Miskolc Mathematical Notes
Vol. 24 (2023), No. 1, pp. 447–455

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.1651

CAYLEY LINE GRAPHS OF TRANSITIVE GROUPOIDS

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Received 1 May, 2015

Abstract. A groupoid is a small category with inverses. Adding appropriate colors to the edges of the line graph of a transitive groupoid creates a Cayley line graph of the groupoid. The groupoid of partial automorphisms of the Cayley line graph is isomorphic to a semidirect product of the original groupoid. Using the trivial coloring to build the Cayley line graph makes the semidirect product trivial, hence the groupoid of partial automorphisms of this Cayley line graph is isomorphic to the original groupoid.

2010 *Mathematics Subject Classification:* 20L05; 05C25

Keywords: groupoid, line graph, Cayley color graph, partial automorphism, semidirect product

1. Introduction

The Cayley color graph of a group is an important object both in group theory and in graph theory. A comprehensive reference can be found for example in [6]. The line graph of the Cayley color graph is studied in [2–4]. In particular, it is shown in [4] that the edges of the line graph of a Cayley color graph of a group can be colored so that the automorphism group of the resulting color digraph is isomorphic to a semidirect product of the group.

A groupoid is a small category with inverses. The notion of Cayley color graph of a group is extended to inverse semigroups and groupoids in [9], where it is shown that the groupoid of partial automorphisms of the Cayley color graph of a transitive groupoid is isomorphic to the original groupoid. In this paper we study the line graph of the Cayley color graph of a transitive groupoid and extend the above mentioned result of [4] to transitive groupoids.

In Section 2 we recall definitions and results from [9] concerning the Cayley color graph of a groupoid and give a few examples. In Section 3 we define a Cayley line graph as a special edge coloring of the line graph of the Cayley graph. The Cayley line graph might not be unique, we can color the edges more than one way. In Section 4 we show that the groupoid of partial automorphisms of the Cayley line

Partially supported by DIMACS, Center for Discrete Mathematics and Theoretical Computer Science, Rutgers University.

graph is isomorphic to a semidirect product of the original groupoid. The semidirect product depends on the choice of the coloring, but the coloring always can be chosen trivially to make this semidirect product isomorphic to the original groupoid.

2. PRELIMINARIES

A *groupoid* $\mathcal{G}$ is a set with a subset $\mathcal{G}^{(2)}$ of $\mathcal{G} \times \mathcal{G}$, a product map $(x,y) \mapsto xy\colon \mathcal{G}^{(2)} \to \mathcal{G}$ and an inverse map $x \mapsto x^{-1}\colon \mathcal{G} \to \mathcal{G}$ such that:

(a) $(xy)z = x(yz)$ for all (x,y), $(y,z) \in \mathcal{G}^{(2)}$;
(b) $(x,x^{-1}) \in \mathcal{G}^{(2)}$ for all $x \in \mathcal{G}$;
(c) $x^{-1}(xy) = y$, $(xy)y^{-1} = x$ for all $(x,y) \in \mathcal{G}^{(2)}$.

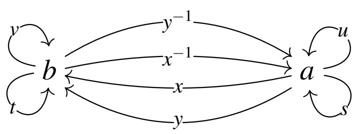
The set $\mathcal{G}^{(2)}$ is called the set of *composable pairs*. The *domain* and *range* maps $d,r\colon \mathcal{G} \to U$ are defined by $d(x) = x^{-1}x$ and $r(x) = xx^{-1}$, where $U = \{xx^{-1} \mid x \in \mathcal{G}\}$ is the set of *units* of $\mathcal{G}$.

A groupoid $\mathcal{G}$ is called *transitive* if for all $u,v \in U$ there is an element $g \in \mathcal{G}$ satisfying $d(x) = u$ and $r(x) = v$. This means $\mathcal{G}$ is connected if viewed as a small category with inverses. Every groupoid is the disjoint union of transitive groupoids, and every transitive groupoid is the direct product of a group G and a trivial groupoid $A \times A$. More precisely, a transitive groupoid $\mathcal{G}$ is isomorphic to a groupoid $A \times G \times A$ where (d,h,c) and (b,g,a) are composable whenever $b = c$, in which case their product is (d,hg,a). The inverse of (b,g,a) is (a,g^{-1},b). The set A can be identified with the unit space $\{(a,e,a) \mid a \in A\}$ of $\mathcal{G}$ where e is the identity of G. The group G is isomorphic to the isotropy subgroup $\mathcal{G}_u^u = \{x \mid d(x) = u = r(x)\}$ for any unit u of $\mathcal{G}$. We only work with finite groupoids. Our references for groupoids are [1, 7, 8].

