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ON THE DEDEKIND SUMS AND ITS NEW RECIPROCITY FORMULA

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Abstract. The main purpose of this paper is using the analytic method and the properties of Dirichlet L-functions to study the computational problem of one kind Dedekind sums, and give a new reciprocity formula for it.

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1. INTRODUCTION

Let q be a natural number and h an integer prime to q. The classical Dedekind sums

$$S(h,q)=\sum_{a=1}^{q}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer,}\end{cases}$$

describes the behaviour of the logarithm of the eta-function (see [6,7]) under modular transformations. Many authors have studied the arithmetical properties of $S(h,q)$ and obtained many interesting results, some of them can be found in [10–12]. For example, Conrey et al [3] studied the mean value distribution of $S(h,k)$, and proved the asymptotic formula

$$\sum_{h=1}^{k}{}'|S(h,k)|^{2m}=f_m(k)\left(\frac{k}{12}\right)^{2m}+O\left(\left(k^{\frac{9}{5}}+k^{2m-1+\frac{1}{m+1}}\right)\cdot\ln^3 k\right), \qquad (1.1)$$

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where $\sum_{h}'$ denotes the summation over all h such that $(k,h)=1$, and

$$\sum_{m=1}^{\infty}\frac{f_m(n)}{n^s}=2\cdot\frac{\zeta^2(2m)}{\zeta(4m)}\cdot\frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\cdot\zeta(s).$$

C. Jia [5] improved the error term in (1) to $O\left(k^{2m-1}\ln^3 k\right)$, if $m\geq 2$.

Walum [9] obtained an identity between the mean square value of $S(h,p)$ and the fourth power mean of Dirichlet L-functions. That is, he proved the following:

$$\sum_{h=1}^{p-1}|S(h,p)|^2=\frac{p^2}{\pi^4(p-1)}\cdot\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^4,$$

where p be an odd prime.

Perhaps the most famous property of Dedekind sums is the reciprocity formula (see references [2,4] and [6]):

$$S(h,k)+S(k,h)=\frac{h^2+k^2+1}{12hk}-\frac{1}{4} \tag{1.2}$$

for all $(h,k)=1$, $h>0$ and $k>0$.

An interesting three term version of (1.2) was also discovered by H. Rademacher and and E. Grosswald [7].

In this paper, as a note of [8], we use the analytic method and the properties of Dirichlet L-functions to study the computational problem of one kind Dedekind sums, and give a new reciprocity formula for $S(k,h)$. That is, we will prove the following:

Theorem 1. *Let h and k are two positive odd numbers with $(k,h)=1$. Then we have the identity*

$$S(2\cdot\overline{k},h)+S(2\cdot\overline{h},k)=\frac{h^2+k^2+4}{24hk}-\frac{1}{4},$$

where integers $\overline{k}$ and $\overline{h}$ satisfying the congruence equation $k\cdot\overline{k}\equiv 1 \bmod h$ and $h\cdot\overline{h}\equiv 1 \bmod k$.

We prove this result by the analytic method and the properties of Dirichlet L-functions, which is distinct from other methods of proving reciprocity formula of Dedekind sums. Whether there exists a direct elementary method to prove this identity is an interesting problem.

2. Preliminaries

To complete the proof of our theorem, we need to prove several lemmas. Hereinafter, we shall use some properties of Dirichlet L-functions, all of these can be found in reference [1], so they will not be repeated here.

Lemma 1. *Let $q > 2$ be an integer, then for any integer a with $(a,q) = 1$, we have the identity*

$$S(a,q) = \frac{1}{\pi^2 q} \sum_{d\,|\,q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a)|L(1,\chi)|^2,$$

where $L(1,\chi)$ denotes the Dirichlet L-function corresponding to the character χ mod *d.*

Proof. See Lemma 2 of [8]. □

Lemma 2. *Let a and q are two positive odd numbers with $(a,q) = 1$. Then we have the identity*

$$S(a,2q) + S(2\cdot a,q) + S(\overline{2}\cdot a,q) = 3S(a,q),$$

where $\overline{2}\cdot 2 \equiv 1 \bmod q$.

