Miskolc Mathematical Notes
Vol. 18 (2017), No. 1, pp. 17–29
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2017.1569

AN EXTENSION OF TOTAL GRAPH OVER A MODULE

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Received 01 March, 2015

Abstract. Let R be a commutative ring with nonzero identity and $U(R)$ its multiplicative group of units. Let M be an R-module where the collection of prime submodules is non-empty and let N_Λ be an arbitrary union of prime submodules. Also, suppose that $c \in U(R)$ such that $c^{-1} = c$. We define the extended total graph of M as a simple graph $T\Gamma_c(M, N_\Lambda)$ with vertex set M, and two distinct elements $x, y \in M$ are adjacent if and only if $x + cy \in N_\Lambda$. In this paper, we will study some graph theoretic results of $T\Gamma_c(M, N_\Lambda)$.

2010 *Mathematics Subject Classification:* 05C25; 13C99

Keywords: total graph, prime submodule

1. Introduction

Let R be commutative ring with $1 \neq 0$, $U(R)$ its multiplicative group of units and $Z(R)$ its set of zero-divisors. A proper submodule N of M is said to be a $prime\ submodule$ if whenever $rm \in N$ for some $r \in R$ and $m \in M$, then either $m \in N$ or $r \in (N :_R M)$. Clearly, if N is a prime submodule of M, then $P = (N :_R M)$ is a prime ideal of R. Let M be an R-module, $T(M)$ its set of torsion elements and $\{N_\lambda\}_{\lambda\in\Omega}$ its set of all prime submodules. The R-module M is said to be $primeless$ if $\Omega = \varnothing$. For a submodule L of an R-module M, the ideal $\{r \in R | rM \subseteq L\}$ and submodule $\{m \in M | rm \subseteq L\}$ will be denoted by $(L :_R M)$ and $(L :_M r)$, respectively. Let $N_\Lambda = \bigcup_{\lambda\in\Lambda} N_\lambda$ be a proper subset of M, and let $H_\Lambda = (N_\Lambda :_R M)$ for $\varnothing \neq \Lambda \subseteq \Omega$. It can be shown that $H_\Lambda = \bigcup_{\lambda\in\Lambda} P_\lambda$.

The total graph of R was introduced by Anderson and Badawi in [4], as the graph with all elements of R as vertices, and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$. Also they introduced in [5] the generalized total graph of R in which $Z(R)$ is extended to H, a $multiplicative-prime$ subset of R, in such away that $ab \in H$ for every $a \in H$ and $b \in R$, and whenever $ab \in H$ for all $a, b \in R$, then either $a \in H$ or $b \in H$. In fact, it is easily seen that H is a multiplicative-prime subset of R if and only if $R \setminus H$ is a saturated multiplicatively closed subset of R. Thus H is a multiplicative-prime subset of R if and only if H is a union of prime ideals of R.

In what follows, we extend the generalized total graph such that the extended total graph of M is a simple graph with vertex set M, and two distinct elements $x, y \in M$ are adjacent if and only if $x + cy \in N_\Lambda$ where $c \in U(R)$ and $c^{-1} = c$. This graph is denoted by $T\Gamma_c(M, N_\Lambda)$. In general, for $A, B \subseteq M$, $T\Gamma_c(A, B)$ is a simple graph with vertices all element of A, and two distinct vertices x and y are adjacent if and only if $x + cy \in B$.

The authors in [7] and [10] generalized the notion of a total graph to an R-module M. They considered the vertex set of a graph $T(\Gamma(M))$ as the elements of M such that two vertices are adjacent if and only if $x + y \in T(M)$. In [3], D. D. Anderson and Sangmin Chun proved that if $M \neq T(M)$, then $T(M)$ is a union of prime submodules of M. Consequently, $T\Gamma_c(M, N_\Lambda)$ is a generalization of $T(\Gamma(M)$ too. Let $c = -1$, then $\overline{T\Gamma_c}(M, N_\Lambda)$, the complement graph of $T\Gamma_c(M, N_\Lambda)$ (i.e., $T\Gamma_c(M, M \setminus N_\Lambda)$), is a Cayley graph, also let $M = R$ and N_Λ be the union of all the maximal ideals of R (i.e., $N_\Lambda = R \setminus U(R)$), then observe that $\overline{T\Gamma_1}(M, N_\Lambda)$ (i.e., $T\Gamma_1(M, M \setminus N_\Lambda)$) is the unit graph of R in the sense of [6] and $\overline{T\Gamma_{-1}}(M, N_\Lambda)$ (i.e., $T\Gamma_{-1}(M, M \setminus N_\Lambda)$) is the unitary Cayley graph in the sense of [2] and [9].

For a proper submodule L of M, $M(L) = \{m \in M | rm \in L$ for some $r \in R \setminus (L :_R M)\}$. In section 2, we will show that if $M \neq M(L)$, then there is $\Lambda \subseteq \Omega$ such that $M(L) = N_\Lambda$. In [1], the authors introduce a generalization of total graph of a module with respect to the set $M(L)$. For $\Lambda \subseteq \Omega$, N_Λ is more general than $M(L)$ since there are R-modules M and $\Lambda \subseteq \Omega$ such that N_Λ is not of the form $M(L)$ for all submodules L of M.

In section 3, we determine some basic properties of extended total graph specifically identifying its regularity and vertex transitivity. Since N_Λ is a union of prime submodules of M, the study of $T\Gamma_c(M, N_\Lambda)$ breaks naturally into two cases depending on whether or not N_Λ is a (prime) submodule of M. In section 4, we study the case when N is a (prime) submodule of M. In the final section, we do the case when N_Λ is not a submodule of M, and we improve Theorem 3.3 in [7] and Theorem 4.3 in [1] by Theorem 14, and Theorem 3.12 in [5] by Theorem 15.