Example 1. If $A = \{a,b\}$ then the transitive groupoid $\mathcal{G} = A \times \mathbb{Z}_2 \times A$ has eight elements

$$\mathcal{G} = \{x,y,x^{-1},y^{-1},u = x^{-1}x, v = xx^{-1}, s = y^{-1}x, t = yx^{-1}\}$$

where $x = (b,0,a)$ and $y = (b,1,a)$. The groupoid can be visualized by an arrow diagram.



A subset Δ of the groupoid $\mathcal{G}$ *generates* $\mathcal{G}$ if every element of $\mathcal{G}$ can be written as a finite product of elements of Δ.

A *color digraph* is a directed graph with possible multiple edges and loops, together with a color function defined on the set of edges.

The *tail* of a vertex v in a color digraph is the set $\operatorname{tail}(v)$ of vertices that can be reached by a finite directed walk starting at v. We say that v is a *head* of its tail. Note that a tail may have more than one head, and a tail contains each of its heads.

If Δ is a set of generators of the groupoid $\mathcal{G}$, then the *Cayley color graph* [9] $D = D_\Delta(\mathcal{G})$ of $\mathcal{G}$ with respect to the generating set Δ is the color digraph with vertices $\mathcal{G}$ and edges

$$\{(x,z) \mid x \in \mathcal{G}, z \in \Delta, (x,z) \in \mathcal{G}^{(2)}\}$$

such that the edge (x,z) connects x to xz and has color z. See [5] for another version of groupoid Cayley graphs.

Example 2. If $\mathcal{G} = A \times \mathbb{Z}_2 \times A$ is the transitive groupoid of Example 1, then $\Delta = \{x, x^{-1}, s\}$ is a generating set and the Cayley color graph has two tails:

$$x^{-1} \underset{x^{-1}}{\overset{x}{\rightleftarrows}} u \underset{s}{\overset{s}{\rightleftarrows}} s \underset{x}{\overset{x^{-1}}{\rightleftarrows}} y^{-1} \qquad\qquad v \underset{x^{-1}}{\overset{x}{\rightleftarrows}} x \underset{s}{\overset{s}{\rightleftarrows}} y \underset{x}{\overset{x^{-1}}{\rightleftarrows}} t$$

If x is a vertex of $D_\Delta(\mathcal{G})$, then $\operatorname{tail}(x) = \{xy \mid y \in \mathcal{G}\}$ and $r(x)$ is the unique unit u of $\mathcal{G}$ for which $\operatorname{tail}(x) = \operatorname{tail}(u)$. Every tail is strongly connected, and every element of a tail is a head of the tail. A tail contains all the elements of the groupoid whose range is a given unit, so the number of tails is equal to the number of units.

A *partial automorphism* of a Cayley color graph $D = D_\Delta(\mathcal{G})$ is a bijection between two tails of the graph that preserves the colors of the edges. Every partial automorphism of D is implemented by a left multiplication by an element of $\mathcal{G}$ and this representation gives an isomorphism between the partial automorphisms of D and $\mathcal{G}$ (see [9]).

If $\mathcal{G} = A \times G \times A$ and T is a tail of D, then the partial automorphisms of D whose domain and range is T form a group isomorphic to G.

A *color permuting partial automorphism* of a Cayley color graph $D_\Delta(\mathcal{G})$ is a bijection α between two tails of $D_\Delta(\mathcal{G})$ and a permutation ρ of Δ such that $\alpha(xz) = \alpha(x)\rho(z)$ for all $x \in \operatorname{dom}(\alpha)$, $(x,z) \in \mathcal{G}^{(2)}$ and $z \in \Delta$.

