



ON WEAK ARMENDARIZ MODULES OVER COMMUTATIVE RINGS

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Received 27 May, 2015

Abstract. Let M be a module over a commutative ring R . In this paper we generalize some annihilator conditions on rings to modules. Denote by $Nil(M)$ the set of all nilpotent elements of M . M is said to be weak Armendariz if $f(x)m(x) = 0$, where $f(x) = \sum_{i=0}^n a_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^k m_j x^j \in M[x] \setminus \{0\}$, then $a_i m_j \in Nil(M)$ for each $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$. We prove that the class of these modules are closed under direct sum, finite product and localization. We also prove that if $\frac{M}{N}$ is weak Armendariz, then so is M . Furthermore, we show that if D -module M is torsion, for a domain D , then M is weak Armendariz if and only if $T(M)$ is weak Armendariz, where $T(M)$ is the torsion submodule of M .

2010 *Mathematics Subject Classification:* 16D10; 13C99; 16N40

Keywords: Armendariz ring, Armendariz module, weak Armendariz ring, weak Armendariz module

1. INTRODUCTION

Throughout this paper all rings are considered to be commutative with a nonzero identity and all modules are unitary unless otherwise stated. Let R be a ring, M an R -module and N a submodule of M . We denote by $(N :_R M)$ the set of all r in R such that $rM \subseteq N$. The *annihilator* of M denoted by $ann_R(M)$ is $(0 :_R M)$. An R -module M is called *faithful* if $ann_R(M) = 0$. $R[x]$ denotes the polynomial ring over the ring R and $M[x]$ the polynomial module over the module M .

An R -module M is called a *multiplication module* if every submodule N of M has the form IM for some ideal of R . In this case $N = (N :_R M)M$, since $I \subseteq (N :_R M)$ and so $N = IM \subseteq (N :_R M)M \subseteq N$. According to [6], a submodule N of M is called *pure* if $IN = N \cap IM$ for every ideal I of R . Pure submodules of multiplication modules are multiplication, too. A submodule N of M is idempotent if $N = (N :_R M)N$ (See [1]). A submodule N of M is *prime* whenever $rm \in N$, for some $r \in R$ and $m \in M$ implies that $m \in N$ or $r \in (N :_R M)$. In this case $P := (N :_R M)$ is a prime ideal of R and N is called a P -prime submodule of M .

We recall that an ideal I of R is *nilpotent* if $I^k = 0$ for positive integer k and an element r of R is nilpotent if $r^k = 0$ for some $k \in \mathbb{N}$. Also we denote by $Nil(R)$

the set of all nilpotent elements of R . According to [2], a submodule N of M is called *nilpotent* if $(N :_R M)^k N = 0$ for some positive integer k . We say that $m \in M$ is *nilpotent* if Rm is a nilpotent submodule of M . By this definition, clearly the zero submodule of M is nilpotent and hence the zero element of M is nilpotent. $Nil(M)$ denotes the set of all nilpotent elements of M . $Nil(M)$ is not necessarily a submodule of M , but if M is faithful, then $Nil(M)$ is a submodule of M , by [2, Theorem 6]. Moreover, if M is a faithful multiplication R -module, then $Nil(M) = Nil(R)M = \cap P$, where P runs over all prime submodules of M . By [2, Proposition 4(2)], if I is a nilpotent ideal of R , then IM is nilpotent in M and the converse is true if M is faithful and also by this Proposition, if $K \subseteq N$ and N is nilpotent in M , then $\frac{N}{K}$ is nilpotent in $\frac{M}{K}$ as an R -module and the converse is true if K is nilpotent in M and M is faithful. By [2, Theorem 3], if N is a direct summand in multiplication module M , then N is multiplication and idempotent.

Let R be an associative ring with identity. R is said to be *Armendariz* if $f(x)g(x) = 0$, where $f(x) = \sum_{i=0}^m a_i x^i, g(x) = \sum_{j=0}^n b_j x^j \in R[x] \setminus \{0\}$, then $a_i b_j = 0$, for each $0 \leq i \leq m$ and $0 \leq j \leq n$ (See [5]).

