AN EXTENSION OF TOTAL GRAPH OVER A MODULE

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Abstract. Let $R$ be a commutative ring with nonzero identity and $U(R)$ its multiplicative group of units. Let $M$ be an $R$-module where the collection of prime submodules is non-empty and let $N_A$ be an arbitrary union of prime submodules. Also, suppose that $c \in U(R)$ such that $c^{-1} = c$. We define the extended total graph of $M$ as a simple graph $T^c(M, N)$ with vertex set $M$, and two distinct elements $x, y \in M$ are adjacent if and only if $x + cy \in N_A$. In this paper, we will study some graph theoretic results of $T^c(M, N)$.

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1. INTRODUCTION

Let $R$ be commutative ring with $1 \neq 0$, $U(R)$ its multiplicative group of units and $Z(R)$ its set of zero-divisors. A proper submodule $N$ of $M$ is said to be a prime submodule if whenever $rm \in N$ for some $r \in R$ and $m \in M$, then either $m \in N$ or $r \in (N :_RM)$. Clearly, if $N$ is a prime submodule of $M$, then $P = (N :_RM)$ is a prime ideal of $R$. Let $M$ be an $R$-module, $T(M)$ its set of torsion elements and $\{N_\lambda\}_{\lambda \in \Omega}$ its set of all prime submodules. The $R$-module $M$ is said to be primeless if $\Omega = \emptyset$. For a submodule $L$ of an $R$-module $M$, the ideal $\{r \in R | rM \subseteq L\}$ and submodule $\{m \in M | rm \subseteq L\}$ will be denoted by $(L :_RM)$ and $(L :_MR)$, respectively. Let $N_A = \bigcup_{\lambda \in A} N_\lambda$ be a proper subset of $M$, and let $H_A = (N_A :_RM)$ for $\emptyset \neq A \subseteq \Omega$. It can be shown that $H_A = \bigcup_{\lambda \in A} P_\lambda$.

The total graph of $R$ was introduced by Anderson and Badawi in [4], as the graph with all elements of $R$ as vertices, and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$. Also they introduced in [5] the generalized total graph of $R$ in which $Z(R)$ is extended to $H$, a multiplicative prime subset of $R$, in such away that $ab \in H$ for every $a \in H$ and $b \in R$, and whenever $ab \in H$ for all $a, b \in R$, then either $a \in H$ or $b \in H$. In fact, it is easily seen that $H$ is a multiplicative-prime subset of $R$ if and only if $R \setminus H$ is a saturated multiplicatively closed subset of $R$. Thus $H$ is a multiplicative-prime subset of $R$ if and only if $H$ is a union of prime ideals of $R$.

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In what follows, we extend the generalized total graph such that the extended total graph of $M$ is a simple graph with vertex set $M$, and two distinct elements $x, y \in M$ are adjacent if and only if $x + cy \in N_A$ where $c \in U(R)$ and $c^{-1} = c$. This graph is denoted by $T\Gamma_c(M, N_A)$. In general, for $A, B \subseteq M$, $T\Gamma_c(A, B)$ is a simple graph with vertices all element of $A$, and two distinct vertices $x$ and $y$ are adjacent if and only if $x + cy \in B$.

The authors in [7] and [10] generalized the notion of a total graph to an $R$-module $M$. They considered the vertex set of a graph $T(\Gamma(M))$ as the elements of $M$ such that two vertices are adjacent if and only if $x + y \in T(M)$. In [3], D. D. Anderson and Sangmin Chun proved that if $M \neq T(M)$, then $T(M)$ is a union of prime submodules of $M$. Consequently, $T\Gamma_c(M, N_A)$ is a generalization of $T(\Gamma(M))$ too. Let $c = -1$, then $\overline{T\Gamma_c(M, N_A)}$, the complement graph of $T\Gamma_c(M, N_A)$ (i.e., $T\Gamma_c(M, M \setminus N_A)$), is a Cayley graph, also let $M = R$ and $N_A$ be the union of all the maximal ideals of $R$ (i.e., $N_A = R \setminus U(R)$), then observe that $\overline{T\Gamma_1(M, N_A)}$ (i.e., $T\Gamma_1(M, M \setminus N_A)$) is the unit graph of $R$ in the sense of [6] and $\overline{T\Gamma_{-1}(M, N_A)}$ (i.e., $T\Gamma_{-1}(M, M \setminus N_A)$) is the unitary Cayley graph in the sense of [2] and [9].

For a proper submodule $L$ of $M$, $M(L) = \{m \in M | rm \in L$ for some $r \in R \setminus L : R M\}$. In section 2, we will show that if $M \neq M(L)$, then there is $\Lambda \subseteq \Omega$ such that $M(L) = N_A$. In [1], the authors introduce a generalization of total graph of a module with respect to the set $M(L)$. For $\Lambda \subseteq \Omega$, $N_A$ is more general than $M(L)$ since there are $R$-modules $M$ and $\Lambda \subseteq \Omega$ such that $N_A$ is not of the form $M(L)$ for all submodules $L$ of $M$.

In section 3, we determine some basic properties of extended total graph specifically identifying its regularity and vertex transitivity. Since $N_A$ is a union of prime submodules of $M$, the study of $T\Gamma_c(M, N_A)$ breaks naturally into two cases depending on whether or not $N_A$ is a (prime) submodule of $M$. In section 4, we study the case when $N$ is a (prime) submodule of $M$. In the final section, we do the case when $N_A$ is not a submodule of $M$, and we improve Theorem 3.3 in [7] and Theorem 4.3 in [1] by Theorem 14, and Theorem 3.12 in [5] by Theorem 15.

