More on contra $\delta$-precontinuous functions

Miguel Caldas, Saeid Jafari, Takashi Noiri, and Marilda Simões
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MIGUEL CALDAS, SAEID JAFARI, TAKASHI NOIRI, AND MARILDA SIMÕES

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Abstract. In [4], Dontchev introduced and investigated a new notion of continuity called contra-continuity. Recently, Jafari and Noiri [8–10] introduced new generalizations of contra-continuity called contra-$\alpha$-continuity, contra-super-continuity and contra-precontinuity. Recently, Ekici and Noiri [6] have introduced a new class of continuity called contra $\delta$-continuity as a generalization of contra-continuity. In this paper, we obtain some more properties of contra $\delta$-precontinuous functions.

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1. INTRODUCTION AND PRELIMINARIES

Recently, Jafari and Noiri have introduced and investigated the notions of contra-precontinuity [10], contra-$\alpha$-continuity [9] and contra-super-continuity [8] as a continuation of research done by Dontchev [4] and Dontchev and Noiri [5] on the interesting notions of contra-continuity and contra-semi-continuity, respectively. Caldas and Jafari [3] introduced and investigated the notion of contra-$\beta$-continuous functions in topological spaces. Raychaudhuri and Mukherjee [15] introduced the notions of $\delta$-preopen sets and $\delta$-almost continuity in topological spaces. The class of $\delta$-preopen sets is larger than one of preopen sets. Recently, by using $\delta$-preopen sets, Ekici and Noiri [6] have introduced the notion of contra $\delta$-precontinuity as a generalization of contra-precontinuity.

In this paper, we obtain the further properties of contra $\delta$-precontinuous functions. Throughout this paper, all spaces $(X, \tau)$ and $(Y, \sigma)$ (or $X$ and $Y$) are topological spaces. A subset $A$ of $X$ is said to be regular open (resp., regular closed) if $A = \text{Int}(	ext{Cl}(A))$ (resp., $A = \text{Cl}(	ext{Int}(A))$) where $\text{Cl}(A)$ and $\text{Int}(A)$ denote the closure and interior of $A$. A subset $A$ of a space $X$ is called preopen [12] (resp., semi-open [11], $\alpha$-open [14], $\beta$-open [1]) if $A \subset \text{Int}(	ext{Cl}(A))$ (resp., $A \subset \text{Cl}(	ext{Int}(A))$, $A \subset \text{Int}(	ext{Cl}(\text{Int}(A)))$, $A \subset \text{Cl}(	ext{Int}(	ext{Cl}(A))))$. The complement of a preopen (resp., semi-open, $\alpha$-open, $\beta$-open) set is said to be preclosed (resp., semi-closed, $\alpha$-closed, $\beta$-closed). The collection of all closed (resp., semi-open, clopen) subsets of $X$ will

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be denoted by $C(X)$ (resp., $SO(X)$, $CO(X)$). We set $C(X, x) = \{ V \in C(X) \mid x \in V \}$ for $x \in X$. We define $CO(X, x)$ in a similar way.

The notion of the $\delta$-closure of $A$ which is denoted by $\delta Cl(A)$ was introduced by Veličko [19] and is widely investigated in the literature. The $\delta$-closure of $A$ is the set $\{ x \in X \mid Int(Cl(U)) \cap A \neq \emptyset \}$ for every open set $U$ containing $x$. If $\delta Cl(A) = A$, then $A$ is said to be $\delta$-closed [19]. The complement of a $\delta$-closed set is said to be $\delta$-open. The union of all $\delta$-open sets contained in $A$ is called the $\delta$-interior of $A$ and is denoted by $\delta Int(A)$. A subset $A$ of a topological space $X$ is said to be $\delta$-preopen [15] if $\delta Cl(A)$. The complement of a $\delta$-preopen set is said to be $\delta$-preclosed. The intersection (union) of all $\delta$-preclosed ($\delta$-preopen) sets containing (contained in) $A$ in $X$ is called the $\delta$-preclosure ($\delta$-preinterior) of $A$ and is denoted by $\delta Cl_p(A)$ (resp., $\delta Int_p(A)$). By $\delta PO(X)$ (resp., $\delta PC(X)$), we denote the collection of all $\delta$-preopen (resp., $\delta$-preclosed) sets of $X$.

