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SPECTRA OF SOME SPECIAL BIPARTITE GRAPHS

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Abstract. Let $G = (P, Q)$ be a bipartite graph and G' be a graph obtained by joining each vertex of P and Q with m and s new vertices respectively. We obtain the characteristic, Laplacian and signless Laplacian polynomial of G'. As an application, we give a simple proof for Csikvari's lemma on eigenvalues of graphs.

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1. Introduction

Throughout the paper $0_{n\times n}$ and I_n are used respectively, for square zero and identity matrix of order n, 0_n and j_n, respectively, stand for length-n column vectors consisting entirely of 0's and 1's. The Kronecker product $A \otimes B$ of two matrices $A(a_{ij})$ and $B(b_{ij})$ of order $m\times n$ and $p\times q$, respectively, is the $mp\times nq$ matrix obtained from A by replacing each element a_{ij} by $a_{ij}B$. This operation has the properties $(A\otimes B)^T = A^T \otimes B^T$ and $(A\otimes B)(C\otimes D) = (AC\otimes BD)$ whenever the products AC and BD exist. The latter implies $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$ for nonsingular matrices A and B. Moreover, if A and B are $n\times n$ and $p\times p$ matrices, then $\det(A\otimes B) = (\det A)^p.(\det B)^n$. $\bar{K}_n$, K_n and $K_{n,m}$ stand for the graph with n isolated vertices, complete graph and complete bipartite graph, respectively. Also we use these two notations

$$T_w = \begin{bmatrix} I_w & 0 \\ 0 & 0_{(n-w)\times(n-w)} \end{bmatrix} \quad S_w = \begin{bmatrix} 0_{(n-w)\times(n-w)} & 0 \\ 0 & I_w \end{bmatrix}$$

where $T_n = S_n = I_n$.

Let G be a graph with the vertex set $\{v_1,\dots,v_n\}$. The adjacency matrix of G is an $(n\times n)$ matrix $A(G)$ whose (i,j)-entry is 1 if v_i is adjacent to v_j and 0, otherwise. The characteristic polynomial of G, denoted by $f_G(\lambda)$, is the characteristic polynomial of $A(G)$. We will write it simply f_G when there is no confusion. The roots of f_G are called the eigenvalues of G. The notation $(a)^{[m]}$, means that multiplicitiy of the root a is m. The Laplacian matrix of G and the signless Laplacian matrix of G are defined as $L(G) = \Delta(G) - A(G)$ and $Q(G) = \Delta(G) + A(G)$, respectively, where $\Delta(G)$ is the diagonal matrix whose entries are the degrees of G. We denote the Laplacian polynomial of G by $f_{L(G)}(\mu)$ and the signless Laplacian polynomial of G by $f_{Q(G)}(\nu)$. Denote the eigenvalues of $A(G)$, $L(G)$ and $Q(G)$, respectively, by

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$$
$$\mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$$
$$\nu_1(G) \leq \nu_2(G) \leq \cdots \leq \nu_n(G).$$

Spectral graph theory is an area of intense study, see for instance [1,2,4–6]. Graph transformations are natural techniques for producing new graphs from old ones, and their spectra have received considerable attention in recent years. In this paper, we find the characteristic, Laplacian and signless Laplacian polynomial of some special bipartite graphs obtained from other bipartite graphs by a specific graph transformation. We describe it as follows. Let $G = (P, Q)$ be a bipartite graph with n vertices which $|P| = p$, $|Q| = n - p$. We join each vertex of P and Q with m and s new vertices, respectively. We add $(n-p)(m-s)$ isolated vertices. This only results adding 0's to the various spectrums, but will simplify our computation. We denote the resulting graph by G_{ms}. By a proper labeling, the adjacency matrix of G_{ms} is

$$A(G_{ms}) = \begin{bmatrix} A(G) & D \\ D^T & 0 \end{bmatrix},$$

where $D = {j_m}^T \otimes I_n - ({0_s}^T, {j_{m-s}}^T) \otimes S_{n-p}$.

