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SHIFTED EULER-SEIDEL MATRICES

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Abstract. In this study defining Shifted Euler-Seidel matrices we generalize the Euler-Seidel matrices method. Owing to this generalization one can investigate any sequences (s_n) which have two term linear recurrences as $s_{m+n} = \alpha s_{m+n-1} + \beta s_{n-1}$ (α and β are real parameters and $n,m \in \mathbb{Z}^+$). By way of illustration, we give some examples related to the Fibonacci p-numbers.

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1. INTRODUCTION

Let (a_n) be a sequence. The Euler-Seidel matrix associated with this sequence is determined recursively by the formula (see [6]):

$$\begin{aligned} a_n^0 &= a_n \quad (n \geq 0) \\ a_n^k &= a_n^{k-1} + a_{n+1}^{k-1} \qquad (n \geq 0,\ k \geq 1)\,. \end{aligned} \tag{1.1}$$

From relation (1.1), it can be seen that the first row and the first column can be transformed into each other via the well known binomial inverse pair as,

$$a_0^n = \sum_{k=0}^{n} \binom{n}{k} a_k^0, \tag{1.2}$$

$$a_n^0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_0^k. \tag{1.3}$$

Also any entry a_n^k can be written in terms of the initial sequence as:

$$a_n^k = \sum_{i=0}^{k} \binom{k}{i} a_{n+i}^0. \tag{1.4}$$

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Proposition 1 (Euler)**.** *Let*

$$a(t) = \sum_{n=0}^{\infty} a_n^0 t^n$$

be the generating function of the initial sequence (a_n^0)*. Then the generating function of the sequence* (a_0^n) *is*

$$\overline{a}(t) = \sum_{n=0}^{\infty} a_0^n t^n = \frac{1}{1-t} a\left(\frac{t}{1-t}\right). \tag{1.5}$$

Proposition 2 (Seidel)**.** *Let*

$$A(t) = \sum_{n=0}^{\infty} a_n^0 \frac{t^n}{n!}$$

be the exponential generating function of the initial sequence (a_n^0)*. Then the exponential generating function of the sequence* (a_0^n) *is*

$$\overline{A}(t) = \sum_{n=0}^{\infty} a_0^n \frac{t^n}{n!} = e^t A(t). \tag{1.6}$$

The Euler-Seidel matrices are useful and rather elementary technique to investigate properties of some special numbers and polynomials. In [4] one can see applications related to the Bernoulli and Euler polynomials. Using the Euler-Seidel matrix authors obtained some properties of the geometric and exponential polynomials and numbers in [3]. Mező and Dil in [12] use the Euler-Seidel method for deriving new identities for the hyperharmonic and r-Stirling numbers. In [7] present authors obtained identities for the generalized second order recurrence relation by using the generalized Euler-Seidel matrix. Barry and Hennessy [1] studied the Euler-Seidel matrix of certain integer sequences, using the binomial transform and the Hankel matrices. For moment sequences, they gave an integral representation of the Euler-Seidel matrix. Chen [2] investigated the summation form of Bernoulli numbers which can form an Euler-Seidel matrix. The upper diagonal elements of this Euler-Seidel matrix are called “the median Bernoulli numbers”. Chen determined the prime divisors of their numerators and denominators also obtained their ordinary generating function. Tutaş [15] defined the period of a Euler–Seidel matrix over a field $\mathbb{F}_p$ with p elements, where p is a prime number and gave applications on the generalized Franel numbers.

There are also similar matrices related to the Euler-Seidel matrices. One of them is the symmetric infinite matrix which is defined by Dil and Mező in [5]. They established this matrix especially to investigate properties of the hyperharmonic numbers. In [8] authors gave some identities for the Fibonacci and incomplete Fibonacci p-numbers via the symmetric matrix method.

The Fibonacci p-numbers had beed discovered by Stakhov while investigating "diagonal sums" of the Pascal triangle (see [13]). In [14] the Fibonacci p-numbers $F_p(n)$ are defined by the following recurrence relation for $n > p$

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \tag{1.7}$$

with initial conditions

$$F_p(0) = 0,\ F_p(n) = 1 \quad (n = 1, 2, \ldots, p),$$

and the Lucas p-numbers $L_p(n)$ are defined by the following recurrence relation for $n > p$

$$L_p(n) = L_p(n-1) + L_p(n-p-1) \tag{1.8}$$

with initial conditions

$$L_p(0) = p+1,\ L_p(n) = 1 \quad (n = 1, 2, \ldots, p).$$

Note that for $p = 1$ the Fibonacci and Lucas p-numbers are reduced to the well-known Fibonacci and Lucas sequences $\{F_n\}$, $\{Ln\}$, respectively.