Let $\Gamma$ be a simple graph. We say that $\Gamma$ is totally disconnected if none of two vertices of $\Gamma$ are adjacent. A subgraph $\Gamma_1$ of $\Gamma$ is an induced subgraph if vertex set of $\Gamma_1$ is contained in vertex set of $\Gamma$ and two vertices of $\Gamma_1$ are adjacent if and only if they are adjacent in $\Gamma$. Throughout this paper, all subgraphs are induced. We say that two subgraphs $\Gamma_1$ and $\Gamma_2$ of $\Gamma$ are disjoint if $\Gamma_1$ and $\Gamma_2$ have no common vertices and no vertex of $\Gamma_1$ (resp., $\Gamma_2$) is adjacent (in $\Gamma$) to any vertex not in $\Gamma_1$ (resp., $\Gamma_2$). If vertex $x$ is an end point of edge $e$, then $x$ and $e$ are called incident. The degree of a vertex $x$ in a graph $\Gamma$, written $\text{deg}_\Gamma(x)$ is the number of edges incident to $x$. Graph $\Gamma$ is called $k$-regular if degree of each vertex of $\Gamma$ is $k$. For vertices $x$ and $y$ of $\Gamma$, we define $d(x, y)$ to be the length of the shortest path from $x$ to $y$ ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The diameter of $\Gamma$ is $\text{diam}(\Gamma) = \sup\{d(x, y) | x$ and $y$ are vertices of $\Gamma\}$. The girth of $\Gamma$, denoted by
is a union of prime submodules of $M$ above theorem, $L$ scalar product, for some $rM$ and hence $rM$ is a submodule of $M$ with $M.L/$ is a proper submodule of $M$. For $A \subseteq M$, let $A^* = A \setminus \{0\}$. As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the integers and integers modulo $n$, respectively.

2. $M(L)$ as a Union of Prime Submodules

In this section, we consider the question of when for an $R$-module $M$, the set $M(L)$ is a union of prime submodules and determine a family of non-primeless $R$-modules. We refer the reader to [1] for some properties concerning $M(L)$. Throughout this section, $L$ is a proper submodule of $M$ over the commutative ring $R$. Let $M(L) = \{m \in M \mid rm \in L \text{ for some } r \in R \setminus (L :_RM)\}$. It is easy to see that $M(L)$ is closed under the multiplication of scalars. However $M(L)$ may not be an additive subgroup of $M$. Let $M = M(L)$, then $M(L)$ may or may not be a union of prime submodules. Also let $M(L)$ be a proper submodule of $M$, then it is a prime submodule of $M$, by [1, Theorem 2.1]. For our main result in this section, we need to the below theorem.

**Theorem 1.** Let $M$ be an $R$-module with $M \neq M(L)$, and let $A = \{L \in M \mid L \text{ is a submodule of } M \text{ with } L \subseteq M(L) \text{ and } L = \bigcup_{h \in A}(L :_M h) \text{ for some } A \subseteq R \}$. Then a maximal element of $A$ is a prime submodule.

**Proof.** Let $L = \bigcup_{h \in A}(L :_M h)$ be a maximal element of $A$. Suppose that $rm \in L$ for some $r \in R$ and $m \in M$ such that $m \notin L$. First, assume that $rh \in R \setminus (L :_RM)$ for every $h \in A$. So each $(L :_M h) \subseteq (L :_M rh)$ and hence $L \subseteq L' = \bigcup_{h \in A}(L :_M rh)$. Also, suppose $l_1, l_2 \in L'$. Then $l_i \in (L :_M rh_i)$ for $i = 1, 2$. So $l_i \in (L :_M h_i) \subseteq L$ and hence $l_1 + l_2 \in L$. Thus $l_1 + l_2 \in (L :_M rh_i)$ for some $h_j \in A$; so $l_1 + l_2 \in (L :_M rh_i) \subseteq L'$. Since $L'$ is clearly closed under scalar product, $L'$ is a submodule of $M$ with $L' \subseteq M(L)$. Now by the maximality of $L$, $L = L'$, so $rm \in L$ implies $rm \in (L :_M h)$ for some $h \in A$. Hence $m \in (L :_M rh) \subseteq L' = L$; a contradiction. Thus $rh \in (L :_RM)$ for some $h \in A$ and hence $rM \subseteq (L :_M h) \subseteq L$. Therefore $L$ is a prime submodule of $M$. □

**Theorem 2.** Let $M$ be an $R$-module with $M \neq M(L)$, then $M(L)$ is a union of prime submodules.

**Proof.** Let $l \in M(L)$, $A_l = \{L \in M(L) \mid L \subseteq M(L) \}$, and $L = \bigcup_{h \in A}(L :_M h)$ for some $A \subseteq R$}, and $r \in L$ where $r \in R \setminus (L :_RM)$. Then $l \in (L :_M r)$; thus $A_l \neq \emptyset$. By Zorn’s Lemma, $A_l$ has a maximal element $L$. By above theorem, $L$ is a prime submodule of $M(L)$. Therefore $M(L) = \bigcup_{l \in M(L)} L$ is a union of prime submodules of $M$. □
In the next corollary, we determine a family of non-primeless R-modules.

**Corollary 1.** Let $M$ be an R-module such that $M \neq M(L)$ for some proper submodule $L$ of $M$, then $M$ is not primeless.

**Proof.** This is clear from above theorem.

3. **Basic properties of the extended total graph**

The basic properties of the extended total graph are given below, independent of whether or not $N_A$ is a submodule of $M$. Since $(c+1)(c-1) = 0 \in H_A = \bigcup_{\lambda \in A} P_{\lambda}$, either $c + 1 \in H_A$ or $c - 1 \in H_A$. First, we determine properties of $T\Gamma_c(M, N_A)$ for some $c \in U(R)$.

**Theorem 3.** Let there is $c \in U(R)$ where $c \neq 1, -1$ and $c^2 = 1$.

1. If $x + y \in N_A$, then either $x + cy \in N_A$ or $x - cy \in N_A$.
2. If $c + 1 \notin H_A$, then $T\Gamma_c(M, N_A) \cong T\Gamma_1(M, N_A)$.
3. If $c - 1 \notin H_A$, then $T\Gamma_c(M, N_A) \cong T\Gamma_{-1}(M, N_A)$.

**Proof.**

(1) Let $x + y \in N_{\lambda}$ for some $\lambda \in A$, then $(c-1)(x-cy) \in N_{\lambda}$. Let $x - cy \notin N_{\lambda}$, then $c - 1 \notin P_{\lambda}$ hence $x + y + (c-1)y = x + cy \in N_{\lambda}$.

(2) Let $c + 1 \notin H_A$ and $x + y \in N_A$, then $x + cy \in N_A$ since $(c+1)(x+cy) \in N_A$. Also if $x + cy \in N_A$, then $x + y \in N_A$, since $(c+1)(x+y) \in N_A$ and $c+1 \notin H_A$.