We obtain the characteristic, Laplacian and signless Laplacian polynomial of G_{ms}. Let $G = (P, Q)$ be a bipartite graph with n vertices which $|P| = p$, $|Q| = n - p$. We join each vertex of P and Q with m and s new vertices respectively. From now on, we denote this graph by G'. In Section 2 we give some preliminaries. In Section 3 we obtain the characteristic polynomial of G_{ms}. The results on G_{ms} enable us to obtain the characteristic polynomial of G'. As an application, we give a simple proof for Csikvari's lemma on eigenvalues of graphs ([3] Lemma 2.8). Finally, we construct some graphs whose all nonzero eigenvalues are of the form $\pm\sqrt{m \pm \sqrt{m}}$ for any natural number m. In Section 4 we obtain Laplacian and signless Laplacian polynomials of G_{ms} and G'.

2. Preliminaries

Lemma 1 (Schur complement [1])**.** *Let A be an $n \times n$ matrix partitioned as*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square matrices. If A_{11} and A_{22} are invertible, then

$$\det\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \det(A_{22})\det(A_{11} - A_{12}{A_{22}}^{-1}A_{21})$$

$$= \det(A_{11})\det(A_{22} - A_{21}{A_{11}}^{-1}A_{12}).$$

Notation 1. Let $G = (P, Q)$ be a bipartite graph with n vertices such that $|P| = p$, $|Q| = n - p$. Also C is a matrix of order $p \times (n-p)$ such that

$$A(G) = \begin{pmatrix} 0 & C \\ C^T & 0 \end{pmatrix}.$$

For

$$f_G(\lambda) = \det\begin{pmatrix} \lambda I_p & -C \\ -C^T & \lambda I_{n-p} \end{pmatrix},$$

with two methods of Schur complement we have:

$$f_G(\lambda) = \lambda^{n-2p}\det(\lambda^2 I_p - CC^T) = \lambda^{n-2p} g_G(\lambda)$$

$$= \lambda^{2p-n}\det(\lambda^2 I_{n-p} - C^T C) = \lambda^{2p-n} q_G(\lambda).$$

where $g_G(\lambda) = \det(\lambda^2 I_p - CC^T)$ and $q_G(\lambda) = \det(\lambda^2 I_{n-p} - C^T C)$. It is clear that if $n = 2p$, then $f_G(\lambda) = g_G(\lambda) = q_G(\lambda)$.

3. Characteristic polynomial of G_{ms}

Theorem 1. *Let $G = (P, Q)$ be a bipartite graph with n vertices such that $|P| = p$, $|Q| = n - p$. Then*

$$f_{G_{ms}}(\lambda) = \begin{cases} \lambda^{mn-n+2p}(\lambda^2 - s)^{n-2p} g_G\left(\sqrt{\frac{(\lambda^2-m)(\lambda^2-s)}{\lambda^2}}\right) & n \geq 2p \\ \lambda^{mn+n-2p}(\lambda^2 - m)^{2p-n} q_G\left(\sqrt{\frac{(\lambda^2-m)(\lambda^2-s)}{\lambda^2}}\right) & n \leq 2p. \end{cases}$$

Proof. Let $D = {j_m}^T \otimes I_n - ({0_s}^T, {j_{m-s}}^T) \otimes S_{n-p}$. By using of Schur complement we have

$$f_{G_{ms}}(\lambda) = \det\begin{pmatrix} \lambda I_n - A(G) & -D \\ -D^T & \lambda I_{nm} \end{pmatrix}$$

$$= \lambda^{mn} \det((\lambda I_n - A(G)) - D(\lambda I_{mn})^{-1} D^T)$$
$$= \lambda^{mn} \det((\lambda I_n - A(G)) - \frac{1}{\lambda} D D^T)$$
$$= \lambda^{mn-n} \det((\lambda^2 I_n - \lambda A(G)) - D D^T).$$

It is easy to see that $DD^T = mI_n - (m-s)S_{n-p}$. Since G is bipartite there is a matrix C of order $p \times (n-p)$ such that

$$f_{G_{ms}}(\lambda) = \lambda^{mn-n} \det\begin{pmatrix} (\lambda^2 - m)I_p & -\lambda C \\ -\lambda C^T & (\lambda^2 - s)I_{n-p} \end{pmatrix}.$$

There are two cases:

CASE 1. Let $n \geq 2p$. In this case, let $(\lambda^2 - s)I_{n-p}$ be invertible. Then by Schur complement, we have

$$f_{G_{ms}}(\lambda) = \lambda^{mn-n}(\lambda^2 - s)^{n-p} \det((\lambda^2 - m)I_p - (\lambda C((\lambda^2 - s)I_{n-p})^{-1} \lambda C^T))$$
$$= \lambda^{mn-n}(\lambda^2 - s)^{n-p} \det((\lambda^2 - m)I_p - \frac{\lambda^2}{(\lambda^2 - s)}(CC^T))$$
$$= \lambda^{mn-n}(\lambda^2 - s)^{n-p}(\frac{\lambda^2}{(\lambda^2 - s)})^p \det(\frac{(\lambda^2 - m)(\lambda^2 - s)}{\lambda^2} I_p - (CC^T)).$$

By Notation 1 we obtain

$$f_{G_{ms}}(\lambda) = \lambda^{mn-n+2p}(\lambda^2 - s)^{n-2p} g_G(\sqrt{\frac{(\lambda^2 - m)(\lambda^2 - s)}{\lambda^2}}).$$

CASE 2. Let $n \leq 2p$. In this case, let $(\lambda^2 - m)I_p$ be invertible. Then

$$
\begin{aligned}
f_{G_{ms}}(\lambda) &= \lambda^{mn-n}(\lambda^2-m)^p \det((\lambda^2-s)I_{n-p}-(\lambda C^T((\lambda^2-m)I_p)^{-1}\lambda C)) \\
&= \lambda^{mn-n}(\lambda^2-m)^p \det((\lambda^2-s)I_{n-p}-\frac{\lambda^2}{(\lambda^2-m)}(C^TC)) \\
&= \lambda^{mn-n}(\lambda^2-m)^p (\frac{\lambda^2}{(\lambda^2-m)})^{n-p} \det(\frac{(\lambda^2-s)(\lambda^2-m)}{\lambda^2}I_{n-p}-(C^TC)).
\end{aligned}
$$

By Notation 1 we obtain

$$
f_{G_{ms}}(\lambda) = \lambda^{mn+n-2p}(\lambda^2-m)^{2p-n} q_G(\sqrt{\frac{(\lambda^2-s)(\lambda^2-m)}{\lambda^2}}).
$$

□

Corollary 1. *For graph G' we have*

$$
f_{G'}(\lambda) = \begin{cases} \lambda^{n(s-1)+p(m-s+2)}(\lambda^2-s)^{n-2p} g_G(\sqrt{\frac{(\lambda^2-m)(\lambda^2-s)}{\lambda^2}}) & n \geq 2p \\ \\ \lambda^{n(s+1)+p(m-s-2)}(\lambda^2-m)^{2p-n} q_G(\sqrt{\frac{(\lambda^2-m)(\lambda^2-s)}{\lambda^2}}) & n \leq 2p. \end{cases}
$$

Proof. The result follows by removing $(n-p)(m-s)$ number of λ's from $f_{G_{ms}}(\lambda)$. □

For $s=0$, we obtain a nice result which is due to Csikvari ([3] Lemma 2.8).

Corollary 2. *For graph G', let s=0. Then*

$$
f_{G'}(\lambda) = \begin{cases} \lambda^{n+p(m-2)} g_G(\sqrt{\lambda^2-m}) & n \geq 2p \\ \\ \lambda^{n+p(m-2)}(\lambda^2-m)^{2p-n} q_G(\sqrt{\lambda^2-m}) & n \leq 2p. \end{cases}
$$

Corollary 3. *All nonzero eigenvalues of the graph G' are of the form $\pm\sqrt{m}$, $\pm\sqrt{s}$ and $\pm\sqrt{\frac{m+s+{\lambda_i}^2 \pm \sqrt{(m+s+{\lambda_i}^2)^2-4ms}}{2}}$, where λ_i for $i=1,\ldots,t$ are all positive eigenvalues of G.*

Proof. It is clear from $\lambda^2-m=0$, $\lambda^2-s=0$ and $\frac{(\lambda^2-m)(\lambda^2-s)}{\lambda^2}={\lambda_i}^2$. □

