique*, vol. 353, no. 11, pp. 973–978, 2015.
[10] J. Sokół and A. Wiśniowska-Wajnryb, "On certain problem in the class of k-starlike functions," *Comput. Math. Appl.*, vol. 62, pp. 4733–4741, 2011, doi: 10.1016/j.camwa.2011.10.064.

Authors' addresses

Ravinder Krishna Raina
M.P. University of Agriculture and Technology, Department of Mathematics, 10/11 Ganpati Vihar, Opposite Sector 5, Udaipur 313002, Rajasthan, India
E-mail address: rkraina_7@hotmail.com

Janusz Sokół
University of Rzeszów, Faculty of Mathematics and Natural Sciences, ul. Prof. Pigonia 1, 35-310 Rzeszów, Poland
E-mail address: jsokol@ur.edu.pl