Let Γ be a simple graph. We say that Γ is $totally\ disconnected$ if none of two vertices of Γ are adjacent. A subgraph Γ_1 of Γ is an $induced\ subgraph$ if vertex set of Γ_1 is contained in vertex set of Γ and two vertices of Γ_1 are adjacent if and only if they are adjacent in Γ. Throughout this paper, all subgraphs are induced. We say that two subgraphs Γ_1 and Γ_2 of Γ are $disjoint$ if Γ_1 and Γ_2 have no common vertices and no vertex of Γ_1 (resp., Γ_2) is adjacent (in Γ) to any vertex not in Γ_1 (resp., Γ_2). If vertex x is an end point of edge e, then x and e are called incident. The degree of a vertex x in a graph Γ, written $deg_\Gamma(x)$ is the number of edges incident to x. Graph Γ is called $k-regular$ if degree of each vertex of Γ is k. For vertices x and y of Γ, we define $d(x, y)$ to be the length of the shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The $diameter$ of Γ is $diam(\Gamma) = sup\{d(x, y) | x$ and y are vertices of $\Gamma\}$. The $girth$ of Γ, denoted by

$gr(\Gamma)$, is the length of a shortest cycle in Γ ($gr(\Gamma) = \infty$ if Γ contains no cycles). We denote the complete graph on n vertices by K^n and the complete bipartite graph on m and n vertices by $K^{m,n}$. We will call a $K^{1,n}$, a $stargraph$. A graph Γ is called $vertex\ transitive$ if for every two vertices x and y there exists $\tau \in Aut(\Gamma)$ such that $\tau(x) = y$.

Throughout, all rings R are commutative with $1 \neq 0$, and M is an R-module with at least one prime submodule . For $A \subseteq M$, let $A^* = A \setminus \{0\}$. As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the integers and integers modulo n, respectively.

2. $M(L)$ AS A UNION OF PRIME SUBMODULES

In this section, we consider the question of when for an R-module M, the set $M(L)$ is a union of prime submodules and determine a family of non-primeless R-modules. We refer the reader to [1] for some properties concerning $M(L)$. Throughout this section, L is a proper submodule of M over the commutative ring R. Let $M(L) = \{m \in M | rm \in L \text{ for some } r \in R \setminus (L :_R M)\}$. It is easy to see that $M(L)$ is closed under the multiplication of scalars. However $M(L)$ may not be an additive subgroup of M. Let $M = M(L)$, then $M(L)$ may or may not be a union of prime submodules. Also let $M(L)$ be a proper submodule of M, then it is a prime submodule of M, by [1, Theorem 2.1]. For our main result in this section, we need to the below theorem.

Theorem 1. *Let M be an R-module with $M \neq M(L)$, and let $\mathcal{A} = \{L_\Delta | L_\Delta$ is a submodule of M with $L_\Delta \subseteq M(L)$, and $L_\Delta = \bigcup_{h\in\Delta}(L :_M h)$ for some $\Delta \subset R\}$. Then a maximal element of $\mathcal{A}$ is a prime submodule.*

Proof. Let $L_\Delta = \bigcup_{h\in\Delta}(L :_M h)$ be a maximal element of $\mathcal{A}$. Suppose that $rm \in L_\Delta$ for some $r \in R$ and $m \in M$ such that $m \notin L_\Delta$. First, assume that $rh \in R \setminus (L :_R M)$ for every $h \in \Delta$. So each $(L :_M h) \subseteq (L :_M rh)$ and hence $L_\Delta \subseteq L'_\Delta = \bigcup_{h\in\Delta}(L :_M rh)$. Also, suppose $l_1, l_2 \in L'_\Delta$. Then $l_i \in (L :_M rh_i)$ for $i = 1, 2$. So $rl_i \in (L :_M h_i) \subseteq L_\Delta$ and hence $rl_1 + rl_2 \in L_\Delta$. Thus $rl_1 + rl_2 \in (L :_M h_j)$ for some $h_j \in \Delta$; so $l_1 + l_2 \in (L :_M rh_j) \subseteq L'_\Delta$. Since L'_Δ is clearly closed under scalar product, L'_Δ is submodule of M with $L'_\Delta \subseteq M(L)$. Thus by the maximality of L_Δ, $L_\Delta = L'_\Delta$. Now $rm \in L_\Delta$ implies $rm \in (L :_M h)$ for some $h \in \Delta$. Hence $m \in (L :_M rh) \subseteq L'_\Delta = L_\Delta$; a contradiction. Thus $rh \in (L :_R M)$ for some $h \in \Delta$ and hence $rM \subseteq (L :_M h) \subseteq L_\Delta$. Therefore L_Δ is a prime submodule of M. □

Theorem 2. *Let M be an R-module with $M \neq M(L)$, then $M(L)$ is a union of prime submodules.*

Proof. Let $l \in M(L)$, $\mathcal{A}_l = \{L_\Delta | L_\Delta$ is a submodule of M, $l \in L_\Delta \subseteq M(L)$, and $L_\Delta = \bigcup_{h\in\Delta}(L :_M h)$ for some $\Delta \subset R\}$, and $rl \in L$ where $r \in R \setminus (L :_R M)$. Then $l \in (L :_M r)$; thus $\mathcal{A}_l \neq \varnothing$. By Zorn's Lemma, $\mathcal{A}_l$ has a maximal element L_l. By above theorem, L_l is a prime submodule of $M(L)$. Therefore $M(L) = \bigcup_{l\in M(L)} L_l$ is a union of prime submodules of M. □

In the next corollary, we determine a family of non-primeless R-modules.

Corollary 1. *Let M be an R-module such that $M \neq M(L)$ for some proper submodule L of M, then M is not primeless.*

Proof. This is clear from above theorem. □

3. Basic properties of the extended total graph

The basic properties of the extended total graph are given below, independent of whether or not N_Λ is a submodule of M. Since $(c+1)(c-1) = 0 \in H_\Lambda = \bigcup_{\lambda\in\Lambda} P_\lambda$, either $c+1 \in H_\Lambda$ or $c-1 \in H_\Lambda$. First, we determine properties of $T\Gamma_c(M, N_\Lambda)$ for some $c \in U(R)$.

Theorem 3. *Let there is $c \in U(R)$ where $c \neq 1, -1$ and $c^2 = 1$.*

(1) *If $x + y \in N_\Lambda$, then either $x + cy \in N_\Lambda$ or $x - cy \in N_\Lambda$.*
(2) *If $c + 1 \notin H_\Lambda$, then $T\Gamma_c(M, N_\Lambda) \cong T\Gamma_1(M, N_\Lambda)$.*
(3) *If $c - 1 \notin H_\Lambda$, then $T\Gamma_c(M, N_\Lambda) \cong T\Gamma_{-1}(M, N_\Lambda)$.*

Proof. (1) Let $x + y \in N_\lambda$ for some $\lambda \in \Lambda$, then $(c-1)(x-cy) \in N_\lambda$. Let $x - cy \notin N_\lambda$, then $c - 1 \in P_\lambda$ hence $x + y + (c-1)y = x + cy \in N_\lambda$.