In [10] the Pell p-numbers $P_p(n)$ are defined by the following recurrence relations for $n > p$

$$P_p(n) = 2P_p(n-1) + P_p(n-p-1) \tag{1.9}$$

with initial conditions

$$P_p(0) = 0,\ P_p(n) = 2^{n-1} \quad (n = 1, 2, \ldots p)$$

and the Pell-Lucas p-numbers $Q_p(n)$ are defined by the following recurrence relations for $n > p$

$$Q_p(n) = 2Q_p(n-1) + Q_p(n-p-1) \tag{1.10}$$

with initial conditions

$$Q_p(0) = p+1,\ Q_p(n) = 2^n\ (n = 1, 2, 3, \ldots, p).$$

Note that for $p = 1$ the Pell and Pell-Lucas p-numbers are reduced to the well-known Pell and Pell-Lucas sequences $\{P_n\}$, $\{Q_n\}$, respectively.

The generalized bivariate Fibonacci p-polynomials $F_{p,n}(a,b)$ and the generalized bivariate Lucas p-polynomials $L_{p,n}(a,b)$ are defined (see [16]) the recursion for $p \geq 1$

$$F_{p,n}(a,b) = aF_{p,n-1}(a,b) + bF_{p,n-p-1}(a,b); \quad n > p \tag{1.11}$$

with

$$F_{p,0}(a,b) = 0,\ F_{p,n}(a,b) = a^{n-1}\ for\ n = 1,\ 2,\ \ldots p$$

and

$$L_{p,n}(a,b) = aL_{p,n-1}(a,b) + bL_{p,n-p-1}(a,b); \quad n > p \tag{1.12}$$

with

$$L_{p,0}(a,b) = p+1,\ L_{p,n}(a,b) = a^n\ for\ n = 1,\ 2,\ \ldots p$$

When $a = b = 1$, $F_{p,n}(a,b) = F_p(n)$ and $L_{p,n}(a,b) = L_p(n)$.

For $p \geq 1, n \geq 1$, the incomplete bivariate Fibonacci and Lucas p-polynomials are defined as

$$F_{p,n}^{k}(a,b) = \sum_{j=0}^{k} \binom{n-jp-1}{j} a^{n-j(p+1)-1} b^{j}; \quad 0 \leq k \leq \left\lfloor \frac{n-1}{p+1} \right\rfloor. \tag{1.13}$$

and

$$L_{p,n}^{k}(a,b) = \sum_{j=0}^{k} \frac{n}{n-jp} \binom{n-jp}{j} a^{n-j(p+1)} b^{j}; \quad 0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor. \tag{1.14}$$

Moreover the following properties of the incomplete bivariate Fibonacci and Lucas p-polynomials are given in [8] as

$$\sum_{j=0}^{h} \binom{h}{j} b^{h-j} a^{j} F_{p,n+p(j-1)}^{k+j}(a,b) = F_{p,n+(p+1)h-p}^{k+h}(a,b); \; 0 \leq k \leq \frac{n-h-p-1}{p+1} \tag{1.15}$$

and

$$\sum_{j=0}^{h} \binom{h}{j} b^{h-j} a^{j} L_{p,n+p(j-1)}^{k+j}(a,b) = L_{p,n+(p+1)h-p}^{k+h}(a,b); \; 0 \leq k \leq \frac{n-p-h}{p+1}. \tag{1.16}$$

2. Shifted Euler-Seidel matrices with two parameters

So far we have mentioned about the Euler-Seidel matrix and its applications. But this method is useful for only linear sequences which have a recurrence related to two consecutive terms. Here we generalize this method to the sequence having linear recurrence with an arbitrary "gap". Owing to this generalization one can investigate for example the Fibonacci p-numbers or get informations about such subsequences indexed by multiples of some natural numbers

$$\sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^{r} F_{rm}.$$

Let us consider a given sequence $(a_n)_{n\geq 0}$. We define "Shifted Euler-Seidel matrix with two parameters" corresponding to this sequence recursively by the formulae

$$a_n^0 = a_n \quad (n \geq 0), \tag{2.1}$$
$${}_m a_n^k = x_m a_n^{k-1} + y_m a_{n+m}^{k-1} \quad (n \geq 0, k \geq 1 \text{ and } m \text{ is a fixed positive integer})$$

where ${}_m a_n^k$ represents the kth row and nth column entry (here the left below index m shows that all entries depend on m) and x, y are real paramaters; i.e;

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x_m a_n^{k-1} & \therefore\!\!\rightarrow & y_m a_{n+m}^{k-1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & {}_m a_n^k & \swarrow & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

The following proposition gives the relation between an arbitrary entry of the matrix and the initial sequence.

Proposition 3. *We have*

$$_m a_n^k = \sum_{i=0}^{k} \binom{k}{i} x^{k-i} y^i a_{n+im}^0 \tag{2.2}$$

where m is a fixed positive integer.