**Lemma 1** ([2,15,17]). The following properties holds for the $\delta$-preclosure of a set in a space $X$:

1. Arbitrary union (intersection) of $\delta$-preopen ($\delta$-preclosed) sets in $X$ is $\delta$-preopen (resp., $\delta$-preclosed).
2. $A$ is $\delta$-preclosed in $X$ iff $A = \delta Cl_p(A)$.
3. $\delta Cl_p(A) \subset \delta Cl_p(B)$ whenever $A \subset B(\subset X)$.
4. $\delta Cl_p(A)$ is $\delta$-preclosed in $X$.
5. $\delta Cl_p(\delta Cl_p(A)) = \delta Cl_p(A)$.
6. $\delta Cl_p(A) = \{ x \in X \mid U \cap A \neq \emptyset \}$ for every $\delta$-preopen set $U$ containing $x$.
7. $\delta Cl_p(A) = A \cup Cl(\delta Int(A))$.
8. If $A$ is $\delta$-open, then $\delta Cl_p(A) = Cl(A)$.
9. If $Y \subset X$ is $\delta$-open and $U \in \delta PO(Y)$, then $U \in \delta PO(X)$.
10. $U \cap V \in \delta PO(U)$ if $U$ is $\delta$-open and $V \in \delta PO(X)$.

**Definition 1.** A function $f : X \to Y$ is said to be contra $\delta$-precontinuous [6] (resp., $\delta$-almost continuous [15]) if $f^{-1}(V)$ is $\delta$-preclosed (resp., $\delta$-preopen) in $X$ for each open set $V$ of $Y$.

**Definition 2.** Let $A$ be a subset of a space $(X, \tau)$. The set $\cap \{ U \in \tau \mid A \subset U \}$ is called the kernel of $A$ [13] and is denoted by $ker(A)$.

**Lemma 2** (Jafari and Noiri [8]). The following properties hold for subsets $A$, $B$ of a space $X$:

1. $x \in ker(A)$ if and only if $A \cap F \neq \emptyset$ for any $F \in C(X, x)$.
2. $A \subset ker(A)$ and $A = ker(A)$ if $A$ is open in $X$.
3. If $A \subset B$, then $ker(A) \subset ker(B)$.

2. **Contra $\delta$-precontinuous functions**

**Theorem 1.** The following assertions are equivalent for a function $f : X \to Y$:
Lemma 2, there exists continuous and hence inverse image of every open set is continuous are independent concepts.

Therefore, \( f(U) \subseteq F \).

Thus, \( x \notin \delta Cl_p(A) \) for any \( x \in f^{-1}(F) \). Therefore, \( f^{-1}(F) \cap \delta Cl_p(A) = \emptyset \) and hence \( F \cap f(\delta Cl_p(A)) = \emptyset \).

(4) \( \Rightarrow \) (5) : Let \( B \) be any subset of \( Y \). Then, by (4) and Lemma 2, we have \( f(\delta Cl_p(f^{-1}(B))) \subseteq ker(f(f^{-1}(B))) \subseteq ker(B) \) and therefore \( \delta Cl_p(f^{-1}(B)) \subseteq f^{-1}(ker(B)) \).

(5) \( \Rightarrow \) (1) : Let \( V \) be any open set of \( Y \). Then, by virtue of Lemma 2, we have \( \delta Cl_p(f^{-1}(V)) \subseteq f^{-1}(ker(V)) = f^{-1}(V) \) and \( \delta Cl_p(f^{-1}(V)) = f^{-1}(V) \). This shows that \( f^{-1}(V) \) is \( \delta \)-preclosed in \( X \).

The following two examples show that \( \delta \)-almost continuous and contra \( \delta \)-precontinuous are independent concepts.