Let $H = \{\pi \in \operatorname{Aut}(\mathcal{G}) \mid \pi(\Delta) = \Delta\}$ be the group containing the automorphisms of $\mathcal{G}$ preserving Δ. Let $\iota\colon H \to \operatorname{Aut}(\mathcal{G})$ be the canonical embedding. Recall [8] that the semidirect product $\mathcal{G} \times_\iota H$ is the groupoid with operations

$$(x_1,\pi_1)(x_2,\pi_2) = (x_1\pi_1(x_2), \pi_1\pi_2) \quad \text{and} \quad (x,\pi)^{-1} = (\pi^{-1}(x^{-1}), \pi^{-1})$$

whenever x_1 and $\pi_1(x_2)$ are composable. The map $(y,\rho) \mapsto \beta_{(y,\rho)}$ where $\beta_{(y,\rho)}(x) = y\rho(x)$ is an isomorphism between the semidirect product $\mathcal{G} \times_\iota H$ and the groupoid $\operatorname{PAut}^*(D)$ of color permuting partial automorphisms of $D_\Delta(\mathcal{G})$.

3. Cayley Line Graphs

Definition 1. Let $D = D_\Delta(\mathcal{G})$ be a Cayley color graph and π_z be a permutation of Δ for all $z \in \Delta$ satisfying $\pi_z \circ \pi_w = \pi_{\pi_z(w)}$ for all $w, z \in \Delta$. The *Cayley line graph* $L = L_\pi(D)$ with respect to π is the color digraph with vertices $\{(x,z) \mid x \in \mathcal{G}, z \in \Delta, (x,z) \in \mathcal{G}^{(2)}\}$ and edges

$$\{((x,z),w) \mid x \in \mathcal{G} \text{ and } z,w \in \Delta \text{ and } (x,z),(z,\pi_z(w)) \in \mathcal{G}^{(2)}\}$$

such that the edge $((x,z),w)$ connects vertex (x,z) to vertex $(xz,\pi_z(w))$ and has color w.

The correspondence can be visualized by the following diagram where the color is written next to the edge, and the name is written on the edge:

$$
\begin{array}{ll}
\text{groupoid } \mathcal{G}: & \bullet \xleftarrow{\;x\;} \bullet \xleftarrow{\quad z_1 \quad} \bullet \xleftarrow{\quad z_2 \quad} \bullet \\
\text{Cayley graph } D: & x \xrightarrow{\quad z_1 \quad} xz_1 \xrightarrow{\quad z_2 \quad} xz_1z_2 \\
\text{Cayley line graph } L: & (x,z_1) \xrightarrow[w=\pi_{z_1}^{-1}(z_2)]{((x,z_1),w)} (xz_1,z_2)
\end{array}
$$

Example 3. If $\Delta = \{z_0,\ldots,z_{n-1}\}$ then both of the definitions

(a) $\pi_{z_i} = \mathrm{id}$;
(b) $\pi_{z_i}(z_j) = z_{i+j \bmod n}$;

satisfy $\pi_z \circ \pi_w = \pi_{\pi_z(w)}$ for all $w,z \in \Delta$.

Example 4. Let $\mathcal{G} = \{x,y,u,v\}$ be the trivial groupoid, that is, $y = x^{-1}$, $v = xy$, and $u = yx$. If $\Delta = \{x,y\}$ then the Cayley graph is:

$$
u \underset{x}{\overset{y}{\rightleftarrows}} y \qquad\qquad v \underset{y}{\overset{x}{\rightleftarrows}} x
$$

If $\pi_x = \mathrm{id}$ and $\pi_y = (x\ \ y)$ is the cycle permutation as in Example 3(b), then the Cayley line graph is:

$$
(y,x) \underset{y}{\overset{y}{\rightleftarrows}} (u,y) \qquad\qquad (x,y) \underset{y}{\overset{y}{\rightleftarrows}} (v,x)
$$