Proof. By induction on $n+k$. $\square$

Corollary 1.

$$_m a_0^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i a_{im}^0 \tag{2.3}$$

and

$$a_{nm}^0 = \frac{1}{y^n} \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} \ _m a_0^i. \tag{2.4}$$

2.1. *Generating Functions*

In this subsection we give connections between the generating functions of the initial sequences and the first column entries of the Shifted Euler-Seidel matrix with two parameters.

2.1.1. *Ordinary generating functions*

Proposition 4. *For the real parameters x and y the following relation holds:*

$$\overline{_m a_{x,y}}(t) = \frac{1}{1-xt} \ {}_{\pi m} a_{x,y}\left(\frac{yt}{1-xt}\right) \tag{2.5}$$

where

$$\overline{_m a_{x,y}}(t) = \sum_{n=0}^{\infty} {}_m a_0^n t^n \quad \textit{and} \quad {}_{\pi m} a_{x,y}(t) = \sum_{n=0}^{\infty} a_{nm}^0 t^n.$$

Proof. Considering (2.3) we write

$$\overline{{}_{m}a_{x,y}}(t)=\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^{r}a_{rm}^{0}\right)t^{n}.$$

By changing the order of the above sums we get

$$\overline{{}_{m}a_{x,y}}(t)=\sum_{r=0}^{\infty}(yt)^{r}a_{rm}^{0}\sum_{n=0}^{\infty}\binom{n+r}{r}(xt)^{n}$$
$$=\frac{1}{1-xt}\sum_{r=0}^{\infty}a_{rm}^{0}\left(\frac{yt}{1-xt}\right)^{r}.$$

In the last step we used the Newton binomial formula. □

Applications

If we take $a_n^0=F_n$ in (2.5), then we have

$$\pi_{m}a_{x,y}(t)=\sum_{n=0}^{\infty}F_{nm}t^{n}.$$

By using the following well known identity (see [11, p.230])

$$\sum_{n=0}^{\infty}F_{nm}t^{n}=\frac{F_{m}t}{1-L_{m}t+(-1)^{m}t^{2}},$$

we have

$$\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^{r}F_{rm}\right)t^{n}=\frac{yF_{m}t}{(1-xt)^{2}-yt(1-xt)L_{m}+(-1)^{m}(yt)^{2}}$$

which is the ordinary generating function of the sequence

$$\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^{r}F_{rm}.$$

Remark 1. Special case of this relation for $x=y=1$ obtained by Hoggatt in [9].

Similarly, using Binet's formula we obtain generating function for every m-th terms of the Lucas numbers as:

$$\sum_{n=0}^{\infty}L_{nm}t^{n}=\frac{L_{m}t}{1-L_{m}t+(-1)^{m}t^{2}}. \tag{2.6}$$

Considering this generating function and setting $a_n^0=L_n$ in (2.5) we get

$$\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^{i}L_{im}\right)t^{n}=\frac{yL_{m}t}{(1-xt)^{2}-yt(1-xt)L_{m}+(-1)^{m}(yt)^{2}}.$$

2.1.2. *Exponential generating functions*

The following proposition also provides the conection between the exponential generating functions of the initial sequence and the first column entries of the Shifted Euler-Seidel matrix with two parameters.

Proposition 5. *The following relation holds:*

$$\overline{{}_{m}A}_{x,y}(t) = e^{xt}\ {}_{m\pi}A(yt) \tag{2.7}$$

where

$$\overline{{}_{m}A}_{x,y}(t) = \sum_{n=0}^{\infty} {}_{m}a_{0}^{n}\frac{t^n}{n!} \quad \textit{and} \quad {}_{m\pi}A_{x,y}(t) = \sum_{n=0}^{\infty} a_{nm}^{0}\frac{t^n}{n!}.$$

Proof. Using (2.3) we get

$$\begin{aligned}\overline{{}_{m}A}_{x,y}(t) &= \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^{r}a_{rm}^{0}\right)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{a_{rm}^{0}x^{n-r}y^{r}}{(n-r)!r!}t^{n}.\end{aligned}$$

Hence we have

$$\overline{{}_{m}A}_{x,y}(t) = \left(\sum_{n=0}^{\infty}a_{rn}^{0}\frac{(yt)^n}{n!}\right)\times\left(\sum_{n=0}^{\infty}\frac{(xt)^n}{n!}\right)$$

and this completes the proof. □

The above results on generating functions enable us to transfer informations of the sequence $(a_n)_n$ to the subsequence $(a_{nm})_n$, and vice versa.