(2) Let $c+1 \notin H_\Lambda$ and $x + y \in N_\Lambda$, then $x + cy \in N_\Lambda$ since $(c+1)(x+cy) \in N_\Lambda$. Also if $x + cy \in N_\Lambda$, then $x + y \in N_\Lambda$, since $(c+1)(x+y) \in N_\Lambda$ and $c + 1 \notin H_\Lambda$.
(3) Let $c-1 \notin H_\Lambda$ and $x - y \in N_\Lambda$, then $x + cy \in N_\Lambda$ since $(c-1)(x+cy) \in N_\Lambda$. Also if $x + cy \in N_\Lambda$, then $x - y \in N_\Lambda$, since $(c-1)(x-y) \in N_\Lambda$ and $c - 1 \notin H_\Lambda$.

□

Remark 1. Suppose that $x, y \in M$ are adjacent in $T\Gamma_1(M, N_\Lambda)$, then x and cy are adjacent in $T\Gamma_c(M, N_\Lambda)$, if $x \neq cy$.

Theorem 4. *Let M be an R-module with $|N_\Lambda| = \alpha$ and $m \in M$. If $(c+1)m \notin N_\Lambda$, then $deg_{T\Gamma_c(M,N_\Lambda)}(m) = \alpha$; otherwise, $deg_{T\Gamma_c(M,N_\Lambda)}(m) = \alpha - 1$. In particular, $c + 1 \in H_\Lambda$ if and only if $T\Gamma_c(M, N_\Lambda)$ is a $(\alpha - 1)$-regular graph.*

Proof. There is a unique $x = n - cm \in M$ for every $n \in N_\Lambda$. Hence m is adjacent to x unless $x = m$. If $(c+1)m \notin N_\Lambda$, then $n - cm \neq m$ for every $n \in N_\Lambda$. Therefore $deg_{T\Gamma_c(M,N_\Lambda)}(m) = \alpha$. Otherwise, let $(c+1)m \in N_\Lambda$, then $deg_{T\Gamma_c(M,N_\Lambda)}(m) = \alpha - 1$ since m cannot be adjacent to itself. In particular, if $c + 1 \in H_\Lambda$, then $(c+1)m \in N_\Lambda$ and $deg_{T\Gamma_c(M,N_\Lambda)}(m) = \alpha - 1$ for every $m \in M$. Now suppose that $T\Gamma_c(M, N_\Lambda)$ is a $(\alpha-1)$-regular graph, $c + 1 \notin H_\Lambda$ and $m' \in M \setminus N_\Lambda$, then $deg_{T\Gamma_c(M,N_\Lambda)}(m') = \alpha$ since $(c+1)m' \notin N_\Lambda$ for every $m' \in M \setminus N_\Lambda$, a contradiction. so $c + 1 \in H_\Lambda$. □

Theorem 5. *If $c - 1 \notin H_\Lambda$, then $T\Gamma_c(M, N_\Lambda)$ is vertex transitive.*

Proof. For each $m \in M$ the mapping $\tau_m : x \longmapsto x+m$ is a permutation of the elements of M. Let $c-1 \notin H_\Lambda$, then $x+cy \in N_\Lambda$ if and only if $x+m+c(y+m) \in N_\Lambda$ since $(c-1)(x+m+c(y+m)) = (c-1)(x+cy) \in N_\Lambda$. Hence τ_m is an automorphism of $T\Gamma_c(M,N_\Lambda)$. The permutations τ_m form a subgroup of the automorphism group of $T\Gamma_c(M,N_\Lambda)$. This subgroup acts transitively on the vertices of $T\Gamma_c(M,N_\Lambda)$ because for any two vertices m and m', the automorphism $\tau_{m-m'}$ maps m to m'. □

The previous theorem gave an important property of $T\Gamma_c(M,N_\Lambda)$. The vertex transitivity identifies some properties of graph such as the edge connectivity. For further investigation on vertex transitivity and edge connectivity, see [8]. Hence if $c-1 \notin H_\Lambda$ and $T\Gamma_c(M,N_\Lambda)$ is connected (i.e., $M =< N_\Lambda >$ by Theorem 12), then the edge connectivity of $T\Gamma_c(M,N_\Lambda)$ is equal to $\alpha-1$.

Example 1. Let $M = \mathbb{Z}_3 \times \mathbb{Z}_3$, $R = \mathbb{Z}_{12}$, $c = 5$, $N_1 = \mathbb{Z}_3 \times 0$, $N_2 = 0 \times \mathbb{Z}_3$, and $N_\Lambda = N_1 \cup N_2$, then by Theorem 3(3), $T\Gamma_5(M,N_\Lambda) \cong T\Gamma_{11}(M,N_\Lambda)$. $T\Gamma_5(M,N_\Lambda)$ is a 4-regular graph by Theorem 4. Also, by Theorem 5, $T\Gamma_5(M,N_\Lambda)$ is vertex transitive.

4. The case when N_Λ is a submodule of M

We know that $N_\Lambda = \bigcup_{\lambda \in \Lambda} N_\lambda$ is a proper submodule of M if and only if N_Λ is a prime submodule of M. Moreover $P_\lambda = (N_\lambda :_R M)$ and $H_\Lambda = (N_\Lambda :_R M)$ are prime ideals of R.

Theorem 6. *Let M be an R-module such that N_Λ is a proper submodule of M. Then $T\Gamma_c(N_\Lambda,N_\Lambda)$ is a complete subgraph of $T\Gamma_c(M,N_\Lambda)$ and $T\Gamma_c(N_\Lambda,N_\Lambda)$ is disjoint from $T\Gamma_c(M \setminus N_\Lambda,N_\Lambda)$.*

Proof. It is clear that $T\Gamma_c(N_\Lambda,N_\Lambda)$ is a complete subgraph of $T\Gamma_c(M,N_\Lambda)$. Suppose that $y \in M \setminus N_\Lambda$ and $x \in N_\Lambda$ are adjacent. Then $x+cy \in N_\Lambda$ and $cy \in N_\Lambda$ hence $c \in H_\Lambda$ which is a contradiction since $c \in U(R)$. Therefore, $T\Gamma_c(N_\Lambda,N_\Lambda)$ is disjoint from $T\Gamma_c(M \setminus N_\Lambda,N_\Lambda)$. □

Definition 1. By a slice of $T\Gamma_c(M,N_\Lambda)$ we mean a subgraph of $T\Gamma_c(M,N_\Lambda)$ with a vertex set as the form $x+N_\Lambda$, denoted by S_x, for some $x \in M$.