Example 5. Let D be the Cayley color graph of Example 2, $\pi_x = \mathrm{id}$, $\pi_{x^{-1}} = (x\ \ x^{-1}\ \ s)$, and $\pi_s = (x\ \ s\ \ x^{-1})$. The Cayley line graph is:

$$
\begin{tikzcd}
(x^{-1},x) \arrow[r, "s"] \arrow[d, shift left, "x^{-1}"] & (u,s) \arrow[r, "s"] \arrow[d, shift left, "x"] & (s,x^{-1}) \arrow[d, shift left, "s"] & (v,x) \arrow[r, "s"] \arrow[d, shift left, "x^{-1}"] & (x,s) \arrow[r, "s"] \arrow[d, shift left, "x"] & (y,x^{-1}) \arrow[d, shift left, "s"] \\
(u,x^{-1}) \arrow[u, shift left, "s"] & (s,s) \arrow[l, "s"] \arrow[u, shift left, "x"] & (y^{-1},x) \arrow[l, "s"] \arrow[u, shift left, "x^{-1}"] & (x,x^{-1}) \arrow[u, shift left, "s"] & (y,s) \arrow[l, "s"] \arrow[u, shift left, "x"] & (t,x) \arrow[l, "s"] \arrow[u, shift left, "x^{-1}"]
\end{tikzcd}
$$

Throughout this paper let $L = L_\pi(D)$ denote the Cayley line graph of the Cayley color graph $D = D_\Delta(\mathcal{G})$ of the transitive groupoid $\mathcal{G}$.

Lemma 1. *For all $w,z \in \Delta$ we have $\pi_z^{-1} \circ \pi_w = \pi_{\pi_z^{-1}(w)}$.*

Proof. The result follows from the calculation:

$$
\pi_z^{-1} \circ \pi_w = \pi_z^{-1} \circ \pi_{\pi_z(\pi_z^{-1}(w))} = \pi_z^{-1} \circ \pi_z \circ \pi_{\pi_z^{-1}(w)} = \pi_{\pi_z^{-1}(w)}. \qquad \square
$$

Lemma 2. *If the vertex (x_0,δ_0) is connected to the vertex (y,z) in L through n edges with colors $\delta_1,\ldots,\delta_n$, then $z = \pi_{\delta_0} \circ \pi_{\delta_1} \circ \cdots \circ \pi_{\delta_{n-1}}(\delta_n)$.*

Proof. Suppose vertex (x_0, δ_0) is connected to vertex (y,z) through the vertices

$$(x_1, z_1), \ldots, (x_{n-1}, z_{n-1}),$$

so we have the following walk:

$$(x_0, \delta_0) \xrightarrow{\delta_1} (x_1, z_1) \xrightarrow{\delta_2} \cdots \longrightarrow (x_{n-2}, z_{n-2}) \xrightarrow{\delta_{n-1}} (x_{n-1}, z_{n-1}) \xrightarrow{\delta_n} (y,z)$$

Then

$$\begin{aligned} z &= \pi_{z_{n-1}}(\delta_n) = \pi_{\pi_{z_{n-2}}(\delta_{n-1})}(\delta_n) = \pi_{z_{n-2}} \circ \pi_{\delta_{n-1}}(\delta_n) \\ &= \cdots = \pi_{\delta_0} \circ \pi_{\delta_1} \circ \cdots \circ \pi_{\delta_{n-1}}(\delta_n). \end{aligned}$$

□

Definition 2. Define

$$H_L := \{\rho \in \mathrm{Aut}(\mathcal{G}) \mid \rho(\Delta) = \Delta \text{ and } \rho \circ \pi_z = \pi_{\rho(z)} \text{ for all } z \in \Delta\}.$$

Note that H_L is not empty since it contains the identity.

Example 6.

(a) If π_z is the identity for all $z \in \Delta$ as in Example 3(a), then $H_L = \{\mathrm{id}\}$.
(b) In Example 4, $H_L = \{\mathrm{id}, \rho\} \cong \mathbb{Z}_2$ where $\rho = (x \quad y)(u \quad v)$.
(c) In Example 5, $H_L = \{\mathrm{id}\}$.

Lemma 3. *H_L is a subgroup of* $\mathrm{Aut}(\mathcal{G})$.