Applications

Now we apply this method to the Fibonacci numbers by setting $a_n^0 = F_n$. Then considering (see [11])

$$\sum_{n=0}^{\infty}F_{nm}\frac{t^n}{n!} = \frac{e^{\alpha^m t}-e^{\beta^m t}}{\alpha-\beta}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, we get immediately that

$${}_{m\pi}A_{x,y}(t) = \frac{e^{\alpha^m t}-e^{\beta^m t}}{\alpha-\beta}.$$

This equation together with (2.7) gives

$$\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^{r}F_{rm}\right)\frac{t^n}{n!} = \frac{e^{(\alpha^m y+x)t}-e^{(\beta^m y+x)t}}{\alpha-\beta}.$$

Hence we obtain the exponential generating function of the sequence

$$\sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^{r} F_{rm}.$$

With a similar approach we get

$$\sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^{r} L_{rm} \right) \frac{t^n}{n!} = e^{(\alpha^m y+x)t} + e^{(\beta^m y+x)t}.$$

Remark 2. Considering other special sequences as Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers and applying the Shifted Euler-Seidel matrices one can obtain similar results.

2.2. *Applications of the Generalized Bivariate Fibonacci and Lucas p-polynomials*

In this subsection, we obtain some results on the generalized bivariate Fibonacci and Lucas p-polinomials using the Shifted Euler-Seidel matrices with two parameters.

Proposition 6. *For the integers $n \geq 0$, $k \geq 0$ and $p \geq 1$ we have*

$$F_{p,n+k(p+1)}(a,b) = \sum_{i=0}^{k} \binom{k}{i} b^{k-i} a^{i} F_{p,n+ip}(a,b). \tag{2.8}$$

Proof. By setting $x = b$ and $y = a$ in (2.1), we get

$${}_m a_n^k = b\left({}_m a_n^{k-1}\right) + a\left({}_m a_{n+m}^{k-1}\right). \tag{2.9}$$

Let us consider the initial sequences $a_n^0 = F_{p,n}(a,b)$, $n \geq 0$. In view of equations (1.11) and (2.9) by induction on k we obtain:

$$a_n^k = F_{p,n+k(p+1)}(a,b).$$

Using (2.2), we have

$$a_n^k = \sum_{i=0}^{k} \binom{k}{i} b^{k-i} a^{i} F_{p,n+ip}(a,b).$$

Combining these results we get

$$F_{p,n+k(p+1)}(a,b) = \sum_{i=0}^{k} \binom{k}{i} b^{k-i} a^{i} F_{p,n+ip}(a,b).$$

□

We may use (2.3) and (2.4) to conclude that:

Corollary 2.

$$F_{p,n(p+1)}(a,b) = \sum_{i=0}^{n} \binom{n}{i} b^{n-i} a^{i} F_{p,ip}(a,b) \tag{2.10}$$

and

$$F_{p,np}(a,b) = \frac{1}{a^{n}} \sum_{i=0}^{n} \binom{n}{i} (-b)^{n-i} F_{p,i(p+1)}(a,b). \tag{2.11}$$

Similar results for the generalized bivariate Lucas p-polynomials can be given as follows.

Corollary 3.

$$L_{p,n+k(p+1)}(a,b) = \sum_{i=0}^{k} \binom{k}{i} b^{k-i} a^{i} L_{p,n+ip}(a,b), \tag{2.12}$$

$$L_{p,n(p+1)}(a,b) = \sum_{i=0}^{n} \binom{n}{i} b^{n-i} a^{i} L_{p,ip}(a,b) \tag{2.13}$$

and

$$L_{p,np}(a,b) = \frac{1}{a^{n}} \sum_{i=0}^{n} \binom{n}{i} (-b)^{n-i} L_{p,i(p+1)}(a,b). \tag{2.14}$$

2.3. *Applications for the incomplete bivariate Fibonacci and Lucas p-polynomials*

Proposition 7. *For $0 \le k \le \frac{t-p-n-1}{p+1}$ we have*

$$F_{p,t+p(n-1)}^{k+n}(a,b) = \frac{1}{a^{n}} \sum_{i=0}^{n} \binom{n}{i} (-b)^{n-i} F_{p,t+(p+1)i-p}^{k+i}(a,b).$$

Proof. Let choose $a_{np}^{0} = F_{p,t+p(n-1)}^{k+n}(a,b)$ in (2.9). By (2.3) we have

$$a_{0}^{n} = \sum_{i=0}^{n} \binom{n}{i} b^{n-i} a^{i} F_{p,t+p(i-1)}^{k+i}(a,b).$$

From (1.15) we write

$$a_{0}^{n} = F_{p,t+(p+1)n-p}^{k+n}(a,b).$$

Therefore we obtain the dual formula of (1.15) using the equation (2.4) as:

$$F_{p,t+p(n-1)}^{k+n}(a,b) = \frac{1}{a^{n}} \sum_{i=0}^{n} \binom{n}{i} (-b)^{n-i} F_{p,t+(p+1)i-p}^{k+i}(a,b). \tag{2.15}$$

□

Remark 3. We can write similar result for the Lucas p-polynomials as:

$$L_{p,t+p(n-1)}^{k+n}(a,b)=\frac{1}{a^n}\sum_{i=0}^{n}\binom{n}{i}(-b)^{n-i}L_{p,t+(p+1)i-p}^{k+i}(a,b). \tag{2.16}$$

Example 1. In (2.15) and (2.16) by taking $a=2$ and $b=1$ we obtain the following properties of the incomplete Pell and Pell-Lucas p-numbers

$$P_p^{k+n}(t+p(n-1))=\frac{1}{2^n}\sum_{j=0}^{n}\tbinom{n}{j}(-1)^{n-j}P_p^{k+j}(t+(p+1)j-p)$$

and

$$Q_p^{k+n}(t+p(n-1))=\frac{1}{2^n}\sum_{j=0}^{n}\tbinom{n}{j}(-1)^{n-j}Q_p^{k+j}(t+(p+1)j-p),$$

respectively.