(3) Let $c - 1 \notin H_A$ and $x - y \in N_A$, then $x + cy \in N_A$ since $(c-1)(x+cy) \in N_A$. Also if $x + cy \in N_A$, then $x - y \in N_A$, since $(c-1)(x-y) \in N_A$ and $c-1 \notin H_A$.

**Remark 1.** Suppose that $x, y \in M$ are adjacent in $T\Gamma_1(M, N_A)$, then $x$ and $cy$ are adjacent in $T\Gamma_c(M, N_A)$, if $x \neq cy$.

**Theorem 4.** Let $M$ be an R-module with $|N_A| = \alpha$ and $m \in M$. If $(c+1)m \notin N_A$, then $\deg_{T\Gamma_c(M, N_A)}(m) = \alpha$; otherwise, $\deg_{T\Gamma_c(M, N_A)}(m) = \alpha - 1$. In particular, $c + 1 \in H_A$ if and only if $T\Gamma_c(M, N_A)$ is a $(\alpha - 1)$-regular graph.

**Proof.** There is a unique $x = n - cm \in M$ for every $n \in N_A$. Hence $m$ is adjacent to $x$ unless $x = m$. If $(c+1)m \notin N_A$, then $n - cm \neq m$ for every $n \in N_A$. Therefore $\deg_{T\Gamma_c(M, N_A)}(m) = \alpha$. Otherwise, let $(c+1)m \in N_A$, then $\deg_{T\Gamma_c(M, N_A)}(m) = \alpha - 1$ since $m$ cannot be adjacent to itself. In particular, if $c + 1 \in H_A$, then $(c+1)m \in N_A$ and $\deg_{T\Gamma_c(M, N_A)}(m) = \alpha - 1$ for every $m \in M$. Now suppose that $T\Gamma_c(M, N_A)$ is a $(\alpha - 1)$-regular graph, $c + 1 \notin H_A$ and $m' \in M \setminus N_A$, then $\deg_{T\Gamma_c(M, N_A)}(m') = \alpha$ since $(c+1)m' \notin N_A$ for every $m' \in M \setminus N_A$, a contradiction. so $c + 1 \in H_A$.

**Theorem 5.** If $c - 1 \notin H_A$, then $T\Gamma_c(M, N_A)$ is vertex transitive.
Proof. For each \( m \in M \) the mapping \( \tau_m : x \mapsto x + m \) is a permutation of the elements of \( M \). Let \( c - 1 \notin H_A \), then \( x + cy \in N_A \) if and only if \( x + m + c(y + m) \in N_A \) since \((c - 1)(x + m + c(y + m)) = (c - 1)(x + cy) \in N_A \). Hence \( \tau_m \) is an automorphism of \( TG_c(M, N_A) \). The permutations \( \tau_m \) form a subgroup of the automorphism group of \( TG_c(M, N_A) \). This subgroup acts transitively on the vertices of \( TG_c(M, N_A) \) because for any two vertices \( m \) and \( m' \), the automorphism \( \tau_{m - m'} \) maps \( m \) to \( m' \).

The previous theorem gave an important property of \( TG_c(M, N_A) \). The vertex transitivity identifies some properties of graph such as the edge connectivity. For further investigation on vertex transitivity and edge connectivity, see \cite{8}. Hence if \( c - 1 \notin H_A \) and \( TG_c(M, N_A) \) is connected (i.e., \( M = \Phi N_A \) by Theorem 12), then the edge connectivity of \( TG_c(M, N_A) \) is equal to \( \alpha - 1 \).

Example 1. Let \( M = Z_3 \times Z_3, R = Z_{12}, c = 5, N_1 = Z_3 \times 0, N_2 = 0 \times Z_3, \) and \( N_A = N_1 \cup N_2, \) then by Theorem 3(3), \( TG_5(M, N_A) \) is a 4-regular graph by Theorem 4. Also, by Theorem 5, \( TG_5(M, N_A) \) is vertex transitive.

4. THE CASE WHEN \( N_A \) IS A SUBMODULE OF \( M \)

We know that \( N_A = \bigcup_{\lambda \in A} N_\lambda \) is a proper submodule of \( M \) if and only if \( N_A \) is a prime submodule of \( M \). Moreover \( P_\lambda = (N_\lambda :_R M) \) and \( H_A = (N_A :_R M) \) are prime ideals of \( R \).

Theorem 6. Let \( M \) be an \( R \)-module such that \( N_A \) is a proper submodule of \( M \). Then \( TG_c(N_A, N_A) \) is a complete subgraph of \( TG_c(M, N_A) \) and \( TG_c(N_A, N_A) \) is disjoint from \( TG_c(M \setminus N_A, N_A) \).

Proof. It is clear that \( TG_c(N_A, N_A) \) is a complete subgraph of \( TG_c(M, N_A) \). Suppose that \( y \in M \setminus N_A \) and \( x \in N_A \) are adjacent. Then \( x + cy \in N_A \) and \( cy \in N_A \) hence \( c \in H_A \) which is a contradiction since \( c \in U(R) \). Therefore, \( TG_c(N_A, N_A) \) is disjoint from \( TG_c(M \setminus N_A, N_A) \).

Definition 1. By a slice of \( TG_c(M, N_A) \) we mean a subgraph of \( TG_c(M, N_A) \) with a vertex set as the form \( x + N_A \), denoted by \( S_x \), for some \( x \in M \).

It is easy to see that a slice of \( TG_c(M, N_A) \) is an induced subgraph. So, two slices \( S_x \) and \( S_y \) are the same is and only if \( x - y \in N_A \).

The next theorem gives a complete description of \( TG_c(M, N_A) \). It also shows that non-isomorphic modules may have isomorphic total graphs. We allow \( \alpha \) and \( \beta \) to be infinite cardinals.

Theorem 7. Let \( M \) be an \( R \)-module such that \( N_A \) is a proper submodule of \( M \), and let \( H_A = (N_A :_R M), |N_A| = \alpha \) and \( |M/N_A| = \beta \).