Example 1. The identity function on the real line (with the usual topology) is continuous and hence \( \delta \)-almost continuous but not contra \( \delta \)-precontinuous, since the preimage of each singleton fails to be \( \delta \)-preopen.

Example 2. Let \( X = \{a, b\} \) be the Sierpinski space endowed with the topology \( \tau = \{\emptyset, \{a\}, X\} \). Let \( f:X \rightarrow X \) be defined by \( f(a) = b \) and \( f(b) = a \). Since the inverse image of every open set is \( \delta \)-preclosed, then \( f \) is contra \( \delta \)-precontinuous, but \( f^{-1}(\{a\}) \) is not \( \delta \)-preopen in \( (X, \tau) \). Therefore \( f \) is not \( \delta \)-almost continuous.

Definition 3. A function \( f:X \rightarrow Y \) is said to be contra-continuous [4] (resp., contra-\( \alpha \)-continuous [9], contra-precontinuous [10], contra-semi-continuous [5], contra-\( \beta \)-continuous [3]) if, for each open set \( V \) of \( Y \), \( f^{-1}(V) \) is closed (resp., \( \alpha \)-closed, preclosed, semi-closed, \( \beta \)-closed) in \( X \).

For the functions defined above, we have the following implications:

\[
\begin{array}{cccc}
A & \Rightarrow & B & \Rightarrow & C & \Rightarrow & D \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C & \Rightarrow & F
\end{array}
\]
The meaning of symbols here is as follows: $A$ = contra-continuity, $B$ = contra-$\alpha$-continuity, $C$ = contra-precontinuity, $D$ = contra $\delta$-precontinuity, $E$ = contra-semi-continuity, and $F$ = contra-$\beta$-continuity.

It should be mentioned that none of these implications is reversible as shown by the examples stated below.

Example 3 (Jafari and Noiri [9]). Let $X = \{a, b, c\}$. Put $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is contra-$\alpha$-continuous but not contra-continuous.

Lemma 3 (Caldas et al. [7]). Let $A$ be a subset of $(X, \tau)$. Then the following properties hold:

1. If $A$ is preopen in $(X, \tau)$, then it is $\delta$-preopen in $(X, \tau)$.
2. $A$ is $\delta$-preopen in $(X, \tau)$ if and only if it is preopen in $(X, \tau_3)$.  
3. $A$ is $\delta$-precontinuous in $(X, \tau)$ if and only if it is preclosed in $(X, \tau_3)$.

Since $\text{Cl}(A) \subset \delta\text{Cl}(A)$ for any subset $A$ of $X$, therefore, every contra-precontinuous is contra-$\delta$-precontinuous but not conversely as following example shows.

Example 4 ([5]). A contra-semi-continuous function need not be contra-precontinuous. Let $f : R \to R$ be the function $f(x) = [x]$, where $[x]$ is the Gaussian symbol. If $V$ is a closed subset of the real line, its preimage $U = f^{-1}(V)$ is the union of the intervals of the form $[n, n + 1], n \in Z$; hence $U$ is semi-open being union of semi-open sets. But $f$ is not contra-precontinuous, because $f^{-1}(0.5, 1.5) = [1, 2)$ is not preclosed in $R$.

Example 5 ([5]). A contra-precontinuous function need not be contra-semi-continuous. Let $X = \{a, b\}$, $\tau = \{\emptyset, X\}$ and $\sigma = \{\emptyset, \{a\}, X\}$. The identity function $f : (X, \tau) \to (Y, \sigma)$ is contra-precontinuous as only the trivial subsets of $X$ are open in $(X, \tau)$. However, $f^{-1}(\{a\}) = \{a\}$ is not semi-closed in $(X, \tau)$; hence $f$ is not contra-semi-continuous.