Proof. If $\sigma, \rho \in H_L$ then for all $z \in \Delta$ we have $\sigma\rho \circ \pi_z = \sigma \circ \pi_{\rho(z)} = \pi_{\sigma(\rho(z))}$, so $\sigma\rho \in H_L$. Also,

$$\rho^{-1} \circ \pi_z = \rho^{-1} \circ \pi_{\rho(\rho^{-1}(z))} = \rho^{-1}\rho \circ \pi_{\rho^{-1}(z)} = \pi_{\rho^{-1}(z)}$$

for all $z \in \Delta$. Thus, $\rho^{-1} \in H_L$. □

4. Partial automorphisms

Lemma 4. *If (y,w) and (y,z) are vertices of L, then* $\mathrm{tail}(y,w) = \mathrm{tail}(y,z)$.

Proof. By symmetry, it suffices to show that $(y,z) \in \mathrm{tail}(y,w)$. Write $w^{-1} = \delta_1 \cdots \delta_n$ as a product of elements of Δ. Then (y,z) can be reached from (y,w) along the following walk:

$$(y,w) \to (yw, \delta_1) \to \cdots \to (yw\delta_1 \cdots \delta_{n-1}, \delta_n) \to (y,z)$$

where the color of the last edge is $\pi_{\delta_n}^{-1}(z)$. □

Similar proof shows that $\mathrm{tail}(y,z) = \mathrm{tail}(r(y), \delta)$ for some $\delta \in \Delta$. It is easy to see that if u and v are different elements of U, then $\mathrm{tail}(u, \delta)$ and $\mathrm{tail}(v, \mu)$ are not the same. Thus, the tails of L are of the form $\mathrm{tail}(u, \delta)$, where u is a unit of $\mathcal{G}$ and δ is an arbitrary element of Δ whose range is u.

Definition 3. A *partial automorphism* of L is a bijection α between two tails of L that preserves the colors of the edges, that is,

$$\pi_{\alpha(x,z)_2}(w) = \alpha(xz, \pi_z(w))_2$$

for all $((x,z),w)$ satisfying $(x,z) \in \mathrm{dom}(\alpha)$. The set of partial automorphisms of L is denoted by $\mathrm{PAut}(L)$.

The mapping can we visualized by:

$$(x,z) \xrightarrow{w} (xz, \pi_z(w)) \qquad\qquad \alpha(x,z) \xrightarrow{w} \alpha(xz, \pi_z(w))$$

Lemma 5. *For all $(y,\rho) \in \mathcal{G} \times_\iota H_L$, the map $\alpha_{(y,\rho)} \colon \mathrm{tail}(\rho^{-1}(y^{-1}),\delta) \to \mathrm{tail}(y,\delta_0)$ defined by*

$$\alpha_{(y,\rho)}(x,z) = (y\rho(x), \rho(z))$$

is a partial automorphism of L. Furthermore $\alpha_{(y,\rho)^{-1}} = \alpha_{(y,\rho)}^{-1}$.

Proof. If $((x,z),w)$ is an edge with $(x,z) \in \mathrm{dom}(\alpha_{(y,\rho)})$, then

$$\pi_{\alpha_{(y,\rho)}(x,z)_2}(w) = \pi_{\rho(z)}(w) = \rho \circ \pi_z(w) = \alpha_{(y,\rho)}(xz, \pi_z(w))_2.$$

So $\alpha_{(y,\rho)}$ preserves the colors of the edges.

If $(x,z) \in \mathrm{tail}(y,\delta_0)$ then $(x,z) = (y\delta_0 \cdots \delta_n, z)$ for some $\delta_0, \dots, \delta_n \in \Delta$. Since $r(y)$ and δ_0 are composable, $(\rho^{-1}(\delta_0 \cdots \delta_n), \rho^{-1}(z))$ is an element of $\mathrm{tail}(\rho^{-1}(y^{-1}),\delta)$ that is mapped to (x,z) by $\alpha_{(y,\rho)}$. Thus, $\alpha_{(y,\rho)}$ is surjective. It is easy to check that $\alpha_{(y,\rho)}$ is injective.