3. Tables

In this section we summarize similar results that we obtained in the previous sections.

a	b	p	$F_{p,n}(a,b)$
a	b	1	Bivariate Fibonacci polynomials $F_n(a,b)$
a	1	p	Fibonacci p-polynomials $F_{p,n}(a)$
a	1	1	Fibonacci polynomials $f_n(a)$
1	1	p	Fibonacci p-numbers $F_p(n)$
1	1	1	Fibonacci numbers F_n
$2a$	b	p	Bivariate Pell p-polynomials $F_{p,n}(2a,b)$
$2a$	b	1	Bivariate Pell polynomials $F_n(2a,b)$
$2a$	1	p	Pell p-polynomials $P_{p,n}(a)$
$2a$	1	1	Pell polynomials $P_n(a)$
2	1	1	Pell numbers P_n
$2a$	-1	1	Second kind Chebyshev polynomials $U_{n-1}(a)$
a	$2b$	p	Bivariate Jacobsthal p-polynomials $F_{p,n}(a,2b)$
a	$2b$	1	Bivariate Jacobsthal polynomials $F_n(a,2b)$
1	$2b$	1	Jacobsthal polynomials $J_n(b)$
1	2	1	Jacobsthal numbers J_n

Sequence	For $F_{p,n+k(p+1)}(a,b)=\sum_{i=0}^{k}\binom{k}{i}b^{k-i}a^{i}F_{p,n+ip}(a,b)$, particular cases are
$F_n(a,b)$	$F_{n+2k}(a,b)=\sum\limits_{i=0}^{k}\binom{k}{i}b^{k-i}a^{i}F_{n+i}(a,b)$
$F_{p,n}(a)$	$F_{p,n+k(p+1)}(a)=\sum\limits_{i=0}^{k}\binom{k}{i}a^{i}F_{p,n+ip}(a)$
$f_n(a)$	$f_{n+2k}(a)=\sum\limits_{i=0}^{k}\binom{k}{i}a^{i}f_{n+i}(a);\quad F_{n+2k}=\sum\limits_{i=0}^{k}\binom{k}{i}F_{n+i}$
$F_p(n)$	$F_p(n+k(p+1))=\sum\limits_{i=0}^{k}\binom{k}{i}F_p(n+ip)$
$F_{p,n}(2a,b)$	$F_{p,n+k(p+1)}(2a,b)=\sum\limits_{i=0}^{k}\binom{k}{i}b^{k-i}(2a)^{i}F_{p,n+ip}(2a,b)$
$F_n(2a,b)$	$F_{n+2k}(2a,b)=\sum\limits_{i=0}^{k}\binom{k}{i}b^{k-i}(2a)^{i}F_{n+i}(2a,b)$
$P_{p,n}(a)$	$P_{p,n+k(p+1)}(a)=\sum\limits_{i=0}^{k}\binom{k}{i}(2a)^{i}P_{p,n+ip}(a)$
$P_n(a)$	$P_{n+2k}(a)=\sum\limits_{i=0}^{k}\binom{k}{i}(2a)^{i}P_{n+i}(a);\quad P_{n+2k}=\sum\limits_{i=0}^{k}\binom{k}{i}2^{i}P_{n+i}$
$U_{n-1}(a)$	$U_{n+2k-1}(a)=\sum\limits_{i=0}^{k}\binom{k}{i}(-1)^{k-i}(2a)^{i}U_{n+i-1}(a)$
$F_{p,n}(a,2b)$	$F_{p,n+k(p+1)}(a,2b)=\sum\limits_{i=0}^{k}\binom{k}{i}(2b)^{k-i}a^{i}F_{p,n+ip}(a,2b)$
$F_n(a,2b)$	$F_{n+2k}(a,2b)=\sum\limits_{i=0}^{k}\binom{k}{i}(2b)^{k-i}a^{i}F_{n+i}(a,2b)$
$J_n(b)$	$J_{n+2k}(b)=\sum\limits_{i=0}^{k}\binom{k}{i}(2b)^{k-i}J_{n+i}(b);\quad J_{n+2k}=\sum\limits_{i=0}^{k}\binom{k}{i}2^{k-i}J_{n+i}$