An R -module M is called *Armendariz* if $f(x)m(x) = 0$, where $f(x) = \sum_{i=0}^n a_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^k m_j x^j \in M[x]$ implies that $a_i m_j = 0$ for every $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$ (See [5]). A ring R is called *weak Armendariz* if for given $f(x) = \sum_{i=0}^m a_i x^i, g(x) = \sum_{j=0}^n b_j x^j \in R[x] \setminus \{0\}$, $f(x)g(x) = 0$ implies that $a_i b_j \in Nil(R)$ for each $0 \leq i \leq m$ and $0 \leq j \leq n$ (the converse is obviously true). It is obvious that Armendariz rings are weak Armendariz and that subrings of (weak) Armendariz rings are still (weak) Armendariz (See [3]).

In this paper we generalize the concept of weak Armendariz rings defined on associative rings to modules over commutative rings and introduce weak Armendariz modules. An R -module M is called *weak Armendarize* if $f(x)m(x) = 0$ implies that $a_i m_j \in Nil(M)$, for each $0 \leq i \leq n$ and $0 \leq j \leq k$, where $f(x) = \sum_{i=0}^n a_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^k m_j x^j \in M[x]$. Note that commutative rings are always weak Armendariz.

2. WEAK ARMENDARIZ MODULES

In this section, we study some properties of weak Armendariz modules and investigate the relations between weak Armendariz rings and weak Armendariz modules.

Definition 1. Let R be a ring. An R -module M is called *weak Armendarize* if $f(x)m(x) = 0$ implies that $a_i m_j \in Nil(M)$, for each $0 \leq i \leq n$ and $0 \leq j \leq k$, where $f(x) = \sum_{i=0}^n a_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^k m_j x^j \in M[x] \setminus \{0\}$.

Proposition 1. Let R be a ring. Then every Armendariz R -module M is weak Armendariz and the converse is true if M has no nonzero nilpotent elements.

Corollary 1. *Let M be a finitely generated faithful multiplication R -module. Then the $\frac{R}{\text{Nil}(R)}$ -module $\frac{M}{\text{Nil}(M)}$ is Armendariz if and only if it is weak Armendariz.*

Proof. First note that $\text{Nil}(M)$ is a submodule of M and $\text{Nil}(M) = \text{Nil}(R)M$, by [2, Theorem 6]. So the $\frac{R}{\text{Nil}(R)}$ -module $\frac{M}{\text{Nil}(M)}$ is well-defined. According to [2, Corollary 7], $\frac{M}{\text{Nil}(M)}$ has no nonzero nilpotent elements. Now the result follows from Proposition 1. \square

Proposition 2. *The direct limit of every direct system of weak Armendariz modules is weak Armendariz.*

Proof. Let I be a directed set and $\{M_i; i \in I\}$ a direct system of weak Armendariz R -modules. Set $M = \lim_{\substack{\longrightarrow \\ i \in I}} M_i$. We know that $M = \bigcup_{i \in I} M_i = \sum_{i \in I} M_i$, by [7, Example 5.32]. Let $m(x) = \sum_{j=0}^n m_j x^j \in M[x]$ and $f(x) = \sum_{k=0}^m a_k x^k \in R[x]$, with $f(x)m(x) = 0$. Hence there exists $t \in I$ such that $m_j \in M_t$, for each $j = 0, 1, \dots, n$. So $m(x) \in M_t[x]$. Since M_t is weak Armendariz, $a_k m_j \in \text{Nil}(M_t)$ and hence there exists positive integer n_{kj} such that $(R(a_k m_j) :_R M_t)^{n_{kj}} R(a_k m_j) = 0$, for every $k = 0, 1, \dots, m$ and $j = 0, 1, \dots, n$. But $(R(a_k m_j) :_R M) \subseteq (R(a_k m_j) :_R M_t)$. Therefore, $(R(a_k m_j) :_R M)^{n_{kj}} R(a_k m_j) = 0$, for each $k = 0, 1, \dots, m$ and $j = 0, 1, \dots, n$. Hence M is a weak Armendariz R -module. \square

Corollary 2. *If every finitely generated submodule of an R -module M is weak Armendariz, then M is weak Armendariz.*

Proof. We know every module is a direct limit of its finitely generated submodules, by [7, Example 5.32]. Now the result is trivial by Proposition 2. \square

Definition 2. A ring R is said to be *strongly weak Armendariz* if every R -module is weak Armendariz.

Corollary 3. *A ring R is strongly weak Armendariz if and only if every finitely generated R -module is weak Armendariz.*

Proof. \Rightarrow It is trivial.