It is easy to see that a slice of $T\Gamma_c(M,N_\Lambda)$ is an induced subgraph. So, two slices S_x and S_y are the same is and only if $x-y \in N_\Lambda$.
The next theorem gives a complete description of $T\Gamma_c(M,N_\Lambda)$. It also shows that non-isomorphic modules may have isomorphic total graphs. We allow α and β to be infinite cardinals.

Theorem 7. *Let M be an R-module such that N_Λ is a proper submodule of M, and let $H_\Lambda = (N_\Lambda :_R M)$, $|N_\Lambda| = \alpha$ and $|M/N_\Lambda| = \beta$.*

(1) *If $c+1 \in H_\Lambda$, then $T\Gamma_c(M \setminus N_\Lambda,N_\Lambda)$ is the union of $\beta-1$ disjoint K^α's.*

(2) *If $c+1 \notin H_\Lambda$, then $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is the union of $(\beta-1)/2$ disjoint $K^{\alpha,\alpha}$'s.*

Proof. (1) Assume that $c+1 \in H_\Lambda$, and let $x \in M \setminus N_\Lambda$. Then the slice S_x is a complete subgraph of $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ since $(x+n_1)+c(x+n_2) = (c+1)x+n_1+n_2 \in N_\Lambda$ for all $n_1, n_2 \in N_\Lambda$ since $c+1 \in H_\Lambda$ and N_Λ is a submodule of M. Note that distinct slices form disjoint subgraphs of $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ since if $x+n_1$ and $y+n_2$ are adjacent for some $y \in M \setminus N_\Lambda$ and $n_1, n_2 \in N_\Lambda$, then $x+cy = (x+n_1)+c(y+n_2)-(n_1+cn_2) \in N_\Lambda$ if and only if $x-y = (x+cy)-(c+1)y \in N_\Lambda$ since $c+1 \in H_\Lambda$ and N_Λ is a submodule of M. Then $S_x = S_y$, a contradiction. Thus $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is the union of $\beta-1$ disjoint subgraphs S_x, each of which is a K^α, where $\alpha = |N_\Lambda| = |x+N_\Lambda|$.

(2) Assume that $c+1 \notin H_\Lambda$, and let $x \in M \setminus N_\Lambda$. Then no two distinct elements in S_x are adjacent, suppose not. So, $(x+n_1)+c(x+n_2) \in N_\Lambda$ for $n_1, n_2 \in N_\Lambda$. This implies that $(c+1)x \in N_\Lambda$ hence $c+1 \in H_\Lambda$ since N_Λ is a prime submodule of M, a contradiction. On the other hand, two slices S_x and S_{-cx} are disjoint and each vertex of S_{-cx} is adjacent to all vertex of S_x. Thus $S_x \cup S_{-cx}$ is a complete bipartite subgraph of $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$. Furthermore, if $x+n_1$ is adjacent to $y+n_2$ for some $y \in M \setminus N_\Lambda$ and $n_1, n_2 \in N_\Lambda$, then $y+cx \in N_\Lambda$ as in part(1) above, and hence $y+N_\Lambda = -cx+N_\Lambda$. Thus $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is the union of $(\beta-1)/2$ disjoint subgraphs $S_x \cup S_{-cx}$, each of which is $K^{\alpha,\alpha}$, where $\alpha = |N_\Lambda| = |x+N_\Lambda|$

□

From the above theorem, one can easily deduce when $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is a complete or connected graph. The next theorem determines when $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is either complete or connected.

Theorem 8. *Let M be an R-module such that N_Λ is a proper submodule of M, and let $H_\Lambda = (N_\Lambda :_R M)$.*

(1) *Let $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ be a complete graph, then either $|M/N_\Lambda| = |M| = 3$ or $|M/N_\Lambda| = 2$. Its converse is true when either $|M/N_\Lambda| = |M| = 3$ and $c+1 \notin (0 :_R M)$ or $|M/N_\Lambda| = 2$.*

(2) *Let $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ be a connected graph, then either $|M/N_\Lambda| = 3$ or $|M/N_\Lambda| = 2$. Its converse is true when either $|M/N_\Lambda| = 3$ and $c-1 \in H_\Lambda$ or $|M/N_\Lambda| = 2$.*

(3) *$T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ (and hence $T\Gamma_c(N_\Lambda, N_\Lambda)$ and $T\Gamma_c(M, N_\Lambda)$) is a totally disconnected graph if and only if $N_\Lambda = \{0\}$ and $c+1 \in (0 :_R M)$.*

Proof. Let $|M/N_\Lambda| = \beta$ and $\alpha = |N_\Lambda|$

(1) Let $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ be a complete graph. Then by Theorem 7, $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is a single K^α or $K^{1,1}$. If $c+1 \in H_\Lambda$, then $\beta-1 = 1$ thus $\beta = 2$,

and hence $|M/N_\Lambda| = 2$. If $c+1 \notin H_\Lambda$, then $\alpha = 1$ and $(\beta-1)/2 = 1$. Thus $N_\Lambda = \{0\}$ and $\beta = 3$; hence $|M/N_\Lambda| = |M| = 3$. Conversely, suppose first that $M/N_\Lambda = \{N_\Lambda, x+N_\Lambda\}$, where $x \notin N_\Lambda$. Since $cx \notin N_\Lambda$, so, $x+N_\Lambda = -cx+N_\Lambda$ which implies that $(c+1)x \in N_\Lambda$. Let $m, m' \in M \setminus N_\Lambda$. Then $m+x, m'+x \in N_\Lambda$ since none of $m+x+N_\Lambda$ and $m'+x+N_\Lambda$ are equal to $x+N_\Lambda$ so $m+cm' = (m+x)+c(m'+x)-(c+1)x \in N_\Lambda$ since N_Λ is a submodule of M. Thus $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is complete. Next, suppose that $|M/N_\Lambda| = |M| = 3$; hence $H_\Lambda = (0 :_R M)$ since $N_\Lambda = \{0\}$. By Theorem 7, $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is a complete graph since $c+1 \notin (0 :_R M)$.

(2) Let $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ be connected. Then by Theorem 7, $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is a single K^α or $K^{\alpha,\alpha}$. If $c+1 \in H_\Lambda$, then $\beta-1 = 1$, and hence $|M/N_\Lambda| = 2$. If $c+1 \notin H_\Lambda$, then $(\beta-1)/2 = 1$, and hence $|M/N_\Lambda| = 3$. Conversely, by part (1) above if $|M/N_\Lambda| = 2$ then $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is complete and so it is connected. Suppose that $|M/N_\Lambda| = 3$ and $c-1 \in H_\Lambda$. First, we show that $c+1 \notin H_\Lambda$. Suppose not. Let $c+1 \in H_\Lambda$ and $M/N_\Lambda = \{N_\Lambda, x+N_\Lambda, y+N_\Lambda\}$, where $x, y \notin N_\Lambda$; it is easy to see that $x+cy \in N$. This yields that x and y are adjacent, a contradiction, by the proof of Theorem 7(1). Thus $c+1 \notin H_\Lambda$. Therefore, $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is the complete bipartite graph $K^{\alpha,\alpha}$, by Theorem 7(2).