If $(x,z) \in \mathrm{dom}(\alpha_{(y,\rho)}^{-1})$ then

$$\begin{aligned}\alpha_{(y,\rho)^{-1}}(x,z) &= \alpha_{(\rho^{-1}(y^{-1}),\rho^{-1})}(x,z) = (\rho^{-1}(y^{-1})\rho^{-1}(x), \rho^{-1}(z)) \\ &= (\rho^{-1}(y^{-1}x), \rho^{-1}(z)) = \alpha_{(y,\rho)}^{-1}(x,z).\end{aligned}$$

□

Lemma 6. *If $\alpha \in \mathrm{PAut}(L)$, $\mathrm{dom}(\alpha) = \mathrm{tail}(u,\delta)$ with $u \in U$, $\delta \in \Delta$, and $(t,z) \in \mathrm{dom}(\alpha)$, then $\alpha(t,z)_2 = \pi_{\alpha(u,\delta)_2}(\pi_\delta^{-1}(z))$.*

Proof. Since (u,δ) is connected to (t,z) through edges say with colors $\delta_1, \dots, \delta_n$. By Lemma 2, $z = \pi_\delta \circ \pi_{\delta_1} \circ \cdots \circ \pi_{\delta_{n-1}}(\delta_n)$, and so $\pi_\delta^{-1}(z) = \pi_{\delta_1} \circ \cdots \circ \pi_{\delta_{n-1}}(\delta_n)$. Since α preserves the colors of the edges, $\alpha(u,\delta)$ is connected to $\alpha(t,z)$ through edges with colors $\delta_1, \dots, \delta_n$, so

$$\alpha(t,z)_2 = \pi_{\alpha(u,\delta)_2} \circ \pi_{\delta_1} \circ \cdots \circ \pi_{\delta_{n-1}}(\delta_n) = \pi_{\alpha(u,\delta)_2}(\pi_\delta^{-1}(z)). \qquad □$$

Lemma 7. *If $\alpha \in \mathrm{PAut}(L)$ and $(x,z),(x,w) \in \mathrm{dom}(\alpha)$, then $\alpha(x,z)_1 = \alpha(x,w)_1$.*

Proof. Since L has no sources, there is a vertex (y,δ) of L that is connected to both (x,z) and (x,w) by single edges. So $\alpha(x,z)_1 = \alpha(y,\delta)_1\alpha(y,\delta)_2 = \alpha(x,w)_1$. □

Proposition 1. *Every partial automorphism α of L is $\alpha_{(y,\rho)}$ for some $(y,\rho) \in \mathcal{G} \times_\iota H_L$.*

Proof. Let u be the unique unit of $\mathcal{G}$ such that $\mathrm{tail}(u,\delta) = \mathrm{dom}(\alpha)$ for some $\delta \in \Delta$. By the previous lemma,

$$y := \alpha(u,\delta)_1$$

is independent of the choice of δ.

Let $z \in \Delta$. Since $\mathcal{G}$ is transitive, we can find $t \in \mathcal{G}$ such that $u = r(tz)$. The definition

$$\rho(z) := \alpha(t,z)_2$$

is independent of the choice of t since by Lemma 6, $\alpha(t,z)_2 = \pi_{\alpha(u,\delta)_2}(\pi_\delta^{-1}(z))$.

We now extend ρ to $\mathcal{G}$. If $x \in \mathcal{G}$ then $x = z_1 \cdots z_n$ for some $z_1, \ldots, z_n \in \Delta$. Define

$$\rho(x) = \rho(z_1) \cdots \rho(z_n).$$

We need to check that this definition is independent of the choice of the z_i's. Suppose we also have $x = w_1 \cdots w_m$ for some $w_1, \ldots, w_m \in \Delta$. Choose $t \in \mathcal{G}$ and $\mu \in \Delta$ such that $u = r(tx)$ and $(x,\mu) \in \mathcal{G}^{(2)}$. Then we have the following walks in L

$$\begin{tikzcd}[column sep=small]
(y,\delta) \arrow[r] \arrow[dr] & \bullet\cdots\bullet \arrow[r] & (t,z_1) \arrow[r] & \bullet\cdots\bullet \arrow[r] & (tz_1\cdots z_{n-1}, z_n) \arrow[dr] & \\
& \bullet\cdots\bullet \arrow[r] & (t,w_1) \arrow[r] & \bullet\cdots\bullet \arrow[r] & (tw_1\cdots w_{m-1}, w_m) \arrow[r] & (tx,\mu)
\end{tikzcd}$$