\Leftarrow We know that every R -module M is a direct limit of its finitely generated submodules. Now the result follows from the assumption and Proposition 2. \square

Proposition 3. *A ring R is weak Armendariz if and only if it is weak Armendariz as an R -module.*

Proof. We know the R -module R is multiplication. So $\text{Nil}(R) = \text{Nil}(R)R$, by [2, Theorem 6]. Thus the result is obvious. \square

Proposition 4. *Every pure submodule of a weak Armendariz multiplication module is weak Armendariz.*

Proof. Let N be a pure submodule of the weak Armendariz multiplication module M and $f(x)m(x) = 0$ for $f(x) = \sum_{i=0}^n a_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^k m_j x^j \in N[x] \setminus \{0\}$. Clearly, N is a multiplication submodule of M . Since $m(x) \in M[x]$ and M is weak Armendariz, $a_i m_j \in Nil_R(M)$. By [2, Proposition 4], $(R(a_i m_j) :_R M)$ is a nilpotent ideal of R and so $(R(a_i m_j) :_R M) \subseteq Nil(R)$, for each $0 \leq i \leq n$ and $0 \leq j \leq k$. By [2, Theorem 6], $Nil_R(M) = Nil(R)M$ and $Nil_R(N) = Nil(R)N$, since M and N are multiplication. Since N is a pure submodule of M and $R(a_i m_j) = (R(a_i m_j) :_R M)M$, $a_i m_j \in N \cap (R(a_i m_j) :_R M)M = (R(a_i m_j) :_R M)N \subseteq Nil(R)N = Nil_R(N)$, for each $0 \leq i \leq n$ and $0 \leq j \leq k$. Thus N is weak Armendariz. \square

Corollary 4. *A multiplication R -module M is weak Armendariz if and only if every finitely generated pure submodule of M is weak Armendariz.*

Proof. We know that every module is a direct limit of its finitely generated submodules, by [7, Example 5.32]. Now the result follows from Proposition 2 and 4. \square

Proposition 5. *Let M be a multiplication R -module. Then the following statements are equivalent:*

- (1) M is weak Armendariz;
- (2) eM and $(1-e)M$ are weak Armendariz R -modules for every idempotent element e of R .
- (3) eM and $(1-e)M$ are weak Armendariz R -modules for some idempotent element e of R .

Proof. $1 \Rightarrow 2$: Let e be an idempotent element of R . So $M = eM \oplus (1-e)M$. It means that eM and $(1-e)M$ are direct summands in M and so both of them are multiplication and idempotent submodules of M , by [2, Theorem 3]. Therefore, eM and $(1-e)M$ are pure submodules of M and hence they are weak Armendariz, by Proposition 4.

$2 \Rightarrow 3$: It is trivial.

$3 \Rightarrow 1$: Assume that there exists an idempotent element e of R such that eM and $(1-e)M$ are weak Armendariz R -modules. Let $f(x) = \sum_{i=0}^m a_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^n m_j x^j \in M[x] \setminus \{0\}$, with $f(x)m(x) = 0$. Put $f_1(x) = ef(x)$, $f_2(x) = (1-e)f(x)$, $m_1(x) = em(x)$ and $m_2(x) = (1-e)m(x)$. Then we have $0 = f(x)m(x) = f_1(x)m_1(x) = f_2(x)m_2(x)$. By assumption, $ea_i m_j \in Nil(eM)$ and $(1-e)a_i m_j \in Nil((1-e)M)$, for every $i = 0, 1, \dots, m$ and $j = 0, 1, \dots, n$.

So there exist positive integers k_{ij} and t_{ij} such that

$$(R(ea_i m_j) :_R eM)^{k_{ij}} R(ea_i m_j) = 0$$

and

$$(R((1-e)a_i m_j) :_R (1-e)M)^{t_{ij}} R((1-e)a_i m_j) = 0$$

for each $0 \leq i \leq m$ and $0 \leq j \leq n$. Set $k = \max\{k_{ij}, t_{ij}; 0 \leq i \leq m, 0 \leq j \leq n\}$. Note that $m = em + (1-e)m$, for each element $m \in M$, and so $Rm = R(em) + R((1-e)m)$. Hence, we have

$$\begin{aligned} (R(a_i m_j) :_R M)^k R(a_i m_j) &= (R(a_i m_j) :_R M)^k [R(ea_i m_j) + R((1-e)a_i m_j)] \\ &= (R(a_i m_j) :_R M)^k R(ea_i m_j) \\ &\quad + (R(a_i m_j) :_R M)^k R((1-e)a_i m_j) \\ &\subseteq (R(ea_i m_j) :_R eM)^k R(ea_i m_j) \\ &\quad + (R((1-e)a_i m_j) :_R (1-e)M)^k R((1-e)a_i m_j) \\ &= 0 \end{aligned}$$