(3) $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is totally disconnected if and only if it be a disjoint union of K^1's. so by Theorem 7, $|N_\Lambda| = 1$. Further, since m and $-cm$ are adjacent for all $m \in M$, it follows that $m = -cm$, hence $c+1 \in (0 :_R M)$.

□

The next theorem gives a more explicit description of the diameter of $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ when N_Λ is a proper submodule of M.

Theorem 9. *Let M be an R-module such that N_Λ is a proper submodule of M, and let $H_\Lambda = (N_\Lambda :_R M)$.*

(1) *$diam(T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) = 0$ if and only if $|M| = 2$.*
(2) *$diam(T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) = 1$ if and only if either $N_\Lambda \neq \{0\}$ and $|M/N_\Lambda| = 2$ or $|M| = 3$ and $c+1 \notin (0 :_R M)$.*
(3) *$diamT\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) = 2$ if and only if $N_\Lambda \neq \{0\}$, $|M/N_\Lambda| = 3$ and $c-1 \in H_\Lambda$.*
(4) *Otherwise, $diam(T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) = \infty$.*

Proof. It is clear by the proof of the Theorem 8. □

The next theorem describes the girth of $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ and $T\Gamma_c(M, N_\Lambda)$ when N_Λ is a proper submodule of M.

Theorem 10. *Let M be an R-module such that N_Λ is a proper submodule of M, and let $H_\Lambda = (N_\Lambda :_R M)$.*

(1) (a) *$gr(T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) = 3$ if and only if $c+1 \in H_\Lambda$ and $|N_\Lambda| \geq 3$.*

(b) $gr(T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) = 4$ *if and only if* $c+1 \notin H_\Lambda$ *and* $|N_\Lambda| \geq 2$.
(c) *Otherwise,* $gr(T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) = \infty$.

(2) (a) $gr(T\Gamma_c(M, N_\Lambda)) = 3$ *if and only if* $|N_\Lambda| \geq 3$.
(b) $gr(T\Gamma_c(M, N_\Lambda)) = 4$ *if and only if* $c+1 \notin H_\Lambda$ *and* $|N_\Lambda| = 2$.
(c) *Otherwise,* $gr(T\Gamma_c(M, N_\Lambda)) = \infty$.

Proof. Apply Theorem 7 and Theorem 6. □

Example 2. Let $M = \mathbb{Z}_2 \times \mathbb{Z}_3$, $R = \mathbb{Z}_{12}$ and $N_\Lambda = \mathbb{Z}_2 \times 0$, then $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is the union of 2 disjoint K^2's, if $c = 5$ or 11, by Theorem 7(1) and $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is a $K^{2,2}$, if $c = 1$ or 7, by Theorem 7(2). Now, if $c = 1$ or 7, then $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is connected with $diam T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) = 2$ and $gr(T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) = 4$ by Theorem 8(2), 9(3) and 10(1)(b), respectively, and $gr(T\Gamma_c(M, N_\Lambda)) = 4$ by Theorem 10(2)(b).

5. The case when N_Λ is not a submodule of M

In this section, we consider the remaining case when N_Λ is not a submodule of M, this implies that $|N_\Lambda| \geq 3$. So there are distinct $x, y \in N^*_\Lambda$ such that $x + cy \in M \setminus N_\Lambda$. In this case, we show that $T\Gamma_c(N_\Lambda, N_\Lambda)$ is always connected but never complete. Moreover $T\Gamma_c(N_\Lambda, N_\Lambda)$ and $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ are never disjoint subgraphs of $T\Gamma_c(M, N_\Lambda)$. We first show that $T\Gamma_c(M, N_\Lambda)$ is connected when $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is connected. However, we give an example to show that the converse fails.

Theorem 11. *Let M be an R-module and N_Λ a union of prime submodules of M that is not a submodule of M.*

(1) *$T\Gamma_c(N_\Lambda, N_\Lambda)$ is connected with $diam(T\Gamma_c(N_\Lambda, N_\Lambda)) = 2$.*
(2) *Some vertices of $T\Gamma_c(N_\Lambda, N_\Lambda)$ are adjacent to a vertex of $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$. In particular, the subgraphs $T\Gamma_c(N_\Lambda, N_\Lambda)$ and $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ of $T\Gamma_c(M, N_\Lambda)$ are not disjoint.*
(3) *If $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is connected, then $T\Gamma_c(M, N_\Lambda)$ is connected.*

Proof. (1) Every $x \in N^*_\Lambda$ is adjacent to 0. Thus $x - 0 - y$ is a path in $T\Gamma_c(N_\Lambda, N_\Lambda)$ of length two between any two distinct $x, y \in N^*_\Lambda$. Moreover, there are nonadjacent $x, y \in N^*_\Lambda$ since N_Λ is not a submodule of M; so $diam(T\Gamma_c(N_\Lambda, N_\Lambda)) = 2$.

(2) Since N_Λ is not a submodule of M, there are distinct $x, y \in N^*_\Lambda$ such that $x + cy \in M \setminus N_\Lambda$. Then $-y \in N^*_\Lambda$ and $x + cy \in M \setminus N_\Lambda$ are adjacent vertices in $T\Gamma_c(M, N_\Lambda)$ since $x + cy - cy = x \in N_\Lambda$. The "in particular" statement is clear.

(3) By part (1) above, it suffices to show that there is a path from x to y in $T\Gamma_c(M, N_\Lambda)$ for every $x \in N_\Lambda$ and $y \in M \setminus N_\Lambda$. By part (2) above, there are adjacent vertices u and v in $T\Gamma_c(N_\Lambda, N_\Lambda)$ and $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$, respectively. Since $T\Gamma_c(N_\Lambda, N_\Lambda)$ is connected, there is a path from x to

u in $T\Gamma_c(N_\Lambda, N_\Lambda)$; and since $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is connected, there is a path from v to y in $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$, then there is a path from x to y in $T\Gamma_c(M, N_\Lambda)$ since u and v are adjacent in $T\Gamma_c(M, N_\Lambda)$. Thus $T\Gamma_c(M, N_\Lambda)$ is connected.

