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SOME NEW INVERSE-TYPE HILBERT-PACHPATTE INEQUALITIES

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Abstract. Inverses of some new inequalities similar to the Hilbert–Pachpatte inequality are established.

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1. INTRODUCTION

In recent years several authors [2, 3, 5–10] have given considerable attention to the Hilbert inequalities and Hilbert type inequalities and their various generalizations. In particular, in 1988, B. G. Pachpatte [6] proved some new inequalities similar to Hilbert's inequality [4, p. 226], The main purpose of this paper is to establish their inverses.

2. MAIN RESULTS

Our main results are given in the following theorems.

Theorem 1. *Let* $0 < p \leq 1, 0 < q \leq 1$ *and* $\{a_m\}$ *and* $\{b_n\}$ *be two nonnegative sequences of real numbers defined for* $m = 1, 2, \ldots, k$ *and* $n = 1, 2, \ldots, r$*, where* k *and* r *are the natural numbers and define* $A_m = \sum_{s=1}^{m} a_s$ *and* $B_n = \sum_{t=1}^{n} b_t$*. Then*

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for $3n-2m>0$

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^pB_n^q}{3n-2m}\geq pqk^{-2}r^3\left(\sum_{m=1}^{k}(k-m+1)\left(a_mA_m^p\right)^{-\frac{11}{3}}\right)^3 \times\left(\sum_{n=1}^{r}(r-n+1)\left(b_nB_n^{q-1}\right)^{-\frac{1}{2}}\right)^{-2}. \quad (1)$$

Proof. By using the following inequality (see [4, p. 39])

$$rx^{r-1}(x-y)\leq x^r-y^r,\qquad 0<r\leq 1,$$

where x and y are positive real numbers, we obtain that

$$\sum_{m=0}^{k-1}pA_{m+1}^{p-1}\left(A_{m+1}-A_m\right)\leq\sum_{m=0}^{k-1}\left(A_{m+1}^p-A_m^p\right),$$

that is

$$A_k^p\geq p\sum_{m=1}^{k}a_mA_m^{p-1}. \quad (2)$$

From (2) and in view of Hölder's inequality [4, p. 24], we have

$$\frac{A_m^pB_n^q}{3n-2m}\geq pq\left(3n-2m\right)^{-1}\left(\sum_{s=1}^{m}a_sA_s^{p-1}\right)\left(\sum_{t=1}^{n}b_tB_t^{q-1}\right)$$
$$\geq pq(3n-2m)^{-1}n^3m^{-2}\left(\sum_{s=1}^{m}\left(a_sA_s^{p-1}\right)^{\frac{1}{3}}\right)^3 \times\left(\sum_{t=1}^{n}\left(b_tB_t^{q-1}\right)^{-\frac{1}{2}}\right)^{-2}. \quad (3)$$

Using the following inequality [1, p. 15]

$$m^{\frac{1}{h}}n^{\frac{1}{l}}\geq\frac{m}{h}+\frac{n}{l}, \quad (4)$$

where $m>0$, $n>0$, $h^{-1}+l^{-1}=1$, and $h<1$, we easily get that

$$n^3m^{-2}\geq 3n-2m. \quad (5)$$

Taking the sum on both sides of (3) over n from 1 to r first and then the sum over m from 1 to k and in view of inequality (5) and using again Hölder's inequality, we obtain

$$\begin{aligned}
\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q}{3n-2m} &\geq pq\left(\sum_{m=1}^{k}\left(\sum_{s=1}^{m}\left(a_s A_s^{p-1}\right)^{\frac{1}{3}}\right)^3\right) \\
&\quad\times\left(\sum_{n=1}^{r}\left(\sum_{t=1}^{n}\left(b_t B_t^{q-1}\right)^{-\frac{1}{2}}\right)^{-2}\right) \\
&\geq pqk^{-2}r^3\left(\sum_{m=1}^{k}\left(\sum_{s=1}^{m}\left(a_s A_s^{p-1}\right)^{\frac{1}{3}}\right)\right)^3\left(\sum_{n=1}^{r}\left(\sum_{t=1}^{n}\left(b_t B_t^{q-1}\right)^{-\frac{1}{2}}\right)\right)^{-2} \\
&= pqk^{-2}r^3\left(\sum_{s=1}^{k}\left(a_s A_s^{p-1}\right)^{\frac{1}{3}}\left(\sum_{m=s}^{k}1\right)\right)^3\left(\sum_{t=1}^{r}\left(b_t B_t^{q-1}\right)^{-\frac{1}{2}}\left(\sum_{n=t}^{r}1\right)\right)^{-2} \\
&= pqk^{-2}r^3\left(\sum_{s=1}^{k}(k-s+1)\left(a_s A_s^{p-1}\right)^{\frac{1}{3}}\right)^3 \\
&\quad\times\left(\sum_{t=1}^{r}(r-t+1)\left(b_t B_t^{q-1}\right)^{-\frac{1}{2}}\right)^{-2} \\
&= pqk^{-2}r^3\left(\sum_{m=1}^{k}(k-m+1)\left(a_m A_m^{p-1}\right)^{\frac{1}{3}}\right)^3 \\
&\quad\times\left(\sum_{n=1}^{r}(r-n+1)\left(b_n B_n^{q-1}\right)^{-\frac{1}{2}}\right)^{-2},
\end{aligned}$$

which completes the proof. □

Remark 1. Inequality (1) is just an inverse of the following Inequality A, which was proved by Pachpatte [6]:

Inequality A.

$$\begin{aligned}
\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q}{m+n} &\leq \frac{1}{2}pq\,(kr)^{\frac{1}{2}}\left(\sum_{m=1}^{k}(k-m+1)\left(a_m A_m^{p-1}\right)^2\right)^{\frac{1}{2}} \\
&\quad\times\left(\sum_{n=1}^{r}(r-n+1)\left(b_n B_n^{q-1}\right)^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Theorem 2. *Let* $\{a_m\}, \{b_n\}, A_m, B_n$ *be as defined in Theorem 1. Let* $\{p_m\}$ *and* $\{q_n\}$ *be two positive sequences for* $m = 1, 2, \ldots, k$ *and* $n = 1, 2, \ldots, r$*, where* k *and* r *are natural numbers and put* $P_m = \sum_{s=1}^{m} p_s, Q_n = \sum_{t=1}^{n} q_t$*. Let* ϕ *and* ψ *be two real-valued nonnegative, concave, and supermultiplicative functions** *defined on* $\mathbb{R}_+$*. Then for* $3n - 2m > 0$

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{3n-2m} \geq M'(k,r)\left(\sum_{m=1}^{k}\left(p_m\phi\left(\frac{a_m}{p_m}\right)\right)^{\frac{1}{3}}(k-m+1)\right)^{3} \times\left(\sum_{n=1}^{r}\left(q_n\psi\left(\frac{b_n}{q_n}\right)\right)^{-\frac{1}{2}}(r-n+1)\right)^{-2}, \quad (6)$$

where

$$M'(k,r) = \left(\sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\right)^{-\frac{1}{2}}\right)^{-2}\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^{\frac{1}{3}}\right)^{3}.$$

Proof. From the hypotheses and by Jensen's inequality and Hölder's inequality, we obtain

$$\begin{aligned}\frac{\phi(A_m)\psi(B_n)}{3n-2m} &\geq (3n-2m)^{-1} \\ &\quad\times\phi(P_m)\psi(Q_n)\phi\left(\frac{\sum_{s=1}^{m} p_s a_s/p_s}{\sum_{s=1}^{m} p_s}\right)\psi\left(\frac{\sum_{t=1}^{n} q_t b_t/q_t}{\sum_{t=1}^{n} q_t}\right) \\ &\geq (3n-2m)^{-1}\frac{\phi(P_m)}{P_m}\frac{\psi(Q_n)}{Q_n}\sum_{s=1}^{m} p_s\phi\left(\frac{a_s}{p_s}\right)\sum_{t=1}^{n} q_t\phi\left(\frac{b_t}{q_t}\right) \\ &\geq (3n-2m)^{-1}n^3m^{-2}\frac{\phi(P_m)}{P_m}\frac{\psi(Q_n)}{Q_n}\left(\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^{\frac{1}{3}}\right)^{3} \\ &\quad\times\left(\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^{-\frac{1}{2}}\right)^{-2}\end{aligned}$$

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* f is said to be a supermultiplicative function if $f(xy) \geq f(x)f(y)$ for $x, y \in \mathbb{R}_+ := [0, +\infty)$.

Taking the sum over n from 1 to r first and then the sum over m from 1 to k and using (5), in view of Hölder's inequality, we have