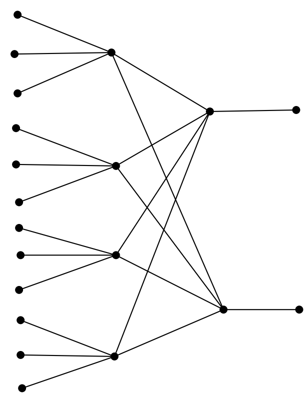
Example 1. Let $G = K_{4,2}$, $m = 3$ and $s = 1$. Then $f_G(\lambda) = \lambda^2(\underbrace{\lambda^2(\lambda^2-8)}_{q_G(\lambda)})$ and $8 = 2p > n = 6$. We have

$$\begin{aligned} f_{G'}(\lambda) &= \lambda^{12}(\lambda^2-3)^2(\frac{(\lambda^2-3)(\lambda^2-1)}{\lambda^2})(\frac{(\lambda^2-3)(\lambda^2-1)}{\lambda^2}-8) \\ &= \lambda^8(\lambda^2-3)^3(\lambda^2-1)(\lambda^4-12\lambda^2+3). \end{aligned}$$

So the eigenvalues of the graph G' are

$$\{(0)^{[8]}, (\sqrt{3})^{[3]}, (-\sqrt{3})^{[3]}, (1)^{[1]}, (-1)^{[1]}, (\sqrt{6+\sqrt{33}})^{[1]}, (-\sqrt{6+\sqrt{33}})^{[1]}$$

$$, (\sqrt{6-\sqrt{33}})^{[1]}, (-\sqrt{6-\sqrt{33}})^{[1]}\}.$$



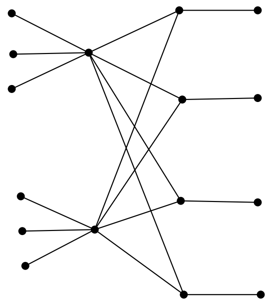
Example 2. Let $G = K_{2,4}$, $m = 3$ and $s = 1$. Then $f_G(\lambda) = \lambda^2(\underbrace{\lambda^2(\lambda^2-8)}_{g_G(\lambda)})$ and $4 = 2p < n = 6$. We have

$$\begin{aligned} f_{G'}(\lambda) &= \lambda^{8}(\lambda^2-1)^2(\frac{(\lambda^2-3)(\lambda^2-1)}{\lambda^2})(\frac{(\lambda^2-3)(\lambda^2-1)}{\lambda^2}-8) \\ &= \lambda^4(\lambda^2-1)^3(\lambda^2-3)(\lambda^4-12\lambda^2+3). \end{aligned}$$

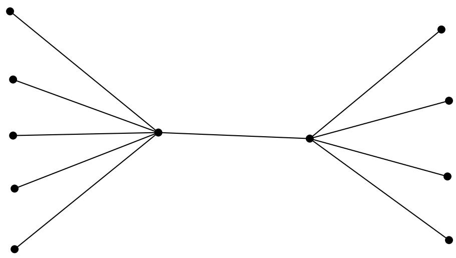
So the eigenvalues of the graph G' are

$$\{(0)^{[4]}, (1)^{[3]}, (-1)^{[3]}, (\sqrt{3})^{[1]}, (-\sqrt{3})^{[1]}, (\sqrt{6+\sqrt{33}})^{[1]}, (-\sqrt{6+\sqrt{33}})^{[1]}$$

$$, (\sqrt{6-\sqrt{33}})^{[1]}, (-\sqrt{6-\sqrt{33}})^{[1]}\}.$$



Example 3. **(Graphs with eigenvalues $\pm\sqrt{m\pm\sqrt{m}}$.)** For any positive integer m, let $G = K_2$ and $s = m-1$. Then by using Corollary 3, $G^{'}$ has eigenvalue 0 with multiplicity $2m-3$ and eigenvalues $\sqrt{m+\sqrt{m}}$, $\sqrt{m-\sqrt{m}}$, $-\sqrt{m+\sqrt{m}}$ and $-\sqrt{m-\sqrt{m}}$ all with multiplicity 1. The graph below has eigenvalue zero with multiplicity 7, and $\sqrt{5+\sqrt{5}}$, $\sqrt{5-\sqrt{5}}$, $-\sqrt{5+\sqrt{5}}$ and $-\sqrt{5-\sqrt{5}}$, all with multiplicity 1.



4. LAPLACIAN AND SIGNLESS LAPLACIAN POLYNOMIAL OF G_{ms}

In this section we assume that G is an r-regular bipartite graph with $|G| = n = 2k$ where k is the size of each part and G' is obtained from G by joining each vertex of one part to m and each vertex of another part to s new vertices. We denote the Laplacian polynomial of G' by $f_{L(G')}(\mu)$ and the signless Laplacian polynomial of G' by $f_{Q(G')}(\nu)$.