1. If \( c + 1 \in H_A \), then \( TG_c(M \setminus N_A, N_A) \) is the union of \( \beta - 1 \) disjoint \( K^\alpha \)'s.
(2) If \(c + 1 \notin H_A\), then \(T \Gamma_c(M \setminus N_A, N_A)\) is the union of \((\beta - 1)/2\) disjoint \(K^{\alpha, \beta}\)s.

Proof. (1) Assume that \(c + 1 \in H_A\), and let \(x \in M \setminus N_A\). Then the slice \(S_x\) is a complete subgraph of \(T \Gamma_c(M \setminus N_A, N_A)\) since \((x + n_1) + c(x + n_2) = (c + 1)x + n_1 + n_2 \in N_A\) for all \(n_1, n_2 \in N_A\) since \(c + 1 \in H_A\) and \(N_A\) is a submodule of \(M\). Note that distinct slices form disjoint subgraphs of \(T \Gamma_c(M \setminus N_A, N_A)\) since \(x + n_1 + y + n_2\) are adjacent for some \(y \in M \setminus N_A\) and \(n_1, n_2 \in N_A\), then \(x + cy = (x + n_1) + c(y + n_2) - (n_1 + cn_2) \in N_A\) and only if \(x - y = (x + cy) - (c + 1)y \in N_A\) since \(c + 1 \in H_A\) and \(N_A\) is a submodule of \(M\). Then \(S_x = S_y\), a contradiction. Thus \(T \Gamma_c(M \setminus N_A, N_A)\) is the union of \(\beta - 1\) disjoint subgraphs \(S_x\), each of which is a \(K^{\alpha}\), where \(\alpha = |N_A| = |x + N_A|\).

(2) Assume that \(c + 1 \notin H_A\), and let \(x \in M \setminus N_A\). Then no two distinct elements in \(S_x\) are adjacent, suppose not. So, \((x + n_1) + c(x + n_2) \in N_A\) for \(n_1, n_2 \in N_A\). This implies that \((c + 1)x \in N_A\) hence \(c + 1 \in H_A\) since \(N_A\) is a prime submodule of \(M\), a contradiction. On the other hand, two slices \(S_x\) and \(S_{-cx}\) are disjoint and each vertex of \(S_{-cx}\) is adjacent to all vertex of \(S_x\). Thus \(S_x \cup S_{-cx}\) is a complete bipartite subgraph of \(T \Gamma_c(M \setminus N_A, N_A)\). Furthermore, if \(x + n_1\) is adjacent to \(y + n_2\) for some \(y \in M \setminus N_A\) and \(n_1, n_2 \in N_A\), then \(y + cx \in N_A\) as in part (1) above, and hence \(y + N_A = -cx + N_A\). Thus \(T \Gamma_c(M \setminus N_A, N_A)\) is the union of \((\beta - 1)/2\) disjoint subgraphs \(S_x \cup S_{-cx}\), each of which is \(K^{\alpha, \beta}\), where \(\alpha = |N_A| = |x + N_A|\).

\[ \square \]

From the above theorem, one can easily deduce when \(T \Gamma_c(M \setminus N_A, N_A)\) is a complete or connected graph. The next theorem determines when \(T \Gamma_c(M \setminus N_A, N_A)\) is either complete or connected.

**Theorem 8.** Let \(M\) be an \(R\)-module such that \(N_A\) is a proper submodule of \(M\), and let \(H_A = (N_A : R M)\).

(1) Let \(T \Gamma_c(M \setminus N_A, N_A)\) be a complete graph, then either \(|M/N_A| = |M| = 3\) or \(|M/N_A| = 2\). Its converse is true when either \(|M/N_A| = |M| = 3\) and \(c + 1 \notin (0 : R M)\) or \(|M/N_A| = 2\).

(2) Let \(T \Gamma_c(M \setminus N_A, N_A)\) be a connected graph, then either \(|M/N_A| = 3\) or \(|M/N_A| = 2\). Its converse is true when either \(|M/N_A| = 3\) and \(c + 1 \notin H_A\) or \(|M/N_A| = 2\).

(3) \(T \Gamma_c(M \setminus N_A, N_A)\) (and hence \(T \Gamma_c(N_A, N_A)\) and \(T \Gamma_c(M, N_A)\)) is a totally disconnected graph if and only if \(N_A = \{0\}\) and \(c + 1 \in (0 : R M)\).

Proof. Let \(|M/N_A| = \beta\) and \(\alpha = |N_A|\).

(1) Let \(T \Gamma_c(M \setminus N_A, N_A)\) be a complete graph. Then by Theorem 7, \(T \Gamma_c(M \setminus N_A, N_A)\) is a single \(K^{\alpha}\) or \(K^{1,1}\). If \(c + 1 \in H_A\), then \(\beta - 1 = 1\) thus \(\beta = 2\),
and hence \( |M/N_A| = 2 \). If \( c + 1 \notin H_A \), then \( \alpha = 1 \) and \( (\beta - 1)/2 = 1 \). Thus \( N_A = \{0\} \) and \( \beta = 3 \); hence \( |M/N_A| = |M| = 3 \). Conversely, suppose first that \( M/N_A = \{N_A, x + N_A\} \), where \( x \notin N_A \). Since \( cx \notin N_A \), so \( x + N_A = -cx + N_A \) implies that \( (c + 1)x \in N_A \). Let \( m, m' \in M \setminus N_A \). Then \( m + x, m + x \in N_A \) since none of \( m + x + N_A \) and \( m + x + N_A \) are equal to \( x + N_A \) so \( m + cm' = (m + x) + (m' + x) - (c + 1)x \in N_A \) since \( N_A \) is a submodule of \( M \). Thus \( \Gamma_c(M \setminus N_A, N_A) \) is complete. Next, suppose that \( |M/N_A| = |M| = 3 \); hence \( H_A = (0 : R M) \) since \( N_A = \{0\} \). By Theorem 7, \( \Gamma_c(M \setminus N_A, N_A) \) is a complete graph since \( c + 1 \notin (0 : R M) \).