Sequence	For $F_{p,n(p+1)}(a,b)=\sum_{i=0}^{n}\binom{n}{i}b^{n-i}a^{i}F_{p,ip}(a,b)$, particular cases are
$F_n(a,b)$	$F_{2n}(a,b)=\sum\limits_{i=0}^{n}\binom{n}{i}b^{n-i}a^{i}F_{i}(a,b)$
$F_{p,n}(a)$	$F_{p,n(p+1)}(a)=\sum\limits_{i=0}^{n}\binom{n}{i}a^{i}F_{p,ip}(a,b)$
$f_n(a)$	$f_{2n}(a)=\sum\limits_{i=0}^{n}\binom{n}{i}a^{i}f_{i}(a);\quad F_{2n}=\sum\limits_{i=0}^{n}\binom{n}{i}F_{i}$
$F_p(n)$	$F_p(n(p+1))=\sum\limits_{i=0}^{n}\binom{n}{i}F_p(ip)$
$F_{p,n}(2a,b)$	$F_{p,n(p+1)}(2a,b)=\sum\limits_{i=0}^{n}\binom{n}{i}b^{n-i}(2a)^{i}F_{p,ip}(2a,b)$
$F_n(2a,b)$	$F_{2n}(a,b)=\sum\limits_{i=0}^{n}\binom{n}{i}b^{n-i}(2a)^{i}F_{i}(2a,b)$
$P_{p,n}(a)$	$P_{p,n(p+1)}(a)=\sum\limits_{i=0}^{n}\binom{n}{i}(2a)^{i}P_{p,ip}(a)$
$P_n(a)$	$P_{2n}(a)=\sum\limits_{i=0}^{n}\binom{n}{i}(2a)^{i}P_{i}(a);\quad P_{2n}=\sum\limits_{i=0}^{n}\binom{n}{i}2^{i}P_{i}$
$U_{n-1}(a)$	$U_{2n-1}(a)=\sum\limits_{i=0}^{n}\binom{n}{i}(-1)^{n-i}(2a)^{i}U_{i-1}(a)$
$F_{p,n}(a,2b)$	$F_{p,n(p+1)}(a,2b)=\sum\limits_{i=0}^{n}\binom{n}{i}(2b)^{n-i}a^{i}F_{p,ip}(a,2b)$
$F_n(a,2b)$	$F_{2n}(a,2b)=\sum\limits_{i=0}^{n}\binom{n}{i}(2b)^{n-i}a^{i}F_{i}(a,2b)$
$J_n(b)$	$J_{2n}(b)=\sum\limits_{i=0}^{n}\binom{n}{i}(2b)^{n-i}J_{i}(b);\quad J_{2n}=\sum\limits_{i=0}^{n}\binom{n}{i}2^{n-i}J_{i}$

Sequence	For $F_{p,np}(a,b)=\frac{1}{a^n}\sum_{i=0}^{n}\binom{n}{i}(-b)^{n-i}F_{p,i(p+1)}(a,b)$, particular cases are
$F_n(a,b)$	$F_n(a,b)=\frac{1}{a^n}\sum\limits_{i=0}^{n}\binom{n}{i}(-b)^{n-i}F_{2i}(a,b)$
$F_{p,n}(a)$	$F_{p,np}(a)=\frac{1}{a^n}\sum\limits_{i=0}^{n}\binom{n}{i}(-1)^{n-i}F_{p,i(p+1)}(a)$
$f_n(a)$	$f_n(a)=\frac{1}{a^n}\sum\limits_{i=0}^{n}\binom{n}{i}(-1)^{n-i}f_{2i}(a)$; $F_n=\sum\limits_{i=0}^{n}\binom{n}{i}(-1)^{n-i}F_{2i}$
$F_p(n)$	$F_p(np)=\sum\limits_{i=0}^{n}\binom{n}{i}(-1)^{n-i}F_p(i(p+1))$
$F_{p,n}(2a,b)$	$F_{p,np}(2a,b)=\frac{1}{(2a)^n}\sum\limits_{i=0}^{n}\binom{n}{i}(-b)^{n-i}F_{p,i(p+1)}(2a,b)$
$F_n(2a,b)$	$F_n(2a,b)=\frac{1}{(2a)^n}\sum\limits_{i=0}^{n}\binom{n}{i}(-b)^{n-i}F_{2i}(2a,b)$
$P_{p,n}(a)$	$P_{p,np}(a)=\frac{1}{(2a)^n}\sum\limits_{i=0}^{n}\binom{n}{i}(-1)^{n-i}P_{p,i(p+1)}(a)$
$P_n(a)$	$P_n(a)=\frac{1}{(2a)^n}\sum\limits_{i=0}^{n}\binom{n}{i}(-1)^{n-i}P_{2i}(a)$; $P_n=\frac{1}{2^n}\sum\limits_{i=0}^{n}\binom{n}{i}(-1)^{n-i}P_{2i}$
$U_{n-1}(a)$	$U_{n-1}(a)=\frac{1}{(2a)^n}\sum\limits_{i=0}^{n}\binom{n}{i}U_{2i}(a)$
$F_{p,n}(a,2b)$	$F_{p,np}(a,2b)=\frac{1}{a^n}\sum\limits_{i=0}^{n}\binom{n}{i}(-2b)^{n-i}F_{p,i(p+1)}(a,2b)$
$F_n(a,2b)$	$F_n(a,2b)=\frac{1}{a^n}\sum\limits_{i=0}^{n}\binom{n}{i}(-2b)^{n-i}F_{2i}(a,2b)$
$J_n(b)$	$J_n(b)=\sum\limits_{i=0}^{n}\binom{n}{i}(-2b)^{n-i}J_{2i}(b)$; $J_n=\sum\limits_{i=0}^{n}\binom{n}{i}(-2)^{n-i}J_{2i}$