Example 6 ([6]). Let $R$ be the set of real numbers, $\tau$ be the countable extension topology on $R$, i.e., the topology with subbase $\tau_1 \cup \tau_2$, where $\tau_1$ is the Euclidean topology of $R$ and $\tau_2$ is the topology of countable complements of $R$, and $\sigma$ be the discrete topology of $R$. Define a function $f : (R, \tau) \to (R, \sigma)$ as follows: $f(x) = 1$ if $x$ is rational, and $f(x) = 2$ if $x$ is irrational. Then $f$ is contra-$\delta$-precontinuous but not contra-$\beta$-continuous, because $\{1\}$ is closed in $(R, \sigma)$ and $f^{-1}(\{1\}) = Q$, where $Q$ is the set of rationals, is not $\beta$-open in $(R, \tau)$.

Example 7 ([3]). Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $Y = \{p, q\}$, $\sigma = \{\emptyset, \{p\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = p$ and $f(b) = f(c) = q$. Then $f$ is contra-$\beta$-continuous but not contra-$\delta$-precontinuous since $f^{-1}(\{q\}) = \{b, c\}$ is $\beta$-open but not $\delta$-preopen.

Theorem 2. If a function $f : X \to Y$ is contra $\delta$-precontinuous and $Y$ is regular, then $f$ is $\delta$-almost continuous.
Proof. Let \( x \) be an arbitrary point of \( X \) and \( V \) an open set of \( Y \) containing \( f(x) \). Since \( Y \) is regular, there exists an open set \( W \) in \( Y \) containing \( f(x) \) such that \( \text{Cl}(W) \subset V \). Since \( f \) is contra \( \delta \)-precontinuous, so by Theorem 1 there exists \( U \in \delta\text{PO}(X, x) \) such that \( f(U) \subset \text{Cl}(W) \subset V \). Hence, \( f \) is \( \delta \)-almost continuous. \( \square \)

The converse of Theorem 2 is not true. Example 1 shows that \( \delta \)-almost continuity does not necessarily imply contra \( \delta \)-precontinuity even if the range is regular.

Definition 4. A function \( f: X \to Y \) is said to be:

1. (\( \delta,s \))-preopen if \( f(U) \in \text{SO}(Y) \) for every \( \delta \)-preopen set of \( X \).
2. contra-\( I(\delta,p) \)-continuous if for each \( x \in X \) and each \( F \in C(Y, f(x)) \), there exists \( U \in \delta\text{PO}(X, x) \) such that \( \text{Int}(f(U)) \subset F \).

Theorem 3. If a function \( f: X \to Y \) is contra-\( I(\delta,p) \)-continuous and (\( \delta,s \))-preopen, then \( f \) is contra-\( \delta \)-precontinuous.

Proof. Suppose that \( x \in X \) and \( F \in C(Y, f(x)) \). Since \( f \) is contra-\( I(\delta,p) \)-continuous, there exists \( U \in \delta\text{PO}(X, x) \) such that \( \text{Int}(f(U)) \subset F \). By hypothesis \( f \) is (\( \delta,s \))-preopen, therefore \( f(U) \in \text{SO}(Y) \) and \( f(U) \subset \text{Cl}(\text{Int}(f(U))) \subset F \). This shows that \( f \) is contra-\( \delta \)-precontinuous. \( \square \)

Definition 5. A space \( (X, \tau) \) is said to be:

1. locally (\( \delta,p \))-indiscrete if every \( \delta \)-preopen set of \( X \) is closed in \( X \).
2. \( \delta\text{p} \)-space if every \( \delta \)-preopen set of \( X \) is open in \( X \).
3. \( \delta\text{S} \)-space if and only if every \( \delta \)-preopen subset of \( X \) is semi-open.

The following theorem follows immediately from Definition 5.

Theorem 4. If a function \( f: X \to Y \) is contra-\( \delta \)-precontinuous and \( X \) is a \( \delta\text{S} \)-space (resp., \( \delta\text{p} \)-space, locally (\( \delta,p \))-indiscrete), then \( f \) is contra-semi-continuous (resp., contra-continuous, continuous).