So $a_i m_j \in Nil_R(M)$, for each $0 \leq i \leq m$ and $0 \leq j \leq n$. Therefore, M is weak Armendariz. \square

Theorem 1. *Every finite direct product of weak Armendariz modules is weak Armendariz.*

Proof. Suppose that M_1, M_2, \dots, M_k are weak Armendariz R -modules. Set $M = \prod_{t=1}^k M_t$. Clearly, M is an R -module. Let $f(x)m(x) = 0$, for some $m(x) = \sum_{i=0}^n m_i x^i \in M[x] \setminus \{0\}$ and $f(x) = \sum_{j=0}^p a_j x^j \in R[x] \setminus \{0\}$, where $m_i = (m_{i1}, \dots, m_{ik}) \in M$.

Define $m_t(x) = \sum_{i=0}^n m_{it} x^i \in M_t[x]$, for each $1 \leq t \leq k$. From $f(x)m(x) = 0$ we have $a_0 m_0 = a_1 m_0 + a_0 m_1 = \dots = a_p m_n = 0$. This implies that

$$\begin{aligned} a_0 m_{01} &= \dots = a_0 m_{0k} = 0 \\ a_1 m_{01} + a_0 m_{11} &= \dots = a_1 m_{0k} + a_0 m_{1k} = 0 \\ &\vdots \\ a_p m_{n1} &= \dots = a_p m_{nk} = 0 \end{aligned}$$

This means that $f(x)m_t(x) = 0$ in $M_t[x]$, for every $1 \leq t \leq k$. Since each M_t is weak Armendariz module, we have $a_j m_{it} \in Nil_R(M_t)$ and so there exists positive integer n_{ijt} , for each $i = 0, 1, \dots, n$, $j = 0, 1, \dots, p$ and $0 \leq t \leq k$, such that $(R(a_j m_{it}) : M_t)^{n_{ijt}} R(a_j m_{it}) = 0$ in M_t . If we put $n_{ij} = \max\{n_{ijt}; 1 \leq t \leq k\}$, then

$$\begin{aligned} (R(a_j m_i) : M)^{n_{ij}} R(a_j m_i) &\subseteq \left(\prod_{t=1}^k R(a_j m_{it}) : \prod_{t=1}^k M_t \right)^{n_{ij}} \left(\prod_{t=1}^k R(a_j m_{it}) \right) \\ &= \left(\bigcap_{t=1}^k (R(a_j m_{it}) : M_t) \right)^{n_{ij}} \left(\prod_{t=1}^k R(a_j m_{it}) \right) \end{aligned}$$

$$\begin{aligned}
& \subseteq \bigcap_{t=1}^k ((R(a_j m_{it}) : M_t)^{n_{ij}}) \prod_{t=1}^k (R(a_j m_{it})) \\
& = \prod_{t=1}^k \bigcap_{i=1}^k ((R(a_j m_{it}) : M_t)^{n_{ij}} R(a_j m_{it})) \\
& = 0
\end{aligned}$$

So $a_j \bar{m}_i \in Nil_R(M)$, for each $0 \leq i \leq n$ and $0 \leq j \leq p$. Thus M is a weak Armendariz R -module. \square

Proposition 6. *Let R be a ring and N a nilpotent submodule of a faithful R -module M . If $\frac{M}{N}$ is a weak Armendariz R -module, then M is weak Armendariz.*