$$\begin{aligned}
\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{3n-2m} &\geq \sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\left(\sum_{s=1}^{m}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^{\frac{1}{3}}\right)^{3}\right) \\
&\quad\times\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\left(\sum_{t=1}^{n}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^{-\frac{1}{2}}\right)^{-2}\right) \\
&\geq \left(\sum_{m=1}^{k}\left(\frac{\phi((P_m))}{P_m}\right)^{-\frac{1}{2}}\right)^{-2}\left(\sum_{m=1}^{k}\sum_{s=1}^{m}p_s\phi\left(\frac{a_s}{p_s}\right)^{\frac{1}{3}}\right)^{3} \\
&\quad\times\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^{\frac{1}{3}}\right)^{3}\left(\sum_{n=1}^{r}\sum_{t=1}^{n}q_t\psi\left(\frac{b_t}{q_t}\right)^{-\frac{1}{2}}\right)^{-2} \\
&= M'(k,r)\left(\sum_{s=1}^{k}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^{\frac{1}{3}}\left(\sum_{m=s}^{k}1\right)\right)^{3} \\
&\quad\times\left(\sum_{t=1}^{r}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^{-\frac{1}{2}}\left(\sum_{n=t}^{r}1\right)\right)^{-2} \\
&= M'(k,r)\left(\sum_{s=1}^{k}\left(p_s\phi\left(\frac{a_s}{p_s}\right)\right)^{\frac{1}{3}}(k-s+1)\right)^{3} \\
&\quad\times\left(\sum_{t=1}^{r}\left(q_t\psi\left(\frac{b_t}{q_t}\right)\right)^{-\frac{1}{2}}(r-t+1)\right)^{-2} \\
&= M'(k,r)\left(\sum_{m=1}^{k}\left(p_m\phi\left(\frac{a_m}{p_m}\right)\right)^{\frac{1}{3}}(k-m+1)\right)^{3} \\
&\quad\times\left(\sum_{n=1}^{r}\left(q_n\psi\left(\frac{b_n}{q_n}\right)\right)^{-\frac{1}{2}}(r-n+1)\right)^{-2}.
\end{aligned}$$

The proof is complete. □

Remark 2. Inequality (6) is just an inverse of the following Inequality B, which was proved by Pachpatte [6]:

Inequality B.

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{m+n} \le M(k,r)\left(\sum_{m=1}^{k}(k-m+1)\left(p_m\phi\left(\frac{a_m}{p_m}\right)\right)^2\right)^{\frac{1}{2}} \times\left(\sum_{n=1}^{r}(r-n+1)\left(q_n\psi\left(\frac{b_n}{q_n}\right)\right)^2\right)^{\frac{1}{2}},$$

where

$$M(k,r)=\frac{1}{2}\left(\sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\right)^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^2\right)^{\frac{1}{2}}.$$

Similarly, the following two theorems can also be established.

Theorem 3. *Let $\{a_m\}$ and $\{b_n\}$ be as in Theorem 1 and set $A_m=(1/m)\sum_{s=1}^{m}a_s$ and $B_n=(1/n)\sum_{t=1}^{n}b_t$, for $m=1,2,\ldots,k$ and $n=1,2,\ldots,r$, where k and r are natural numbers. Let ϕ and ψ be two real-valued, nonnegative, and concave functions defined on $\mathbb{R}_+$. Then for $3n-2m>0$,*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{mn}{3n-2m}\phi(A_m)\psi(B_n)\ge k^{-2}r^3\left(\sum_{m=1}^{k}(k-m+1)(\phi(a_m))^{\frac{1}{3}}\right)^3 \times\left(\sum_{n=1}^{r}(r-n+1)(\psi(b_n))^{-\frac{1}{2}}\right)^{-2}. \quad (7)$$

Remark 3. Inequality (7) is just an inverse of the following Inequality C, which was proved by Pachpatte [6]:

Inequality C.

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{mn}{m+n}\phi(A_m)\psi(B_n)\le\frac{1}{2}(kr)^{\frac{1}{2}}\left(\sum_{m=1}^{k}(k-m+1)(\phi(a_m))^2\right)^{\frac{1}{2}} \times\left(\sum_{n=1}^{r}(r-n+1)(\psi(b_n))^2\right)^{\frac{1}{2}}.$$

Theorem 4. *Let $\{a_m\},\{b_n\},\{p_m\},\{q_n\},P_m,Q_n$ be as in Theorem 2 and put $A_m=(1/p_m)\sum_{s=1}^{m}p_sa_s$, $B_n=(1/Q_n)\sum_{t=1}^{n}q_tb_t$, for $m=1,2,\ldots k,n=1,2,\ldots,r$, where k and r are natural numbers. Let ϕ and ψ be as defined in Theorem 3. Then* *for* $3n-2m>0$,

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{P_m Q_n \phi(A_m)\psi(B_n)}{3n-2m} \geq k^{-2}r^{3}\left(\sum_{m=1}^{k}(k-m+1)\left(p_m\phi(a_m)\right)^{\frac{1}{3}}\right)^{3} \times\left(\sum_{n=1}^{r}(r-n+1)\left(q_n\psi(b_n)\right)^{-\frac{1}{2}}\right)^{-2}. \quad (8)$$

Remark 4. Inequality (8) is just an inverse of the following Inequality D, which was proved by Pachpatte [6]:

Inequality D.

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{P_m Q_n \phi(A_m)\psi(B_n)}{m+n} \leq \frac{\sqrt{kr}}{2}\left(\sum_{m=1}^{k}(k-m+1)\left(p_m\phi(a_m)\right)^{2}\right)^{\frac{1}{2}} \times\left(\sum_{n=1}^{r}(r-n+1)\left(q_n\psi(b_n)\right)^{2}\right)^{\frac{1}{2}}.$$

The proofs of Theorems 3 and 4 can be completed by following the same steps as in the proof of Theorem 2 with suitable changes. Here, we omit the details.

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