Recall that a topological space is said to be:

1. (\( \delta,p \))-\( T_2 \) ([16]) if for each pair of distinct points \( x \) and \( y \) in \( X \) there exist \( U \in \delta\text{PO}(X, x) \) and \( V \in \delta\text{PO}(X, y) \) such that \( U \cap V = \emptyset \).
2. Ultra Hausdorff [18] if for each pair of distinct points \( x \) and \( y \) in \( X \) there exist \( U \in \text{CO}(X, x) \) and \( V \in \text{CO}(X, y) \) such that \( U \cap V = \emptyset \).

Theorem 5. If \( X \) is a topological space and for each pair of distinct points \( x_1 \) and \( x_2 \) in \( X \) there exists a map \( f \) of \( X \) into a Urysohn topological space \( Y \) such that \( f(x_1) \neq f(x_2) \) and \( f \) is contra-\( \delta \)-precontinuous at \( x_1 \) and \( x_2 \), then \( X \) is (\( \delta,p \))-\( T_2 \).

Proof. Let \( x_1 \) and \( x_2 \) be any distinct points in \( X \). Then by hypothesis, there is a Urysohn space \( Y \) and a function \( f: X \to Y \), which satisfies the conditions of the theorem. Let \( y_i = f(x_i) \) for \( i = 1,2 \). Then \( y_1 \neq y_2 \). Since \( Y \) is Urysohn, there
exist open neighbourhoods \( U_{y_1} \) and \( U_{y_2} \) of \( y_1 \) and \( y_2 \) respectively in \( Y \) such that \( \text{Cl}(U_{y_1}) \cap \text{Cl}(U_{y_2}) = \emptyset \). Since \( f \) is contra \( \delta \)-precontinuous at \( x_1 \), there exists a \( \delta \)-preopen neighbourhood \( W_{x_1} \) of \( x_1 \) in \( X \) such that \( f(W_{x_1}) \subset \text{Cl}(U_{y_1}) \) for \( i = 1, 2 \). Hence we get \( W_{x_1} \cap W_{x_2} = \emptyset \) because \( \text{Cl}(U_{y_1}) \cap \text{Cl}(U_{y_2}) = \emptyset \). Then \( X \) is \((\delta, p)\)-\( T_2 \).

**Corollary 1.** If \( f \) is a contra \( \delta \)-precontinuous injection of a topological space \( X \) into a Urysohn space \( Y \), then \( X \) is \((\delta, p)\)-\( T_2 \).

**Proof.** For each pair of distinct points \( x_1 \) and \( x_2 \) in \( X \), \( f \) is a contra \( \delta \)-precontinuous function of \( X \) into a Urysohn space \( Y \) such that \( f(x_1) \neq f(x_2) \) because \( f \) is injective. Hence by Theorem 5, \( X \) is \((\delta, p)\)-\( T_2 \).

**Corollary 2.** If \( f \) is a contra \( \delta \)-precontinuous injection of a topological space \( X \) into a Ultra Hausdorff space \( Y \), then \( X \) is \((\delta, p)\)-\( T_2 \).

**Proof.** Let \( x_1 \) and \( x_2 \) be any distinct points in \( X \). Then since \( f \) is injective and \( Y \) is Ultra Hausdorff, \( f(x_1) \neq f(x_2) \), and there exist \( V_1, V_2 \in \text{CO}(Y) \) such that \( f(x_1) \in V_1 \), \( f(x_2) \in V_2 \) and \( V_1 \cap V_2 = \emptyset \). Then \( x_j \in f^{-1}(V_i) \in \delta \text{PO}(X) \) for \( i = 1, 2 \) and \( f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset \). Thus \( X \) is \((\delta, p)\)-\( T_2 \).

**Lemma 4** ([15]). If \( A_i \) is a \( \delta \)-preopen set in a topological space \( X_i \) for \( i = 1, 2, \ldots, n \), then \( A_1 \times \cdots \times A_n \) is also \( \delta \)-preopen in the product space \( X_1 \times \cdots \times X_n \).

**Theorem 6.** Let \( f_1 : X_1 \to Y \) and \( f_2 : X_2 \to Y \) be two functions, where

1. \( Y \) is a Urysohn space,
2. \( f_1 \) and \( f_2 \) are contra \( \delta \)-precontinuous.