Proof. Let $f(x) = \sum_{i=0}^n a_i x^i$ and $m(x) = \sum_{j=0}^k m_j x^j$ are polynomials in $R[x] \setminus \{0\}$ and $M[x] \setminus \{0\}$, respectively, such that $f(x)m(x) = 0$. Set $\bar{m}(x) = \sum_{j=0}^k \bar{m}_j x^j$, where \bar{m}_j denotes the coset of m_j in $\frac{M}{N}$. So $f(x)\bar{m}(x) = 0$ in the R -module $\frac{M}{N}$. Since $\frac{M}{N}$ is weak Armendariz, $a_i \bar{m}_j \in Nil(\frac{M}{N})$, for every $0 \leq i \leq n$ and $0 \leq j \leq k$. Thus $R(a_i \bar{m}_j) = \frac{(a_i m_j)R + N}{N}$ is a nilpotent submodule of $\frac{M}{N}$. But N is nilpotent in M , so $R(a_i m_j) + N$ is nilpotent in M and also $R(a_i m_j) = (R(a_i m_j) + N) \cap R(a_i m_j)$ is nilpotent in M , by [2, Proposition 4]. This means that $a_i m_j \in Nil(M)$, for each $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$ and therefore M is weak Armendariz. \square

Proposition 7. *Let R be a ring and N a submodule of an R -module M such that $\frac{M}{N}$ is a faithful weak Armendariz R -module. Then M is weak Armendariz.*

Proof. Let $f(x) = \sum_{i=0}^m a_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^n m_j x^j \in M[x] \setminus \{0\}$, with $f(x)m(x) = 0$. Set $\bar{m}(x) = \sum_{j=0}^n \bar{m}_j x^j = \sum_{j=0}^n (m_j + N)x^j$. Hence $\bar{m}(x) \in \frac{M}{N}[x]$ and $f(x)\bar{m}(x) = \bar{0} = N$. By assumption and [2, Proposition 4], $a_i \bar{m}_j \in Nil_R(\frac{M}{N})$ and so $(R(a_i \bar{m}_j) :_R \frac{M}{N})$ is a nilpotent ideal of R , for each $0 \leq i \leq m$ and $0 \leq j \leq n$, since $\frac{M}{N}$ is a faithful R -module. But $(R(a_i m_j) :_R M) \subseteq (R(a_i \bar{m}_j) :_R \frac{M}{N})$. Hence, $(R(a_i m_j) :_R M)$ is a nilpotent ideal of R and so $a_i m_j \in Nil_R(M)$, for each $0 \leq i \leq m$ and $0 \leq j \leq n$. Thus, M is a weak Armendariz R -module. \square

Proposition 8. *Let R be a ring and N a nilpotent submodule of a faithful R -module M such that $N \subseteq Nil(M)$. If M is weak Armendariz, then the R -module $\frac{M}{N}$ is weak Armendariz.*

Proof. Suppose that M is a weak Armendariz R -module. Let $f(x) = \sum_{i=0}^n a_i x^i \in R[x] \setminus \{0\}$ and $\bar{m}(x) = \sum_{j=0}^k \bar{m}_j x^j \in (\frac{M}{N})[x] \setminus \{0\}$ satisfy $f(x)\bar{m}(x) = 0$, where $\bar{m}_j = m_j + N$, for each $j = 0, 1, \dots, k$ and for some $m_j \in M$. Therefore, $a_i m_j \in N \subseteq$

$Nil(M)$ and hence $R(a_i m_j)$ is a nilpotent submodule of M , for every $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$. According to [2, Proposition 4], $R(a_i m_j) + N$ is nilpotent in M and $\frac{R(a_i m_j) + N}{N}$ is nilpotent in $\frac{M}{N}$. On the other hand, $\frac{R(a_i m_j) + N}{N} = R(a_i \overline{m_j})$. So $a_i \overline{m_j} \in Nil(\frac{M}{N})$, for every $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$. Thus, $\frac{M}{N}$ is a weak Armendariz R -module. \square

Proposition 9. *Let R be a ring and $\{M_i\}_{i \in I}$ a family of faithful weak Armendariz R -modules. Then the direct sum $M = \coprod_{i \in I} M_i$ is weak Armendariz.*

Proof. Let $f(x) = \sum_{j=0}^n a_j x^j \in R[x]$ and $m(x) = \sum_{k=0}^t m_k x^k \in M[x] \setminus \{0\}$, with $f(x)m(x) = 0$. We can assume that $m_k = (m_{ik})_{i \in I} \in \prod_{i \in I} M_i$. Set $m_i(x) = \sum_{k=0}^t m_{ik} x^k$. It is clear that $f(x)m_i(x) = 0$, for each $i \in I$. Since each M_i is weak Armendariz, $a_j m_{ik} \in Nil(M_i)$, for every $i \in I$, $j = 0, 1, \dots, n$ and $k = 0, 1, \dots, t$. Hence, $(R(a_j m_{ik}) : M_i)$ is a nilpotent ideal of R , by [2, Proposition 4].