(2) Let \( \Gamma_c(M \setminus N_A, N_A) \) be connected. Then by Theorem 7, \( \Gamma_c(M \setminus N_A, N_A) \) is a single \( K^\alpha \) or \( K^{\alpha, \alpha} \). If \( c + 1 \in H_A \), then \( \beta - 1 = 1 \), and hence \( |M/N_A| = 2 \). If \( c + 1 \notin H_A \), then \( (\beta - 1)/2 = 1 \), and hence \( |M/N_A| = 3 \). Conversely, by part (1) above if \( |M/N_A| = 2 \) then \( \Gamma_c(M \setminus N_A, N_A) \) is complete and so it is connected. Suppose that \( |M/N_A| = 3 \) and \( c - 1 \in H_A \). First, we show that \( c + 1 \notin H_A \). Suppose not. Let \( c + 1 \in H_A \) and \( M/N_A = \{N_A, x + N_A, y + N_A\} \), where \( x, y \notin N_A \); it is easy to see that \( x + cy \in N \). This yields that \( x \) and \( y \) are adjacent, a contradiction, by the proof of Theorem 7(1). Thus \( c + 1 \notin H_A \). Therefore, \( \Gamma_c(M \setminus N_A, N_A) \) is the complete bipartite graph \( K^{\alpha, \alpha} \), by Theorem 7(2).

(3) \( \Gamma_c(M \setminus N_A, N_A) \) is totally disconnected if and only if it be a disjoint union of \( K^1 \)'s. so by Theorem 7, \( |N_A| = 1 \). Further, since \( m \) and \( -cm \) are adjacent for all \( m \in M \), it follows that \( m = -cm \), hence \( c + 1 \in (0 : R M) \).

The next theorem gives a more explicit description of the diameter of \( \Gamma_c(M \setminus N_A, N_A) \) when \( N_A \) is a proper submodule of \( M \).

**Theorem 9.** Let \( M \) be an \( R \)-module such that \( N_A \) is a proper submodule of \( M \), and let \( H_A = (N_A : R M) \).

(1) \( \text{diam}(\Gamma_c(M \setminus N_A, N_A)) = 0 \) if and only if \( |M| = 2 \).

(2) \( \text{diam}(\Gamma_c(M \setminus N_A, N_A)) = 1 \) if and only if either \( N_A \neq \{0\} \) and \( |M/N_A| = 2 \) or \( |M| = 3 \) and \( c + 1 \notin (0 : R M) \).

(3) \( \text{diam}(\Gamma_c(M \setminus N_A, N_A)) = 2 \) if and only if \( N_A \neq \{0\} \), \( |M/N_A| = 3 \) and \( c - 1 \in H_A \).

(4) Otherwise, \( \text{diam}(\Gamma_c(M \setminus N_A, N_A)) = \infty \).

**Proof.** It is clear by the proof of the Theorem 8.

The next theorem describes the girth of \( \Gamma_c(M \setminus N_A, N_A) \) and \( \Gamma_c(M, N_A) \) when \( N_A \) is a proper submodule of \( M \).

**Theorem 10.** Let \( M \) be an \( R \)-module such that \( N_A \) is a proper submodule of \( M \), and let \( H_A = (N_A : R M) \).

(1) \( \text{gr}(\Gamma_c(M \setminus N_A, N_A)) = 3 \) if and only if \( c + 1 \in H_A \) and \( |N_A| \geq 3 \).
(b) \( \text{gr}(T \Gamma_c(M \setminus N_A, N_A)) = 4 \) if and only if \( c + 1 \not\in H_A \) and \( |N_A| \geq 2 \).

(c) Otherwise, \( \text{gr}(T \Gamma_c(M \setminus N_A, N_A)) = \infty \).

(2) (a) \( \text{gr}(T \Gamma_c(M, N_A)) = 3 \) if and only if \( |N_A| \geq 3 \).

(b) \( \text{gr}(T \Gamma_c(M, N_A)) = 4 \) if and only if \( c + 1 \not\in H_A \) and \( |N_A| = 2 \).

(c) Otherwise, \( \text{gr}(T \Gamma_c(M, N_A)) = \infty \).

**Proof.** Apply Theorem 7 and Theorem 6. \( \square \)

**Example 2.** Let \( M = \mathbb{Z}_2 \times \mathbb{Z}_3 \), \( R = \mathbb{Z}_{12} \) and \( N_A = \mathbb{Z}_2 \times 0 \), then \( T \Gamma_c(M \setminus N_A, N_A) \) is the union of 2 disjoint \( K^2 \)'s, if \( c = 5 \) or 11, by Theorem 7(1) and \( T \Gamma_c(M \setminus N_A, N_A) \) is a \( K^2.2 \), if \( c = 1 \) or 7, by Theorem 7(2). Now, if \( c = 1 \) or 7, then \( T \Gamma_c(M \setminus N_A, N_A) \) is connected with \( \text{diam} T \Gamma_c(M \setminus N_A, N_A) = 2 \) and \( \text{gr}(T \Gamma_c(M \setminus N_A, N_A)) = 4 \) by Theorem 8(2), 9(3) and 10(1)(b), respectively, and \( \text{gr}(T \Gamma_c(M, N_A)) = 4 \) by Theorem 10(2)(b).

5. The Case When \( N_A \) is Not a Submodule of \( M \)

In this section, we consider the remaining case when \( N_A \) is not a submodule of \( M \), this implies that \( |N_A| \geq 3 \). So there are distinct \( x, y \in N_A \) such that \( x + cy \in M \setminus N_A \).

In this case, we show that \( T \Gamma_c(N_A, N_A) \) is always connected but never complete. Moreover \( T \Gamma_c(N_A, N_A) \) and \( T \Gamma_c(M \setminus N_A, N_A) \) are never disjoint subgraphs of \( T \Gamma_c(M, N_A) \). We first show that \( T \Gamma_c(M, N_A) \) is connected when \( T \Gamma_c(M \setminus N_A, N_A) \) is connected. However, we give an example to show that the converse fails.

**Theorem 11.** Let \( M \) be an \( R \)-module and \( N_A \) a union of prime submodules of \( M \) that is not a submodule of \( M \).

1. \( T \Gamma_c(N_A, N_A) \) is connected with \( \text{diam}(T \Gamma_c(N_A, N_A)) = 2 \).

2. Some vertices of \( T \Gamma_c(N_A, N_A) \) are adjacent to a vertex of \( T \Gamma_c(M \setminus N_A, N_A) \). In particular, the subgraphs \( T \Gamma_c(N_A, N_A) \) and \( T \Gamma_c(M \setminus N_A, N_A) \) of \( T \Gamma_c(M, N_A) \) are not disjoint.