Theorem 2. *Let G be an r-regular bipartite graph with $|G| = n = 2k$ where k is the size of each part. Then*

$$f_{L(G_{ms})}(\mu)$$
$$= \mu^{(m-s)k}(\mu-1)^{ns+(m-s)k} f_G\left(\sqrt{(\mu-m-r-\frac{m}{\mu-1})(\mu-s-r-\frac{s}{\mu-1})}\right)$$

Proof. We have

$$\Delta(G_{ms}) = \begin{bmatrix} (m+r)I_n - (m-s)S_k & 0 \\ 0 & B \end{bmatrix}, \quad B = \begin{bmatrix} I_{sn} & 0 \\ 0 & I_{m-s} \otimes T_k \end{bmatrix},$$

where Δ is diagonal matrix whose entries are the degrees of G_{ms}. Since $L(G_{ms}) = \Delta(G_{ms}) - A(G_{ms})$ we have

$$f_{L(G_{ms})}(\mu) = \det(\mu I - L(G_{ms})) = \det(\mu I - \Delta(G_{ms}) + A(G_{ms}))$$
$$= \det \begin{bmatrix} \mu I_n - (m+r)I_n + (m-s)S_k + A(G) & D \\ D^T & \mu I_{nm} - B \end{bmatrix}.$$

By Schur complement we have

$$f_{L(G_{ms})}(\mu) = \det(\mu I_{mn} - B). \det(\mu I_n - (m+r)I_n + (m-s)S_k + A(G) - D(\mu I_{mn} - B)^{-1} D^T)$$

where $D = {j_m}^T \otimes I_n - ({0_s}^T, {j_{m-s}}^T) \otimes S_{n-p}$. It is easily seen

$$\det(\mu I_{mn} - B) = \mu^{(m-s)k} (\mu - 1)^{ns+(m-s)k}.$$

Now we need to compute $D(\mu I_{mn} - B)^{-1} D^T$. It is easy to see that

$$(\mu I_{mn} - B)^{-1} = \begin{bmatrix} \frac{1}{\mu-1} I_{sn} & 0 \\ 0 & I_{m-s} \otimes E \end{bmatrix},$$

where $E = \frac{1}{\mu-1} T_k + \frac{1}{\mu} S_k$. So

$$D(\mu I_{mn} - B)^{-1} D^T$$

$$= \Big[\underbrace{I_n, \ldots, I_n}_{s times}, \underbrace{T_k, \ldots, T_k}_{m-s times} \Big] \begin{bmatrix} \frac{1}{\mu-1} I_{sn} & 0 \\ 0 & I_{m-s} \otimes E \end{bmatrix} \Big[\underbrace{I_n, \ldots, I_n}_{s times}, \underbrace{T_k, \ldots, T_k}_{m-s times} \Big]^T$$

$$= \Big[\underbrace{\frac{1}{\mu-1} I_n, \ldots, \frac{1}{\mu-1} I_n}_{s times}, \underbrace{\frac{1}{\mu-1} T_k, \ldots, \frac{1}{\mu-1} T_k}_{m-s times} \Big] \Big[\underbrace{I_n, \ldots, I_n}_{s times}, \underbrace{T_k, \ldots, T_k}_{m-s times} \Big]^T$$

$$= \frac{s}{\mu-1} I_n + \frac{m-s}{\mu-1} T_k.$$

Since G is bipartite, there is a matrix C of order $n \times (n-p)$ such that

$$\det(\mu I_n-(m+r)I_n+(m-s)S_k+A(G)-D(\mu I_{mn}-B)^{-1}D^T)$$
$$=\det(\mu I_n-(m+r)I_n+(m-s)S_k+A(G)-(\frac{s}{\mu-1}I_n+\frac{m-s}{\mu-1}T_k))$$
$$=\det\begin{bmatrix}(\mu-m-r-\frac{m}{\mu-1})I_k & C\\ C^T & (\mu-s-r-\frac{s}{\mu-1})I_k\end{bmatrix}.$$