□

Next, we determine an equivalent condition for connectedness of $T\Gamma_c(M, N_\Lambda)$ and compute $diam(T\Gamma_c(M, N_\Lambda))$. As usual, if $A \subseteq M$, then $< A >$ denotes the submodule of M generated by A.

Theorem 12. *Let M be an R-module and N_Λ a union of prime submodule of M. Then $T\Gamma_c(M, N_\Lambda)$ is connected if and only if $M =< N_\Lambda >$ (i.e., $m = n_1 + n_2 + \cdots + n_k$ for every $m \in M$ and for some $n_1, ..., n_k \in N_\Lambda$ and $k \in \mathbb{N}$).*

Proof. Suppose that $T\Gamma_c(M, N_\Lambda)$ is connected, let $m \in M$. Then there is a path $0 - m_1 - \cdots - m_\iota - m$ from 0 to m in $T\Gamma_c(M, N_\Lambda)$. Thus $m_1, m_2 + cm_1, ..., m + cm_\iota \in N_\Lambda$. Hence $m \in < m_1, m_2 + cm_1, ..., m + cm_\iota > \subseteq < N_\Lambda >$; thus $M =< N_\Lambda >$. Conversely, suppose that $M =< N_\Lambda >$. We show that there is a path from 0 to m in $T\Gamma_c(M, N_\Lambda)$ for every $0 \neq m \in M$. By hypothesis, $m = n_1 + n_2 + \cdots + n_k$ for some $n_1, ..., n_k \in N_\Lambda$ and $k \in \mathbb{N}$. Also we define a function, denoted by $f(t)$, which is equal to 0 where t is even and equal to 1 where t is odd. Let $a_0 = 0$ and $a_j = (-1)^{f(k-j)} c^{f(k-j)} (n_1 + \cdots + n_j)$ for every integer j with $1 \leq j \leq k$. Then $a_j + ca_{j+1} \in N_\Lambda$ for every integer j with $0 \leq j \leq k-1$, and thus $0 = a_0 - a_1 - \cdots - a_{k-1} - a_k = m$ is a path from 0 to m in $T\Gamma_c(M, N_\Lambda)$ of length at most k. Now, let $0 \neq u, w \in M$. Then by the preceding argument, there are paths from u to 0 and 0 to w in $T\Gamma_c(M, N_\Lambda)$. Hence there is a path from u to w in $T\Gamma_c(M, N_\Lambda)$; so $T\Gamma_c(M, N_\Lambda)$ is connected. □

In the next theorem and its corollary, we determine a relation between $T\Gamma_c(R, H_\Lambda)$ and $T\Gamma_c(M, N_\Lambda)$, where $H_\Lambda = (N_\Lambda :_R M)$.

Theorem 13. *Let M be an R-module, and N_Λ a union of prime submodules of M where $H_\Lambda = (N_\Lambda :_R M)$. If $T\Gamma_c(R, H_\Lambda)$ is connected, then $T\Gamma_c(M, N_\Lambda)$ is connected as well.*

Proof. Suppose that $T\Gamma_c(R, H_\Lambda)$ is connected and let $m \in M$. Then there is a path $0 - r_1 - \cdots - r_k - 1$ from 0 to 1 in $T\Gamma_c(R, H_\Lambda)$. Then $r_1, r_2 + cr_1, ..., r_k + cr_{k-1}, 1 + cr_k \in H_\Lambda$. Hence $0 - r_1 m - \cdots - r_k m - m$ is a path from 0 to m in $T\Gamma_c(M, N_\Lambda)$. Since all vertices may be connected via 0, $T\Gamma_c(M, N_\Lambda)$ is a connected graph. □

Corollary 2. *Let M be an R-module, and N_Λ a union of prime submodules of M and $H_\Lambda = (N_\Lambda :_R M)$. If $R =< H_\Lambda >$, then $T\Gamma_c(M, N_\Lambda)$ is connected.*

Proof. This follows directly from Theorem 12 and 13. □

Remark 2. According to the proof of the Theorem 13, if $d(0,1) = k$ in $T\Gamma_c(R, H_\Lambda)$, then $d(0, m) \leq k$ in $T\Gamma_c(M, N_\Lambda)$ for every $m \in M$.

In next theorem, we improve Theorem 3.3 of [7] and Theorem 4.3 of [1] without cosidering M as a finitely generated R-module.

Theorem 14. *Let M be an R-module and N_Λ a union of prime submodule of M that is not a submodule of M such that $M =< N_\Lambda >$ (i.e., $T\Gamma_c(M, N_\Lambda)$ is connected). If there is $k \geq 2$ which is a greatest integer i such that $m = n_1 + n_2 + \cdots + n_i$ for every $m \in M$ and for some $n_1, ..., n_i \in N_\Lambda$ where $n_1 + n_2 + \cdots + n_i$ is a shortest representation of the element m, then $diam(T\Gamma_c(M, N_\Lambda)) = k$. Otherwise, $diam(T\Gamma_c(M, N_\Lambda)) = \infty$.*

Proof. Let m and m' be distinct elements in M such that they are not adjacent. We show that there is a path from m to m' in $T\Gamma_c(M, N_\Lambda)$ with length at most k. We define a function, denoted by $f(t)$, which is equal to 0 where t is even and equal to 1 where t is odd. By the proof of Theorem 12, we can consider two path $(m - m') - x_1 - \cdots - x_{k-1} - 0$ and $(m + cm') - y_1 - \cdots - y_{k-1} - 0$ of lengths at most k. Let $a_0 = m$, $a_k = m'$, $a_j = (-1)^{f(k-j)} c^{f(k-j)} m' + x_j f(k-1) + y_j f(k)$ for every integers j with $1 \leq j \leq k-1$. Then $a_j + ca_{j+1} \in N_\Lambda$ for every integers j with $0 \leq j \leq k-1$, and thus $m = a_0 - a_1 - \cdots - a_{k-1} - a_k = m'$ is a path from m to m' in $T\Gamma_c(M, N_\Lambda)$ of length at most k. Now we show that any path from 0 to m in $T\Gamma_c(M, N_\Lambda)$ has length at least k. Suppose that $0 - b_1 - \cdots - b_{\iota-1} - m$ is a path from 0 to m in $T\Gamma_c(M, N_\Lambda)$ of length ι and let $m = n_1 + \cdots + n_k$ be a shortest representation of the element m. Thus $b_1, b_2 + cb_1, ..., b_{\iota-1} + cb_{\iota-2}, m + cb_{\iota-1} \in N_\Lambda$, and hence $m \in < b_1, b_2 + cb_1, ..., b_{\iota-1} + cb_{\iota-2}, m + cb_{\iota-1} >$. Thus $\iota \geq k$ and so the shortest path between 0 and m in $T\Gamma_c(M, N_\Lambda)$ has length k. Therefore $diam(T\Gamma_c(M, N_\Lambda)) = k$. If there is no such k, then we show that $diam(T\Gamma_c(M, N_\Lambda)) = \infty$. Suppose not. Let $diam(T\Gamma_c(M, N_\Lambda)) = t$ where $t < \infty$. Since $k = \infty$, there is $m \in M$ such that $m = n_1 + \cdots + n_{t+1}$ is a shortest representation of the element m. According what is proved, it is contradiction since there is a path from 0 to m in $T\Gamma_c(M, N_\Lambda)$ of length at most t. □