On the other hand, $I_k = \{i \in I; m_{ik} \neq 0\}$ is a finite set and also $R(a_j m_k) \subseteq \prod_{i \in I} R(a_j m_{ik})$, for each $k = 0, 1, \dots, t$. So we have

$$\begin{aligned} (R(a_j m_k) : \prod_{i \in I} M_i) &\subseteq (\prod_{i \in I} R(a_j m_{ik}) : \prod_{i \in I} M_i) \\ &\subseteq (\prod_{i \in I_k} R(a_j m_{ik}) : \prod_{i \in I_k} M_i) \\ &= \bigcap_{i \in I_k} (R(a_j m_{ik}) : M_i) \end{aligned}$$

for each $k = 0, 1, \dots, t$ and $j = 0, \dots, n$. Finite intersection of nilpotent ideals is nilpotent. So $a_j m_k \in Nil(M)$, for every $0 \leq j \leq n$ and $0 \leq k \leq t$. Therefore, M is a weak Armendariz R -module. \square

We recall that if M is an R -module and S a multiplicatively closed subset of R , then $S^{-1}M$ has an $S^{-1}R$ -module structure. Also, M is called S -torsion free if $ms \neq 0$, for every $m \in M$ and $s \in S$.

Lemma 1. *Let R be a ring, S a multiplicatively closed subset of R and M a finitely generated R -module. Let $\frac{m}{s} \in S^{-1}M$, for $m \in M$ and $s \in S$. Then $m \in Nil_R(M)$ if and only if $\frac{m}{s} \in Nil_{S^{-1}R}(S^{-1}M)$.*

Proof. Let $m \in M$ and $k \in \mathbb{N}$. Since M is finitely generated, we have

$$\begin{aligned} S^{-1}((Rm : M)^k Rm) &= S^{-1}((Rm :_R M)^k) S^{-1}(Rm) \\ &= (S^{-1}(Rm :_R M))^k S^{-1}(Rm) \\ &= ((S^{-1}(Rm) :_{S^{-1}R} S^{-1}M)^k S^{-1}(Rm)) \end{aligned}$$

Also, $S^{-1}(Rm) = S^{-1}R(\frac{m}{s})$, for each $s \in S$. So the result is obvious. \square

Proposition 10. *Let R be a ring, S a multiplicatively closed subset of R and M a finitely generated S -torsion free R -module. Then M is weak Armendariz if and only if $S^{-1}M$ is weak Armendariz as an $S^{-1}R$ -module.*

Proof. Let M be a weak Armendariz R -module and $f(x)m(x) = 0$, where $f(x) = \sum_{i=0}^m \alpha_i x^i \in S^{-1}R[x]$ and $m(x) = \sum_{j=0}^n \beta_j x^j \in S^{-1}M[x]$.

We can assume that $\alpha_i = \frac{a_i}{u_i}$ and $\beta_j = \frac{m_j}{v_j}$, for some $a_i \in R$, $m_j \in M$ and $u_i, v_j \in S$. Setting $u = u_1 u_2 \dots u_m$ and $v = v_1 v_2 \dots v_n$, we have

$$\begin{aligned} 0 = f(x)m(x) &= \sum_{i=0}^m \sum_{j=0}^n \alpha_i \beta_j x^{i+j} \\ &= \sum_{i=0}^m \sum_{j=0}^n \frac{a_i m_j}{uv} x^{i+j} \end{aligned}$$

Hence, $\frac{a_i m_j}{uv} = 0$ and so there exists $s_{ij} \in S$ such that $s_{ij}(a_i m_j) = 0$, for each $0 \leq i \leq m$ and $0 \leq j \leq n$. Set $f_1(x) = \sum_{i=0}^m a_i x^i$ and $m_1(x) = \sum_{j=0}^n m_j x^j$. Clearly, $f_1(x) \in R[x] \setminus \{0\}$, $m_1(x) \in M[x] \setminus \{0\}$ and $f_1(x)m_1(x) = 0$. Since M is a weak Armendariz R -module, $a_i m_j \in Nil_R(M)$, for each $0 \leq i \leq m$ and $0 \leq j \leq n$. Now, by Lemma 10, $\frac{(a_i m_j)}{uv} = (a_i m_j)(uv)^{-1} \in Nil_{S^{-1}R}(S^{-1}M)$ and therefore $S^{-1}M$ is weak Armendariz.