a	b	p	$L_{p,n}(a,b)$
a	b	1	Bivariate Lucas polynomials $L_n(a,b)$
a	1	p	Lucas p-polynomials $L_{p,n}(a)$
a	1	1	Lucas polynomials $\ell_n(a)$
1	1	p	Lucas p-numbers $L_p(n)$
1	1	1	Lucas numbers L_n
$2a$	b	p	Bivariate Pell-Lucas p-polynomials $L_{p,n}(2a,b)$
$2a$	b	1	Bivariate Pell-Lucas polynomials $L_n(2a,b)$
$2a$	1	p	Pell-Lucas p-polynomials $Q_{p,n}(a)$
$2a$	1	1	Pell-Lucas polynomials $Q_n(a)$
2	1	1	Pell-Lucas numbers Q_n
$2a$	-1	1	First kind Chebyshev polynomials $T_n(a)$
a	$2b$	p	Bivariate Jacobsthal-Lucas p-polynomials $L_{p,n}(a,2b)$
a	$2b$	1	Bivariate Jacobsthal-Lucas polynomials $L_n(a,2b)$
1	$2b$	1	Jacobsthal polynomials $j_n(b)$
1	2	1	Jacobsthal numbers j_n

Sequence	For $L_{p,n+k(p+1)}(a,b)=\sum_{i=0}^{k}\binom{k}{i}b^{k-i}a^{i}L_{p,n+ip}(a,b)$, particular cases are
$L_n(a,b)$	$L_{n+2k}(a,b)=\sum_{i=0}^{k}\binom{k}{i}b^{k-i}a^{i}L_{n+i}(a,b)$
$L_{p,n}(a)$	$L_{p,n+k(p+1)}(a)=\sum_{i=0}^{k}\binom{k}{i}a^{i}L_{p,n+ip}(a)$
$\ell_n(a)$	$\ell_{n+2k}(a)=\sum_{i=0}^{k}\binom{k}{i}a^{i}\ell_{n+i}(a)$; $L_{n+2k}=\sum_{i=0}^{k}\binom{k}{i}L_{n+i}$
$L_p(n)$	$L_p(n+k(p+1))=\sum_{i=0}^{k}\binom{k}{i}L_p(n+ip)$
$L_{p,n}(2a,b)$	$L_{p,n+k(p+1)}(2a,b)=\sum_{i=0}^{k}\binom{k}{i}b^{k-i}(2a)^{i}L_{p,n+ip}(2a,b)$
$L_n(2a,b)$	$L_{n+2k}(2a,b)=\sum_{i=0}^{k}\binom{k}{i}b^{k-i}(2a)^{i}L_{n+i}(2a,b)$
$Q_{p,n}(a)$	$Q_{p,n+k(p+1)}(a)=\sum_{i=0}^{k}\binom{k}{i}(2a)^{i}Q_{p,n+ip}(a)$
$Q_n(a)$	$Q_{n+2k}(a)=\sum_{i=0}^{k}\binom{k}{i}(2a)^{i}Q_{n+i}(a)$; $Q_{n+2k}=\sum_{i=0}^{k}\binom{k}{i}2^{i}Q_{n+i}$
$T_n(a)$	$T_{n+2k}(a)=\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}(2a)^{i}T_{n+i}(a)$
$L_{p,n}(a,2b)$	$L_{p,n+k(p+1)}(a,2b)=\sum_{i=0}^{k}\binom{k}{i}(2b)^{k-i}a^{i}L_{p,n+ip}(a,2b)$
$L_n(a,2b)$	$L_{n+2k}(a,2b)=\sum_{i=0}^{k}\binom{k}{i}(2b)^{k-i}a^{i}L_{n+i}(a,2b)$
$j_n(b)$	$j_{n+2k}(b)=\sum_{i=0}^{k}\binom{k}{i}(2b)^{k-i}j_{n+i}(b)$; $j_{n+2k}=\sum_{i=0}^{k}\binom{k}{i}2^{k-i}j_{n+i}$