3. If \( T \Gamma_c(M \setminus N_A, N_A) \) is connected, then \( T \Gamma_c(M, N_A) \) is connected.

**Proof.** (1) Every \( x \in N_A \) is adjacent to 0. Thus \( x - 0 - y \) is a path in \( T \Gamma_c(N_A, N_A) \) of length two between any two distinct \( x, y \in N_A \). Moreover, there are nonadjacent \( x, y \in N_A \) since \( N_A \) is not a submodule of \( M \); so \( \text{diam}(T \Gamma_c(N_A, N_A)) = 2 \).

(2) Since \( N_A \) is not a submodule of \( M \), there are distinct \( x, y \in N_A \) such that \( x + cy \in N_A \). Then \( -y \in N_A \) and \( x + cy \in M \setminus N_A \) are adjacent vertices in \( T \Gamma_c(M, N_A) \) since \( x + cy - cy = x \in N_A \). The “in particular” statement is clear.

(3) By part (1) above, it suffices to show that there is a path from \( x \) to \( y \) in \( T \Gamma_c(M, N_A) \) for every \( x \in N_A \) and \( y \in M \setminus N_A \). By part (2) above, there are adjacent vertices \( u \) and \( v \) in \( T \Gamma_c(N_A, N_A) \) and \( T \Gamma_c(M \setminus N_A, N_A) \), respectively. Since \( T \Gamma_c(N_A, N_A) \) is connected, there is a path from \( x \) to
$u$ in $\Gamma_G(M, N_A)$; and since $\Gamma_G(M \setminus N_A, N_A)$ is connected, there is a path from $v$ to $y$ in $\Gamma_G(M \setminus N_A, N_A)$, then there is a path from $x$ to $y$ in $\Gamma_G(M, N_A)$ since $u$ and $v$ are adjacent in $\Gamma_G(M, N_A)$. Thus $\Gamma_G(M, N_A)$ is connected.

Next, we determine an equivalent condition for connectedness of $\Gamma_G(M, N_A)$ and compute $\text{diam}(\Gamma_G(M, N_A))$. As usual, if $A \subseteq M$, then $< A >$ denotes the submodule of $M$ generated by $A$.

**Theorem 12.** Let $M$ be an $R$-module and $N_A$ a union of prime submodule of $M$. Then $\Gamma_G(M, N_A)$ is connected if and only if $M = < N_A >$ (i.e., $m = n_1 + n_2 + \cdots + n_k$ for every $n_1, \ldots, n_k \in N_A$ and $k \in \mathbb{N}$).

**Proof.** Suppose that $\Gamma_G(M, N_A)$ is connected, let $m \in M$. Then there is a path $0 = m_1 - m_2 + \cdots - m_i - m$ from 0 to $m$ in $\Gamma_G(M, N_A)$. Thus $m_1, m_2 + cm_1, \ldots, m + cm_i \in N_A$. Hence $m = n_1 + n_2 + \cdots + n_k > 0 < N_A >$; thus $M = < N_A >$. Conversely, suppose that $M = < N_A >$. We show that there is a path from 0 to $m$ in $\Gamma_G(M, N_A)$ for every $m \neq 0 \in M$. By hypothesis, $m = n_1 + n_2 + \cdots + n_k$ for some $n_1, \ldots, n_k \in N_A$ and $k \in \mathbb{N}$. Also we define a function, denoted by $f(t)$, which is equal to 0 where $t$ is even and equal to 1 where $t$ is odd. Let $a_0 = 0$ and $a_j = (-1)^j f(k-j) c_i f(k-j) (n_1 + \cdots + n_j)$ for every integer $j$ with $1 \leq j \leq k$. Then $a_j + ca_{j+1} \in N_A$ for every integer $j$ with $0 \leq j \leq k - 1$, and thus $0 = a_0 - a_1 - \cdots - a_{k-1} - a_k = m$ is a path from 0 to $m$ in $\Gamma_G(M, N_A)$ of length at most $k$. Now, let $0 \neq u, w \in M$. Then by the preceding argument, there are paths from $u$ to 0 and 0 to $w$ in $\Gamma_G(M, N_A)$. Hence there is a path from $u$ to $w$ in $\Gamma_G(M, N_A)$; so $\Gamma_G(M, N_A)$ is connected.

In the next theorem and its corollary, we determine a relation between $\Gamma_G(R, H_A)$ and $\Gamma_G(N_A, N_A)$, where $H_A = (N_A :_R M)$.

**Theorem 13.** Let $M$ be an $R$-module, and $N_A$ a union of prime submodules of $M$ where $H_A = (N_A :_R M)$. If $\Gamma_G(R, H_A)$ is connected, then $\Gamma_G(M, N_A)$ is connected as well.

**Proof.** Suppose that $\Gamma_G(R, H_A)$ is connected and let $m \in M$. Then there is a path $0 = r_1 - r_2 + \cdots - r_k = 0$ from 0 to $m$ in $\Gamma_G(R, H_A)$. Then $r_1, r_2 + cr_1, \ldots, r_k + cr_{k-1}, 1 + cr_k \in H_A$. Hence $0 = r_1 m - \cdots - r_k m - m$ is a path from 0 to $m$ in $\Gamma_G(M, N_A)$. Since all vertices may be connected via 0, $\Gamma_G(M, N_A)$ is a connected graph.

**Corollary 2.** Let $M$ be an $R$-module, and $N_A$ a union of prime submodules of $M$ and $H_A = (N_A :_R M)$. If $R = < H_A >$, then $\Gamma_G(M, N_A)$ is connected.

**Proof.** This follows directly from Theorem 12 and 13.

**Remark 2.** According to the proof of the Theorem 13, if $d(0, 1) = k$ in $\Gamma_G(R, H_A)$, then $d(0, m) \leq k$ in $\Gamma_G(M, N_A)$ for every $m \in M$.
In next theorem, we improve Theorem 3.3 of [7] and Theorem 4.3 of [1] without considering $M$ as a finitely generated $R$-module.