By Schur complement we have

$$\det\begin{bmatrix}(\mu-m-r-\frac{m}{\mu-1})I_k & C\\ C^T & (\mu-s-r-\frac{s}{\mu-1})I_k\end{bmatrix}$$
$$=(\mu-s-r-\frac{s}{\mu-1})^k\det((\mu-m-r-\frac{m}{\mu-1})I_k-\frac{1}{\mu-s-r-\frac{s}{\mu-1}}CC^T)$$
$$=\det((\mu-m-r-\frac{m}{\mu-1})(\mu-s-r-\frac{s}{\mu-1})I_k-CC^T)$$
$$=f_G(\sqrt{(\mu-m-r-\frac{m}{\mu-1})(\mu-s-r-\frac{s}{\mu-1})}).$$

So we obtain

$$f_{L(G_{ms})}(\mu)$$
$$=\mu^{(m-s)k}(\mu-1)^{ns+(m-s)k}f_G(\sqrt{(\mu-m-r-\frac{m}{\mu-1})(\mu-s-r-\frac{s}{\mu-1})})$$

□

By the same argument of Theorem 2 we have the following theorem.

Theorem 3. *Let G be an r-regular bipartite graph with $|G|=n=2k$ where k is the size of each part. Then*

$$f_{Q(G_{ms})}(\nu)$$
$$=\nu^{(m-s)k}(\nu-1)^{ns+(m-s)k}f_G(\sqrt{(\nu-m-r-\frac{m}{\nu-1})(\nu-s-r-\frac{s}{\nu-1})})$$

Corollary 4. *For graph G' we have*

$$f_{L(G')}(\mu)=(\mu-1)^{ns+(m-s)k}f_G(\sqrt{(\mu-m-r-\frac{m}{\mu-1})(\mu-s-r-\frac{s}{\mu-1})}),$$

$$f_{Q(G')}(\nu)=(\nu-1)^{ns+(m-s)k}f_G(\sqrt{(\nu-m-r-\frac{m}{\nu-1})(\nu-s-r-\frac{s}{\nu-1})}).$$

Proof. The proof follows immediately from Theorems 2 and 3. □

Corollary 5. *For graph G', let $s=0$. Then*

$$f_{L(G')}(\mu)=(\mu-1)^{mk}f_G(\sqrt{(\mu-m-r-\tfrac{m}{\mu-1})(\mu-r)}),$$

$$f_{Q(G')}(\nu)=(\nu-1)^{mk}f_G(\sqrt{(\nu-m-r-\tfrac{m}{\nu-1})(\nu-r)}).$$

Proof. Apply Corollary 4, for $s=0$. □

Example 4. Let $G=K_{3,3}$, $m=3$ and $s=1$. Since $f_G(\lambda)=\lambda^4(\lambda^2-9)$ we have

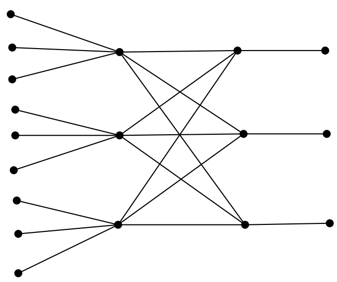
$$f_{L(G')}(\mu)=$$

$$(\mu-1)^{12}(\mu-6-\frac{3}{\mu-1})^2(\mu-4-\frac{1}{\mu-1})^2((\mu-6-\frac{3}{\mu-1})(\mu-4-\frac{1}{\mu-1})-9)$$
$$=(\mu-1)^6(\mu^2-7\mu+3)^2(\mu^2-5\mu+3)^2(\mu^4-12\mu^3+8\mu^2-18\mu)$$
$$=\mu(\mu-1)^6(\mu^2-7\mu+3)^2(\mu^2-5\mu+3)^2(\mu^3-12\mu^2+8\mu-18).$$

So the Laplacian and signless Laplacian eigenvalues of the graph G' are

$$\{(0)^{[1]},(1)^{[6]},(\frac{7+\sqrt{37}}{2})^{[2]},(\frac{7-\sqrt{37}}{2})^{[2]},(\frac{5+\sqrt{13}}{2})^{[2]},(\frac{5-\sqrt{13}}{2})^{[2]}$$
$$,(8.47467)^{[1]},(2.75414)^{[1]},(0.77119)^{[1]}\}.$$

The last three roots are found by $newgraph$.



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