which are mapped by α to the walks

$$\begin{tikzcd}[column sep=small]
\alpha(y,\delta) \arrow[r] \arrow[dr] & \bullet\cdots\bullet \arrow[r] & (\mu_1,\rho(z_1)) \arrow[r] & \bullet\cdots\bullet \arrow[r] & (\mu_n,\rho(z_n)) \arrow[dr] & \\
& \bullet\cdots\bullet \arrow[r] & (\nu_1,\rho(w_1)) \arrow[r] & \bullet\cdots\bullet \arrow[r] & (\nu_m,\rho(w_m)) \arrow[r] & \alpha(tx,\mu)
\end{tikzcd}$$

for some $\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_m \in \mathcal{G}$. Hence, we have

$$\mu_1\rho(z_1)\cdots\rho(z_n) = \alpha(tx,\mu)_1 = \nu_1\rho(w_1)\cdots\rho(w_m).$$

By Lemma 7, $\mu_1 = \nu_1$. So, $\rho(z_1)\cdots\rho(z_n) = \rho(w_1)\cdots\rho(w_m)$.

We show that ρ is an automorphism of $\mathcal{G}$. If $(w,z) \in \mathcal{G}^{(2)}$ then we can find a $t \in \mathcal{G}$ such that $u = r(tw) = r(twz)$. The edge $((t,w),\pi_w^{-1}(z))$ connects vertex (t,w) to (tw,z). Taking the image under α shows that $\alpha(t,w) = (\alpha(t,w)_1, \delta(w))$ is connected to $(\alpha(tw,z)_1, \delta(z))$ by an edge with color $\pi_w^{-1}(z)$, so $(\delta(w),\delta(z)) \in \mathcal{G}^{(2)}$. It follows easily now that ρ is multiplicative. Since $\rho = \pi_{\alpha(u,\delta)_2} \circ \pi_\delta^{-1}$ is a permutation of Δ, $\rho\colon \mathcal{G} \to \mathcal{G}$ is surjective. Since $\mathcal{G}$ is finite, ρ must be injective.

We show that $\rho \in H_L$. Since $\rho(\Delta) = \Delta$, we only need to check that $\rho \circ \pi_z = \pi_{\rho(z)}$ for all $z \in \Delta$. If $u = r(tz)$ for some $t \in \mathcal{G}$, then $\rho(z) = \pi_{\alpha(u,\delta)_2}(\pi_\delta^{-1}(z))$. So,

$$\pi_{\rho(z)} = \pi_{\alpha(u,\delta)_2} \circ \pi_{\pi_\delta^{-1}(z)} = \pi_{\alpha(u,\delta)_2} \circ \pi_\delta^{-1} \circ \pi_z = \rho \circ \pi_z.$$

It remains to show that $\alpha = \alpha_{(y,\rho)}$. To check that α and $\alpha_{(y,\rho)}$ have the same domain, write $u = u_1 \cdots u_n$ as a product of elements of Δ. Then

$$\begin{aligned} y\rho(u) &= \alpha(u,u_1)_1\rho(u_1)\cdots\rho(u_n) \\ &= \alpha(u,u_1)_1\alpha(u,u_1)_2\alpha(uu_1,u_2)_2\cdots\alpha(uu_1\cdots u_{n-1},u_n)_2 \\ &= \alpha(u,u_1)_1 = y \end{aligned}$$

so

$$\begin{aligned} \operatorname{dom}(\alpha_{(y,\rho)}) &= \operatorname{tail}(\rho^{-1}(y^{-1}),\delta) = \operatorname{tail}(r(\rho^{-1}(y^{-1})),\delta) \\ &= \operatorname{tail}(\rho^{-1}(y^{-1}y),\delta) = \operatorname{tail}(u,\delta) = \operatorname{dom}(\alpha). \end{aligned}$$