Remark 3. Let $N_\Lambda = \bigcup_{\lambda \in \Lambda} N_\lambda$ and $M =< N_\Lambda >$, then it is clear that $1 \leq k \leq |\Lambda|$ where k is as mentioned in Theorem 14.

Corollary 3. *Let M be an R-module and N_Λ a union of prime submodule of M such that $T\Gamma_c(M, N_\Lambda)$ is connected. Let $m = n_1 + \cdots + n_k$ be a shortest representation of the element m and k is as mentioned in Theorem 14.*

(1) $diam(T\Gamma_c(M, N_\Lambda)) = d(0, m)$
(2) *If* $diam(T\Gamma_c(M, N_\Lambda)) = k$ *then* $diam(T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) \geq k - 2$.

Proof. (1) This is clear from the proof of Theorem 14.
(2) Since $k = diam(T\Gamma_c(M, N_\Lambda)) = d(0, m)$ by part (1) above, let $0 - b_1 - \cdots - b_{k-1} - m$ be a shortest path from 0 to m in $T\Gamma_c(M, N_\Lambda)$. Clearly $b_1 \in N_\Lambda$. If $b_i \in N_\Lambda$ for some integer i with $2 \leq i \leq k-1$, then the path $0 - b_i -$

$\cdots - b_{k-1} - m$ from 0 to m has length less than k, a contradiction. Thus $b_i \in M \setminus N_\Lambda$ for every integer i with $2 \le i \le k-1$. Hence $b_2 - \cdots - b_{k-1} - m$ is a shortest path from b_2 to m in $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ of length $k-2$. Thus $diam(T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) \ge k-2$.

□

We next investigate the girth of $T\Gamma_c(N_\Lambda, N_\Lambda)$, $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$, and $T\Gamma_c(M, N_\Lambda)$ when N_Λ is not a submodule of M. Recall that $H_\Lambda = (N_\Lambda :_R M) = \bigcup_{\lambda \in \Lambda} P_\lambda$ where N_Λ is proper subset of M with $|N_\Lambda| \ge 3$ and $P_\lambda = (N_\lambda :_R M)$.

Theorem 15. *Let M be an R-module, and let $N_\Lambda = \bigcup_\lambda N_\lambda$ for prime submodules N_λ of M, that is not a submodule of M and $H_\Lambda = (N_\Lambda :_R M)$. Suppose that $m_1 - m_2 - m_3$ is a path of length two in $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ for distinct elements $m_1, m_2, m_3 \in M \setminus N_\Lambda$.*

(1) *If $c+1 \in H_\Lambda$ and $\bigcap_\lambda N_\lambda \ne \{0\}$, then $gr(T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) = 3$.*
(2) *If $(c+1)m_i \ne 0$ for all integers i with $1 \le i \le 3$, then $gr(T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) \le 4$.*

Proof. (1) Suppose that there is a $0 \ne h \in \bigcap_\lambda N_\lambda$. If $m_2 \ne m_1 + h$, then $m_1 - m_2 - (m_1 + h) - m_1$ is a cycle of length three in $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ since $(c+1)m_1 \in N_\Lambda$. Hence, assume that $m_2 = m_1 + h$. Since $(m_1 + h) + cm_3 = m_2 + cm_3 \in N_\Lambda$ and $h \in \bigcap_\lambda N_\lambda$, we have $m_1 + cm_3 \in N_\Lambda$. Thus $m_1 - m_2 - m_3 - m_1$ is a cycle of length three in $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$. Thus $gr(T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) = 3$.

(2) Suppose that $(c+1)m_i \ne 0$ for all integers i with $1 \le i \le 3$. Then $m_i \ne -cm_i$ for every i with $1 \le i \le 3$. There are distinct integer j,k with $1 \le j,k \le 3$ such that they are adjacent and $m_j \ne -cm_k$ since if $m_1 + cm_2 = m_3 + cm_2 = 0$, then $m_1 = m_3$, a contradiction. Thus $m_j - m_k - (-cm_k) - (-cm_j) - m_j$ is a 4-cycle in $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$; so $gr(T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) \le 4$.

□

In the above theorem, we improve the proof of Theorem 3.12(1) in [5]. Also in the part (2) of the above theorem, $0 \ne (c+1)m_i$ can belongs to N_Λ unlike Theorem 3.12(3) in [5].

Corollary 4. *Let M be an R-module, and let $N_\Lambda = \bigcup_\lambda N_\lambda$ for prime submodules N_λ of M, that is not a submodule of M and $H_\Lambda = (N_\Lambda :_R M)$. Suppose that $m_1 - m_2 - m_3$ is a path of length two in $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ for distinct elements $m_1, m_2, m_3 \in M \setminus N_\Lambda$. If $(c+1)m_i = 0$ for some integer i with $1 \le i \le 3$ and $c+1 \notin (0 :_R M)$, then $gr(T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) = 3$.*

Proof. Suppose that $c+1 \notin (0 :_R M)$. Thus $c+1 \ne 0$. Since $m_i \in M \setminus N_\Lambda$ for all integers i with $1 \le i \le 3$ and $(c+1)m_i = 0$ for some i with $1 \le i \le 3$, we have $c+1 \in P_\lambda$ for every P_λ where $P_\lambda = (N_\lambda :_R M)$. Hence $0 \ne c+1 \in \bigcap_\lambda P_\lambda$.