Then the set

\[
\{(x_1, x_2) \mid f_1(x_1) = f_2(x_2)\}
\]

is \( \delta \)-preclosed in the product space \( X_1 \times X_2 \).

**Proof.** Let \( A \) denote the set \( \{(x_1, x_2) \mid f_1(x_1) = f_2(x_2)\} \). In order to show that \( A \) is \( \delta \)-preclosed, we show that \( (X_1 \times X_2) \setminus A \) is \( \delta \)-preopen. Let \( (x_1, x_2) \notin A \). Then \( f_1(x_1) \neq f_2(x_2) \). Since \( Y \) is Urysohn, there exist open \( V_1 \) and \( V_2 \) of \( f_1(x_1) \) and \( f_2(x_2) \) such that \( C(V_1) \cap C(V_2) = \emptyset \). Since \( f_i \) \( (i = 1, 2) \) is contra \( \delta \)-precontinuous, \( f_i^{-1}(C(V_i)) \) is a \( \delta \)-preopen set containing \( x_i \) in \( X_i \) \( (i = 1, 2) \). Hence, by virtue of Lemma 4, \( f_1^{-1}(C(V_1)) \times f_2^{-1}(C(V_2)) \) is \( \delta \)-preopen. Further \( (x_1, x_2) \in f_1^{-1} C(V_1) \times f_2^{-1} C(V_2) \subset (X_1 \times X_2) \setminus A \). It follows that \( (X_1 \times X_2) \setminus A \) is \( \delta \)-preopen. Thus \( A \) is \( \delta \)-preclosed in the product space \( X_1 \times X_2 \).

**Corollary 3.** If \( f : X \to Y \) is contra \( \delta \)-precontinuous and \( Y \) is a Urysohn space, then

\[
A = \{(x_1, x_2) \mid f(x_1) = f(x_2)\}
\]

is \( \delta \)-preclosed in the product space \( X_1 \times X_2 \).
Definition 6. A topological space \(X\) is said to be:

1. \((\delta, p)\)-normal if each pair of non-empty disjoint closed sets can be separated by disjoint \(\delta\)-preopen sets.
2. Ultra normal [18] if each pair of non-empty disjoint closed sets can be separated by disjoint clopen sets.

Theorem 7. If \(f : X \to Y\) is a contra \(\delta\)-precontinuous, closed injection and \(Y\) is ultra normal, then \(X\) is \((\delta, p)\)-normal.

Proof. Let \(F_1\) and \(F_2\) be disjoint closed subsets of \(X\). Since \(f\) is closed and injective, \(f(F_1)\) and \(f(F_2)\) are disjoint closed subsets of \(Y\). Since \(Y\) is ultra normal \(f(F_1)\) and \(f(F_2)\) are separated by disjoint clopen sets \(V_1\) and \(V_2\), respectively. Hence \(F_i \subset f^{-1}(V_i), f^{-1}(V_i) \in \delta PO(X, x)\) for \(i = 1, 2\), and

\[ f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset. \]

Thus, \(X\) is \((\delta, p)\)-normal. \(\Box\)

References


Authors’ addresses

Miguel Caldas  
Universidade Federal Fluminense, Departamento de Matematica Aplicada, Rua Mario Santos Braga, s/n, 24020-140, Niteroi, RJ, Brasil  
*E-mail address*: gmamccs@vm.uff.br

Saeid Jafari  
College of Vestsjaelland South, Herrestraede 11, 4200 Slagelse, Denmark  
*E-mail address*: jafari@stofanet.dk

Takashi Noiri  
Yatsushiro College of Technology, Hirayama shinmachi, Yatsushiro-shi, Kumamoto-ken, 866-8501, Japan  
*E-mail address*: noiri@as.yatsushiro-nct.ac.jp

Marilda Simões  
Università Di Roma “La Sapienza”, Dipartimento Di Matematica “Guido Castelnuovo”, Roma, Italia  
*E-mail address*: simoes@mat.uniroma1.it