Proof. Let χ_2^0 denotes the principal character mod 2. Then for any non-principal character χ mod d with $(2,d) = 1$, note that the identity

$$\begin{aligned}
\left|L\left(1,\chi\chi_2^0\right)\right|^2 &= \left|\prod_p \left(1 - \frac{\chi(p)\chi_2^0(p)}{p}\right)^{-1}\right|^2 = \left|\prod_{p>2} \left(1 - \frac{\chi(p)}{p}\right)^{-1}\right|^2 \\
&= \left|1 - \frac{\chi(2)}{2}\right|^2 \cdot \left|\prod_p \left(1 - \frac{\chi(p)}{p}\right)^{-1}\right|^2 = \left|1 - \frac{\chi(2)}{2}\right|^2 \cdot |L(1,\chi)|^2 \\
&= \left(\frac{5}{4} - \frac{\chi(2)}{2} - \frac{\overline{\chi}(2)}{2}\right) \cdot |L(1,\chi)|^2,
\end{aligned}$$

from Lemma 1 and the properties of Euler function $\phi(n)$ we have

$$\begin{aligned}
S(a,2q) &= \frac{1}{2q\pi^2} \cdot \sum_{d\,|\,2q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a)|L(1,\chi)|^2 \\
&= \frac{1}{2q\pi^2} \cdot \sum_{d\,|\,q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a)|L(1,\chi)|^2 \\
&\quad + \frac{1}{2q\pi^2} \cdot \sum_{d\,|\,q} \frac{4d^2}{\phi(2d)} \sum_{\substack{\chi \bmod 2d \\ \chi(-1)=-1}} \chi(a)|L(1,\chi)|^2 \\
&= \frac{1}{2} \cdot S(a,q) + \frac{2}{\pi^2 q} \cdot \sum_{d\,|\,q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a)\chi_2^0(a)|L(1,\chi\chi_2^0)|^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\cdot S(a,q) \\
&\quad + \frac{2}{\pi^2 q}\cdot\sum_{d\mid q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(a)|L(1,\chi)|^2\cdot\left(\frac{5}{4}-\frac{\chi(2)}{2}-\frac{\overline{\chi}(2)}{2}\right) \\
&= \frac{1}{2}\cdot S(a,q)+\frac{5}{2}\cdot\frac{1}{\pi^2 q}\cdot\sum_{d\mid q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(a)|L(1,\chi)|^2 \\
&\quad -\frac{1}{\pi^2 q}\cdot\sum_{d\mid q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(2a)|L(1,\chi)|^2 \\
&\quad -\frac{1}{\pi^2 q}\cdot\sum_{d\mid q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(\overline{2}a)|L(1,\chi)|^2 \\
&= \frac{1}{2}\cdot S(a,q)+\frac{5}{2}\cdot S(a,q)-S(2\cdot a,q)-S\left(\overline{2}\cdot a,q\right) \\
&= 3S(a,q)-S(2\cdot a,q)-S\left(\overline{2}\cdot a,q\right)
\end{aligned}
$$

or

$$S(a,2q)+S(2\cdot a,q)+S(\overline{2}\cdot a,q)=3S(a,q).$$

This proves Lemma 2. □

3. Proof of Theorem

In this section, we shall complete the proof of our theorem. For any positive odd numbers h and k with $(k,h)=1$, applying Lemma 2 repeatedly we have

$$S(k,2h)+S(2\cdot k,h)+S(\overline{2}\cdot k,h)=3S(k,h) \tag{3.1}$$

and

$$S(h,2k)+S(2\cdot h,k)+S(\overline{2}\cdot h,k)=3S(h,k). \tag{3.2}$$

Adding (3.1) and (3.2), then applying reciprocity formula (1.2) we have

$$
\begin{aligned}
S(k,2h)+S(2h,k)+S(2\cdot k,h)+S(h,2k)+S(\overline{2}\cdot k,h)+S(\overline{2}\cdot h,k)& \\
=3S(k,h)+3S(h,k)&
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{4h^2+k^2+1}{24kh}-\frac{1}{4}+\frac{4k^2+h^2+1}{24kh}-\frac{1}{4}+S(\overline{2}\cdot k,h)+S(\overline{2}\cdot h,k)& \\
=3\left(\frac{h^2+k^2+1}{12hk}-\frac{1}{4}\right)&
\end{aligned}
$$

or

$$S(\overline{2}\cdot k,h)+S(\overline{2}\cdot h,k)=\frac{h^2+k^2+4}{24hk}-\frac{1}{4}. \tag{3.3}$$

Note that if positive integers n and q satisfying $(n,q)=1$, then $S(n,q)=S(\overline{n},q)$, where $\overline{n}$ satisfying the congruence equation $n\cdot\overline{n}\equiv 1 \bmod q$.

Combining this property and (3.3) we may immediately deduce the identity

$$S(2\cdot\overline{k},h)+S(2\cdot\overline{h},k)=\frac{h^2+k^2+4}{24hk}-\frac{1}{4}.$$

This completes the proof of our theorem.

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