**Theorem 14.** Let $M$ be an $R$-module and $N_A$ a union of prime submodule of $M$ that is not a submodule of $M$ such that $M = \langle N_A \rangle$ (i.e., $TT_G(M, N_A)$ is connected). If there is $k \geq 2$ is a greatest integer $i$ such that $m = n_1 + n_2 + \cdots + n_i$ and for every $m \in M$ and for some $n_1, \ldots, n_i \in N_A$ where $n_1 + n_2 + \cdots + n_i$ is a shortest representation of the element $m$, then $diam(TT_G(M, N_A)) = k$. Otherwise, $diam(TT_G(M, N_A)) = \infty$.

**Proof.** Let $m$ and $m'$ be distinct elements in $M$ such that they are not adjacent. We show that there is a path from $m$ to $m'$ in $TT_G(M, N_A)$ with length at most $k$. We define a function, denoted by $f(i)$, which is equal to 0 where $t$ is even and equal to 1 where $t$ is odd. By the proof of Theorem 12, we can consider two path $(m - m') - x_1 - \cdots - x_{k-1} - 0$ and $(m + cm') - y_1 - \cdots - y_{k-1} - 0$ of lengths at most $k$. Let $a_0 = m$, $a_k = m'$, $a_j = (-1)^{f(k-j)} c^{f(k-j)} m + x_j f(k-1) + y_j f(k)$ for every integers $j$ with $1 \leq j \leq k-1$. Then $a_j + ca_j + 1 \in N_A$ for every integers $j$ with $0 \leq j \leq k-1$, and thus $m = a_0 - a_1 - \cdots - a_{k-1} - a_k = m'$ is a path from $m$ to $m'$ in $TT_G(M, N_A)$ of length at most $k$. Now we show that any path from $0$ to $m$ in $TT_G(M, N_A)$ has length at least $k$. Suppose that $0 - b_1 - \cdots - b_{k-1} - m$ is a path from $0$ to $m$ in $TT_G(M, N_A)$ of length $t$ and let $m = n_1 + \cdots + n_k$ be a shortest representation of the element $m$. Thus $b_1, b_2 + cb_1, \ldots, b_{k-1} + cb_{k-2}, m + cb_{k-1} \in N_A$, and hence $m = b_1, b_2 + cb_1, \ldots, b_{k-1} + cb_{k-2}, m + cb_{k-1} >$. Thus $i \geq k$ and so the shortest path between 0 and $m$ in $TT_G(M, N_A)$ has length $k$. Therefore $diam(TT_G(M, N_A)) = k$.

If there is no such $k$, then we show that $diam(TT_G(M, N_A)) = \infty$. Suppose not. Let $diam(TT_G(M, N_A)) = t$ where $t < \infty$. Since $k = \infty$, there is $m \in M$ such that $m = n_1 + \cdots + n_t + 1$ is a shortest representation of the element $m$. According to what is proved, it is contradiction since there is a path from 0 to $m$ in $TT_G(M, N_A)$ of length at most $t$.

**Remark 3.** Let $N_A = \bigcup_{k \in A} N_k$ and $M = \langle N_A \rangle$, then it is clear that $1 \leq k \leq |A|$ where $k$ is as mentioned in Theorem 14.

**Corollary 3.** Let $M$ be an $R$-module and $N_A$ a union of prime submodule of $M$ such that $TT_G(M, N_A)$ is connected. Let $m = n_1 + \cdots + n_k$ be a shortest representation of the element $m$ and $k$ is as mentioned in Theorem 14.

(1) $diam(TT_G(M, N_A)) = d(0, m)$
(2) If $diam(TT_G(M, N_A)) = k$ then $diam(TT_G(M \setminus N_A, N_A)) \geq k - 2$.

**Proof.**
(1) This is clear from the proof of Theorem 14.
(2) Since $k = diam(TT_G(M, N_A)) = d(0, m)$ by part (1) above, let $0 - b_1 - \cdots - b_{k-1} - m$ be a shortest path from 0 to $m$ in $TT_G(M, N_A)$. Clearly $b_1 \in N_A$. If $b_i \in N_A$ for some integer $i$ with $2 \leq i \leq k - 1$, then the path $0 - b_i -
\[ \cdots - b_{k-1} - m \] from 0 to \( m \) has length less than \( k \), a contradiction. Thus \( b_i \in M \setminus N_A \) for every integer \( i \) with \( 2 \leq i \leq k-1 \). Hence \( b_2 - \cdots - b_{k-1} - m \) is a shortest path from \( b_2 \) to \( m \) in \( T\Gamma_c(M \setminus N_A, N_A) \) of length \( k-2 \). Thus \( \text{diam}(T\Gamma_c(M \setminus N_A, N_A)) \geq k-2 \).

We next investigate the girth of \( T\Gamma_c(N_A, N_A) \), \( T\Gamma_c(M \setminus N_A, N_A) \), and \( T\Gamma_c(M, N_A) \) when \( N_A \) is not a submodule of \( M \). Recall that \( H_A = (N_A :_R M) = \bigcup_{\lambda \in \Lambda} P_\lambda \) where \( N_A \) is proper subset of \( M \) with \( |N_A| \geq 3 \) and \( P_\lambda = (N_\lambda :_R M) \).

**Theorem 15.** Let \( M \) be an \( R \)-module, and let \( N_A = \bigcup_{\lambda} N_\lambda \) for prime submodules \( N_\lambda \) of \( M \), that is not a submodule of \( M \) and \( H_A = (N_A :_R M) \). Suppose that \( m_1 - m_2 - m_3 \) is a path of length two in \( T\Gamma_c(M \setminus N_A, N_A) \) for distinct elements \( m_1, m_2, m_3 \in M \setminus N_A \).

1. If \( c + 1 \in H_A \) and \( \bigcap_{\lambda} N_\lambda \neq \{0\} \), then \( \text{gr}(T\Gamma_c(M \setminus N_A, N_A)) = 3 \).
2. If \( (c+1)m_i \neq 0 \) for all integers \( i \) with \( 1 \leq i \leq 3 \), then \( \text{gr}(T\Gamma_c(M \setminus N_A, N_A)) \leq 4 \).