Now suppose that $S^{-1}M$ is a weak Armendariz $S^{-1}R$ -module. Let $g(x) = \sum_{i=0}^m b_i x^i \in R[x] \setminus \{0\}$ and $n(x) = \sum_{j=0}^k n_j x^j \in M[x] \setminus \{0\}$ with $g(x)n(x) = 0$. But $g(x) \in S^{-1}R[x] \setminus \{0\}$ and $n(x) \in S^{-1}M[x] \setminus \{0\}$. So $\frac{b_i n_j}{1} = b_i n_j \in Nil_{S^{-1}R}(S^{-1}M)$, for each $0 \leq i \leq m$ and $0 \leq j \leq k$. Hence, $b_i n_j \in Nil_R(M)$, by Lemma 10. Thus, M is a weak Armendariz R -module. \square

We recall that $R[x, x^{-1}]$ denotes the Laurent polynomial ring over R . For an R -module M , let $M[x, x^{-1}] = \{\sum_i k^n m_i x^i; n, k \in \mathbb{Z}, m_i \in M\}$. $M[x, x^{-1}]$ is an $R[x, x^{-1}]$ -module under the addition operation and the following scalar product operation. For $m(x) = \sum_i m_i x^i \in M[x, x^{-1}]$ and $f(x) = \sum_j a_j x^j \in R[x, x^{-1}]$, then $f(x)m(x) = \sum_k (\sum_{i+j=k} a_j m_i) x^k$ (see [4]).

Corollary 5. *Let M be a finitely generated R -module. Then $M[x]$ is a weak Armendariz $R[x]$ -module if and only if $M[x, x^{-1}]$ is a weak Armendariz $R[x, x^{-1}]$ -module.*

Proof. Set $S = \{1, x, x^2, \dots\}$. Then S is a multiplicatively closed subset of $R[x]$. Also $R[x, x^{-1}] = S^{-1}R[x]$ and $M[x, x^{-1}] = S^{-1}M[x]$. Now the result follows from Proposition 10. \square

Corollary 6. *Let R be a ring and S a multiplicatively closed subset of R . Then the localization $S^{-1}R$ is strongly weak Armendariz if and only if R is strongly weak Armendariz.*

For a commutative domain R and R -module M , the torsion submodule of M is defined by $T(M) = \{x \in M \mid \text{ann}_R(x) \neq 0\}$. In this case, $T(M)$ is a submodule of M , called the torsion part of M . An R -module M is called a *torsion* module if $T(M) = M$.

Proposition 11. *Let D be a domain and M a D -module. If the D -module $T(M)$ is weak Armendariz, then M is also weak Armendariz, too.*

Proof. Let $T(M)$ be weak Armendariz and $f(x)m(x) = 0$, where $f(x) = \sum_{i=0}^n a_i x^i \in D[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^n m_j x^j \in M[x]$. Since $f(x) \neq 0$, we may assume that $a_0 \neq 0$. Also we have $a_0 m_0 = a_0 m_1 + a_1 m_0 = a_0 m_2 + a_1 m_1 + a_2 m_0 = \dots = a_k m_n = 0$. Multiplying $a_0 m_1 + a_1 m_0 = 0$ by a_0 implies that $a_0^2 m_1 = 0$. Similarly, multiplying $a_0 m_2 + a_1 m_1 + a_2 m_0 = 0$ by a_0^2 implies that $a_0^3 m_2 = 0$. Continuing this process, we have $a_0^{j+1} m_j = 0$ and so $m_j \in T(M)$, for every $j = 0, 1, \dots, n$. Hence $m(x) \in T(M)[x]$. Since $T(M)$ is weak Armendariz, $a_i m_j \in \text{Nil}(T(M))$, for every $i = 0, 1, \dots, k$ and $j = 0, 1, \dots, n$. So there exists a positive integer n_{ij} , for every $i = 0, 1, \dots, k$ and $j = 0, 1, \dots, n$, such that $(D(a_i m_j) :_D T(M))^{n_{ij}} D(a_i m_j) = 0$. But $(D(a_i m_j) :_D M) \subseteq (D(a_i m_j) :_D T(M))$. Therefore, $(D(a_i m_j) :_D M)^{n_{ij}} D(a_i m_j) = 0$, for each $0 \leq i \leq k$ and $0 \leq j \leq n$. Thus $a_i m_j \in \text{Nil}(M)$ and M is a weak Armendariz D -module. □