If $(x,z) \in \operatorname{dom}(\alpha)$ then $ux = x = x_1 \cdots x_n$ for some $x_1,\dots,x_n \in \Delta$, so we have

$$\begin{aligned} \alpha_{(y,\rho)}(x,z) &= (y\rho(x),\rho(z)) = (\alpha(u,x_1)_1\rho(x_1)\cdots\rho(x_n),\alpha(x,z)_2) \\ &= (\alpha(u,x_1)_1\alpha(u,x_1)_2\alpha(ux_1,x_2)_2\cdots\alpha(ux_1\cdots x_{n-1},x_n)_2,\alpha(x,z)_2) \\ &= (\alpha(ux_1,x_2)_1\alpha(ux_1,x_2)_2\cdots\alpha(ux_1\cdots x_{n-1},x_n)_2,\alpha(x,z)_2) \\ &= \cdots \\ &= (\alpha(ux_1\cdots x_n,z)_1,\alpha(x,z)_2) = (\alpha(x,z)_1,\alpha(x,z)_2) \\ &= \alpha(x,z). \end{aligned}$$

□

Theorem 1. *If $L = L_\pi(D_\Delta(\mathcal{G}))$ then $\mathcal{G} \times_\iota H_L \cong \operatorname{PAut}(L)$.*

Proof. We saw that the map $(y,\rho) \mapsto \alpha_{(y,\rho)}$ is onto and $\alpha_{(y,\rho)^{-1}} = \alpha_{(y,\rho)}^{-1}$. It is multiplicative since if $(x,\sigma),(y,\rho) \in \mathcal{G} \times_\iota H_L$ are composable and $(x,z) \in \operatorname{dom}(\alpha_{(y,\rho)})$, then

$$\begin{aligned} \alpha_{(x,\sigma)(y,\rho)}(x,z) &= \alpha_{(x\sigma(y),\sigma\rho)}(x,z) = (x\sigma(y)\sigma(\rho(x)),\sigma(\rho(z))) \\ &= (x\sigma(y\rho(x)),\sigma(\rho(z))) = \alpha_{(x,\sigma)}(y\rho(x),\rho(z)) \\ &= \alpha_{(x,\sigma)}\alpha_{(y,\rho)}(x,z). \end{aligned}$$

It remains to show that $(y,\rho) \mapsto \alpha_{(y,\rho)}$ is injective. Suppose $\alpha_{(y,\rho)} = \alpha_{(x,\sigma)}$. There is a unique unit $u \in \mathcal{G}$ such that $(u,\delta) \in \operatorname{dom}(\alpha_{(y,\rho)})$. Hence

$$\begin{aligned} (y,\rho(\delta)) &= (y\rho(u),\rho(\delta)) = \alpha_{(y,\rho)}(u,\delta) \\ &= \alpha_{(x,\sigma)}(u,\delta) = (x\sigma(u),\sigma(\delta)) \\ &= (x,\sigma(\delta)), \end{aligned}$$

so $y = x$. If $z \in \Delta$ and $u = r(tz)$, then

$$\rho(z) = \alpha_{(y,\rho)}(t,z)_2 = \alpha_{(x,\sigma)}(t,z)_2 = \sigma(z).$$

Thus, $(y,\rho) = (x,\sigma)$. □

Corollary 1. *If π_z is the identity for all $z \in \Delta$ as in Example 6(a), then $\operatorname{PAut}(L) \cong \mathcal{G}$.*

Note that H_L is a subgroup of $H = \{\pi \in \mathrm{Aut}(\mathcal{G}) \mid \pi(\Delta) = \Delta\}$ and so $\mathrm{PAut}(L) \cong \mathcal{G} \times_\iota H_L$ is isomorphic to a subgroupoid of the groupoid $\mathrm{PAut}^*(D) \cong \mathcal{G} \times_\iota H$ of color permuting partial automorphisms of D.

Example 7. In Example 4, $H_L = H$ and $\mathcal{G} \times_\iota H_L \cong \mathrm{PAut}(L) \cong \{a,b\} \times \mathbb{Z}_2 \times \{a,b\}$. Note that the Cayley line graph L is not a Cayley color graph since the number of vertices of L is not the same as the number of elements of $\mathrm{PAut}(L)$.

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