Since $c+1 \notin (0:_R M)$, there is a $m \in M$ such that $0 \neq (c+1)m \in \bigcap_\lambda N_\lambda$. Thus $gr(T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) = 3$ by Theorem 15(1). □

Corollary 5. *Let M be an R-module, and let $N_\Lambda = \bigcup_\lambda N_\lambda$ for prime submodules N_λ of M and $H_\Lambda = (N_\Lambda :_R M)$. Suppose that $m_1 - m_2 - m_3$ is a path of length two in $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ for distinct elements $m_1, m_2, m_3 \in M \setminus N_\Lambda$, $c+1 \in H_\Lambda$, $\bigcap_\lambda N_\lambda \neq \{0\}$ and $|N_\Lambda| \geq 3$, then $gr(T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)) = 3$.*

Proof. This follows directly from Theorem 10(1)(a) and Theorem 15(1). □

Theorem 16. *Let M be an R-module and N_Λ a union of prime submodule of M that is not a submodule of M.*

1. *Either $gr(T\Gamma_c(N_\Lambda, N_\Lambda)) = 3$ or $gr(T\Gamma_c(N_\Lambda, N_\Lambda)) = \infty$. Moreover, if one has $gr(T\Gamma_c(N_\Lambda, N_\Lambda)) = \infty$, then $|N_\lambda| = 2$ for any $\lambda \in \Lambda$ where N_λ is a non-zero submodule of M. Also,$T\Gamma_c(N_\Lambda, N_\Lambda)$ is a star graph.*
2. *$gr(T\Gamma_c(M, N_\Lambda)) = 3$ if and only if $gr(T\Gamma_c(N_\Lambda, N_\Lambda)) = 3$.*
3. *$gr(T\Gamma_c(M, N_\Lambda)) \leq 4$.*
4. *$gr(T\Gamma_c(M, N_\Lambda)) = 4$ if and only if $gr(T\Gamma_c(N_\Lambda, N_\Lambda)) = \infty$.*

Proof.
(1) If $n + cn' \in N_\Lambda$ for some distinct $n, n' \in N_\Lambda^*$, then $0 - n - n' - 0$ is a 3-cycle in $T\Gamma_c(N_\Lambda, N_\Lambda)$; so $gr(T\Gamma_c(N_\Lambda, N_\Lambda)) = 3$. Otherwise, $n + cn' \in M \setminus N_\Lambda$ for all distinct $n, n' \in N_\Lambda^*$. So in this case, every $n \in N_\Lambda$ is adjacent to 0, and no two distinct $n, n' \in N_\Lambda^*$ are adjacent. Thus $T\Gamma_c(N_\Lambda, N_\Lambda)$ is a star graph with center 0; so $gr(T\Gamma_c(N_\Lambda, N_\Lambda)) = \infty$. Moreover, let $N_\Lambda = \bigcup_{\lambda \in \Lambda} N_\lambda$ is not a submodule of M where N_λ is a prime submodule of M so $|\Lambda| \geq 2$. Assume that $gr(T\Gamma_c(N_\Lambda, N_\Lambda)) = \infty$. Then $x + cy \in M \setminus N_\Lambda$ for all distinct $x, y \in N_\Lambda^*$, and thus if $N_\lambda \neq \{0\}$, then each $|N_\lambda| = 2$.
(2) It suffices to show that $gr(T\Gamma_c(N_\Lambda, N_\Lambda)) = 3$ when $gr(T\Gamma_c(M, N_\Lambda)) = 3$. Let $(c+1)n \neq 0$ for some $n \in N_\Lambda^*$, then $0 - n - (-cn) - 0$ is a 3-cycle in $T\Gamma_c(N_\Lambda, N_\Lambda)$. Otherwise, $(c+1)n = 0$ for all $n \in N_\Lambda$. Since N_Λ is not a submodule of M, there are distinct elements $n, n' \in N_\Lambda$ such that $n + n' \in M \setminus N_\Lambda$. Then $(c+1)(n+n') = 0$, thus $c+1 \in H_\Lambda$. Let $m - m_1 - m_2 - m$ be a 3-cycle in $T\Gamma_c(M, N_\Lambda)$. Then $n_1 = cm + m_1, n_2 = cm + m_2, m_1 + cm_2 \in N_\Lambda$. Thus $0 - n_1 - n_2 - 0$ is a 3-cycle in $T\Gamma_c(N_\Lambda, N_\Lambda)$; therefore $gr(T\Gamma_c(N_\Lambda, N_\Lambda)) = 3$.
(3) Since N_Λ is not a submodule of M, there are distinct elements $n, n' \in N_\Lambda$ such that $n + n' \in M \setminus N_\Lambda$. Then $0 - (-cn) - n + n' - (-cn') - 0$ is a 4-cycle in $T\Gamma_c(M, N_\Lambda)$.
(4) This follows by parts (1), (2) and (3) above.

□

Example 3. (a) Let $M = \mathbb{Z}[X]$ be an $\mathbb{Z}[X]$-module. Then $N_\Lambda = \mathbb{Z}[X] \setminus \mathbb{Z}^*$ is a union of prime submodule of M, that is not a submodule of M. Thus $T\Gamma_c(N_\Lambda, N_\Lambda)$ is connected with $diam(T\Gamma_c(N_\Lambda, N_\Lambda)) = 2$ by Theorem

11(1). Moreover, by Theorem 12 and 14, $T\Gamma_c(M, N_\Lambda)$ is connected with $diam(T\Gamma_c(M, N_\Lambda)) = 2$ since $z = X + z - X$ for $X, z - X \in N_\Lambda$ for every $z \in \mathbb{Z}^*$. However, $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$ is not connected since there is no path from 1 to 2 in $T\Gamma_c(M \setminus N_\Lambda, N_\Lambda)$. Thus the converse of Theorem 11(3) need not hold.

(b) Let $k \in \mathbb{N}$, $M = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $R = \mathbb{Z}_{2k}$, $N_1 = \mathbb{Z}_2 \times 0 \times 0$, $N_2 = 0 \times \mathbb{Z}_2 \times 0$, $N_3 = 0 \times 0 \times \mathbb{Z}_2$ and $N_\Lambda = \bigcup N_j$ for $1 \leq j \leq 3$, then by Theorem 12 and 14, $T\Gamma_c(M, N_\Lambda)$ is connected with $diam(T\Gamma_c(M, N_\Lambda)) = 3$, note that $(1,1,0)$ is the sum of two elements of N_Λ but $(1,1,1)$ is the sum of three element of N_Λ, and $T\Gamma_c(R, H_\Lambda)$ is disconnected so the converse of Theorem 13 is fails, also $gr(T\Gamma_c(N_\Lambda, N_\Lambda)) = \infty$ and $gr(T\Gamma_c(M, N_\Lambda)) = 4$ by Theorem 16.

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