Theorem 2. *Let R be an integral domain and Q the quotient field of R . If M is a weak Armendariz Q -module, then M is weak Armendariz as an R -module and the converse is true if M is a finitely generated R -module.*

Proof. First note that if $M = 0$, then the result is trivial. Now assume that $M \neq 0$.
 \Rightarrow : Suppose that M is a weak Armendariz Q -module. Let $f(x) = \sum_{i=0}^n a_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^k m_j x^j$ belongs to an $R[x]$ -module $M[x]$ with $f(x)m(x) = 0$. By assumption, $a_i m_j \in \text{Nil}_Q(M)$, for every $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$. M is a faithful Q -module, since $M \neq 0$. So the ideal $(Q(a_i m_j) :_Q M)$ is a nilpotent ideal of Q , by [2, Proposition 4]. But $(R(a_i m_j) :_R M) \subseteq (Q(a_i m_j) :_Q M)$. Thus $(R(a_i m_j) :_R M)$ is a nilpotent ideal of R and so $a_i m_j \in \text{Nil}_R(M)$, for every $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$, by [2, Proposition 4]. Therefore, M is a weak Armendariz R -module.

\Leftarrow : Now, suppose that M is a weak Armendariz R -module. Let $f(x)m(x) = 0$, where $f(x) = \sum_{i=0}^n \alpha_i x^i \in Q[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^k m_j x^j$ belongs to the $Q[x]$ -module $M[x]$. We may assume that $\alpha_i = s_i^{-1} b_i$, for each $i = 0, 1, \dots, n$ and for some $b_i \in R$ and regular element $s_i \in R$. Set $s = s_0 s_1 \dots s_n$. So we can assume that $\alpha_i = s^{-1} a_i$, for each $i = 0, 1, \dots, n$ and for some $a_i \in R$. Then we have

$$0 = f(x)m(x) = \sum_{i=0}^n \sum_{j=0}^k s^{-1} a_i m_j x^{i+j} = s^{-1} f'(x)m(x)$$

where $f'(x) = \sum_{i=0}^n a_i x^i \in R[x] \setminus \{0\}$. Therefore, $f'(x)m(x) = 0$. Since M is a weak Armendariz R -module, $a_i m_j \in Nil_R(M)$. M is a faithful finitely generated R -module, hence the ideal $(R(a_i m_j) :_R M)$ is a nilpotent ideal of R and also $Q(R(a_i m_j) :_R M) = (Q(a_i m_j) :_Q M)$, for each $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$. So $a_i m_j \in Nil_Q(M)$, for every $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, k$. Thus M is a weak Armendariz Q -module. \square

ACKNOWLEDGMENTS

The authors are grateful to the referee of this paper for his/her careful reading and comments.

REFERENCES

- [1] M. M. Ali and D. J. Smith, "Pure submodules of multiplication modules," *Beiträge Algebra Geom.*, vol. 45, no. 1, pp. 61–74, 2004.
- [2] M. Ali, "Idempotent and nilpotent submodules of multiplication modules," *Comm. Algebra.*, vol. 36, pp. 4620–4642, 2008, doi: [10.1080/00927870802186805](https://doi.org/10.1080/00927870802186805).
- [3] Y. C. Jeon, H. K. Kim, Y. Lee, and J. S. Yoon, "On weak armendariz rings," *Bull. Korean Math. Soc.*, vol. 46, no. 1, pp. 135–146, 2009.
- [4] T.-K. Lee and Y. Zhou, "Reduced modules," in *Rings, modules, algebras, and Abelian groups. Proceedings of the algebra conference – Venezia 2002, Venice, Italy, June 3–8, 2002*. New York, NY: Marcel Dekker, 2004, pp. 365–377.
- [5] M. B. Rege and S. Chhawchharia, "Armendariz rings," *Proc. Japan Acad. Ser. A Math. Sci.*, vol. 73A, pp. 14–17, 1997.
- [6] P. Ribenboim, *Algebraic Numbers*, ser. Pure and applied mathematics. Wiley-Interscience, 1972, vol. 27.
- [7] J. J. Rotman, *An Introduction to Homological Algebra*, 2nd ed. Berlin, Springer, 2009. doi: [10.1007/978-0-387-68324-9](https://doi.org/10.1007/978-0-387-68324-9).

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