Sequence	For $L_{p,n(p+1)}(a,b)=\sum_{i=0}^{n}\binom{n}{i}b^{n-i}a^{i}L_{p,ip}(a,b)$, particular cases are
$L_n(a,b)$	$L_{2n}(a,b)=\sum_{i=0}^{n}\binom{n}{i}b^{n-i}a^{i}L_i(a,b)$
$L_{p,n}(a)$	$L_{p,n(p+1)}(a)=\sum_{i=0}^{n}\binom{n}{i}a^{i}L_{p,ip}(a)$
$\ell_n(a)$	$\ell_{2n}(a)=\sum_{i=0}^{k}\binom{k}{i}a^{i}\ell_i(a);\;\; L_{2n}=\sum_{i=0}^{n}\binom{n}{i}L_i$
$L_p(n)$	$L_p(n(p+1))=\sum_{i=0}^{k}\binom{k}{i}L_p(ip)$
$L_{p,n}(2a,b)$	$L_{p,n(p+1)}(2a,b)=\sum_{i=0}^{n}\binom{n}{i}b^{n-i}(2a)^{i}L_{p,ip}(2a,b)$
$L_n(2a,b)$	$L_{2n}(2a,b)=\sum_{i=0}^{n}\binom{n}{i}b^{n-i}(2a)^{i}L_i(2a,b)$
$Q_{p,n}(a)$	$Q_{p,n(p+1)}(a)=\sum_{i=0}^{n}\binom{n}{i}(2a)^{i}Q_{p,ip}(a)$
$Q_n(a)$	$Q_{2n}(a)=\sum_{i=0}^{n}\binom{n}{i}(2a)^{i}Q_i(a);\;\; Q_{2n}=\sum_{i=0}^{n}\binom{n}{i}2^{i}Q_i$
$T_n(a)$	$T_{2n}(a)=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}(2a)^{i}T_i(a)$
$L_{p,n}(a,2b)$	$L_{p,n(p+1)}(a,2b)=\sum_{i=0}^{n}\binom{n}{i}(2b)^{n-i}a^{i}L_{p,ip}(a,2b)$
$L_n(a,2b)$	$L_{2n}(a,2b)=\sum_{i=0}^{n}\binom{n}{i}(2b)^{n-i}a^{i}L_i(a,2b)$
$j_n(b)$	$j_{2n}(b)=\sum_{i=0}^{n}\binom{n}{i}(2b)^{n-i}j_i(b);\;\; j_{2n}=\sum_{i=0}^{n}\binom{n}{i}2^{n-i}j_i$

Sequence	For $L_{p,np}(a,b)=\frac{1}{a^n}\sum_{i=0}^{n}\binom{n}{i}(-b)^{n-i}L_{p,i(p+1)}(a,b)$, particular cases are
$L_n(a,b)$	$L_n(a,b)=\frac{1}{a^n}\sum_{i=0}^{n}\binom{n}{i}(-b)^{n-i}L_{2i}(a,b)$
$L_{p,n}(a)$	$L_{p,np}(a)=\frac{1}{a^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}L_{p,i(p+1)}(a)$
$\ell_n(a)$	$\ell_n(a)=\frac{1}{a^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\ell_{2i}(a)$; $L_n=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}L_{2i}$
$L_p(n)$	$L_p(np)=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}L_p(i(p+1))$
$L_{p,n}(2a,b)$	$L_{p,np}(2a,b)=\frac{1}{(2a)^n}\sum_{i=0}^{n}\binom{n}{i}(-b)^{n-i}L_{p,i(p+1)}(2a,b)$
$L_n(2a,b)$	$L_n(2a,b)=\frac{1}{(2a)^n}\sum_{i=0}^{n}\binom{n}{i}(-b)^{n-i}L_{2i}(2a,b)$
$Q_{p,n}(a)$	$Q_{p,np}(a)=\frac{1}{(2a)^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}Q_{p,i(p+1)}(a)$
$Q_n(a)$	$Q_n(a)=\frac{1}{(2a)^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}Q_{2i}(a)$; $Q_n=\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}Q_{2i}$
$T_n(a)$	$T_n(a)=\frac{1}{(2a)^n}\sum_{i=0}^{n}\binom{n}{i}T_{2i}(a)$
$L_{p,n}(a,2b)$	$L_{p,np}(a,2b)=\frac{1}{a^n}\sum_{i=0}^{n}\binom{n}{i}(-2b)^{n-i}L_{p,i(p+1)}(a,2b)$
$L_n(a,2b)$	$L_n(a,2b)=\frac{1}{a^n}\sum_{i=0}^{n}\binom{n}{i}(-2b)^{n-i}L_{2i}(a,2b)$
$j_n(b)$	$j_n(b)=\sum_{i=0}^{n}\binom{n}{i}(-2b)^{n-i}j_{2i}(b)$; $j_n=\sum_{i=0}^{n}\binom{n}{i}(-2)^{n-i}j_{2i}$

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