**Proof.**

1. Suppose that there is a \( 0 \neq h \in \bigcap_{\lambda} N_\lambda \). If \( m_2 \neq m_1 + h \), then \( m_1 - m_2 - (m_1 + h) - m_1 \) is a cycle of length three in \( T\Gamma_c(M \setminus N_A, N_A) \) since \( (c+1)m_1 \in N_A \). Hence, assume that \( m_2 = m_1 + h \). Since \( (m_1 + h) + cm_3 = m_2 + cm_3 \in N_A \) and \( h \in \bigcap_{\lambda} N_\lambda \), we have \( m_1 + cm_3 \in N_A \). Thus \( m_1 - m_2 - m_3 - m_1 \) is a cycle of length three in \( T\Gamma_c(M \setminus N_A, N_A) \). Thus \( \text{gr}(T\Gamma_c(M \setminus N_A, N_A)) = 3 \).
2. Suppose that \( (c+1)m_i \neq 0 \) for all integers \( i \) with \( 1 \leq i \leq 3 \). Then \( m_i \neq -cm_j \) for every \( i \) with \( 1 \leq i \leq 3 \). There are distinct integer \( j, k \) with \( 1 \leq j, k \leq 3 \) such that they are adjacent and \( m_j \neq -cm_k \) since if \( m_1 + cm_2 = m_3 + cm_2 = 0 \), then \( m_1 = m_3 \), a contradiction. Thus \( m_j - m_k - (-cm_k) - (-cm_j) - m_j \) is a 4-cycle in \( T\Gamma_c(M \setminus N_A, N_A) \); so \( \text{gr}(T\Gamma_c(M \setminus N_A, N_A)) \leq 4 \).

In the above theorem, we improve the proof of Theorem 3.12(1) in [5]. Also in the part (2) of the above theorem, \( 0 \neq (c+1)m_i \) can belongs to \( N_A \) unlike Theorem 3.12(3) in [5].

**Corollary 4.** Let \( M \) be an \( R \)-module, and let \( N_A = \bigcup_{\lambda} N_\lambda \) for prime submodules \( N_\lambda \) of \( M \), that is not a submodule of \( M \) and \( H_A = (N_A :_R M) \). Suppose that \( m_1 - m_2 - m_3 \) is a path of length two in \( T\Gamma_c(M \setminus N_A, N_A) \) for distinct elements \( m_1, m_2, m_3 \in M \setminus N_A \). If \( (c+1)m_i = 0 \) for some integer \( i \) with \( 1 \leq i \leq 3 \) and \( c+1 \notin (0 :_R M) \), then \( \text{gr}(T\Gamma_c(M \setminus N_A, N_A)) = 3 \).

**Proof.** Suppose that \( c+1 \notin (0 :_R M) \). Thus \( c+1 \neq 0 \). Since \( m_i \in M \setminus N_A \) for all integers \( i \) with \( 1 \leq i \leq 3 \) and \( (c+1)m_i = 0 \) for some \( i \) with \( 1 \leq i \leq 3 \), we have \( c+1 \in P_\lambda \) for every \( P_\lambda \) where \( P_\lambda = (N_\lambda :_R M) \). Hence \( 0 \neq c+1 \in \bigcap_{\lambda} P_\lambda \).
Corollary 5. Let $M$ be an $R$-module, and let $N_A = \bigcup \lambda N_\lambda$ for prime submodules $N_\lambda$ of $M$ and $H_A = (N_A :_R M)$. Suppose that $m_1 - m_2 - m_3$ is a path of length two in $TT_c(M \setminus N_A, N_A)$ for distinct elements $m_1, m_2, m_3 \in M \setminus N_A$, $c + 1 \in H_A$, \( \bigcap \lambda N_\lambda \neq \{0\} \) and $|N_A| \geq 3$, then $gr(TT_c(M \setminus N_A, N_A)) = 3$.

Proof. This follows directly from Theorem 10(1)(a) and Theorem 15(1).

Theorem 16. Let $M$ be an $R$-module and $N_A$ a union of prime submodule of $M$ that is not a submodule of $M$.

1. Either $gr(TT_c(N_A, N_A)) = 3$ or $gr(TT_c(N_A, N_A)) = \infty$. Moreover, if one has $gr(TT_c(N_A, N_A)) = \infty$, then $|N_\lambda| = 2$ for any $\lambda \in \Lambda$ where $N_\lambda$ is a non-zero submodule of $M$. Also, $TT_c(N_A, N_A)$ is a star graph.

2. $gr(TT_c(M, N_A)) = 3$ if and only if $gr(TT_c(N_A, N_A)) = 3$.

3. $gr(TT_c(M, N_A)) \leq 4$.

4. $gr(TT_c(M, N_A)) = 4$ if and only if $gr(TT_c(N_A, N_A)) = \infty$.

Proof. (1) If $n + cn' \in N_A$ for some distinct $n, n' \in N_A^*$, then $0 - n - n' - 0$ is a 3-cycle in $TT_c(N_A, N_A)$; so $gr(TT_c(N_A, N_A)) = 3$. Otherwise, $n + cn' \in M \setminus N_A$ for all distinct $n, n' \in N_A^*$. So in this case, every $n \in N_A$ is adjacent to 0, and no two distinct $n, n' \in N_A^*$ are adjacent. Thus $TT_c(N_A, N_A)$ is a star graph with center 0; so $gr(TT_c(N_A, N_A)) = \infty$. Moreover, let $N_A = \bigcup \lambda \subseteq A N_\lambda$ is not a submodule of $M$ where $N_\lambda$ is a prime submodule of $M$ so $|\lambda| \geq 2$. Assume that $gr(TT_c(N_A, N_A)) = \infty$. Then $x + cy \in M \setminus N_A$ for all distinct $x, y \in N_A^*$, and thus if $N_\lambda \neq \{0\}$, then each $|N_\lambda| = 2$.

(2) It suffices to show that $gr(TT_c(N_A, N_A)) = 3$ when $gr(TT_c(M, N_A)) = 3$. Let $(c + 1)n \neq 0$ for some $n \in N_A^*$, then $0 - n - (cn) - 0$ is a 3-cycle in $TT_c(N_A, N_A)$. Otherwise, $(c + 1)n = 0$ for all $n \in N_A$. Since $N_A$ is not a submodule of $M$, there are distinct elements $n, n' \in N_A$ such that $n + n' \in M \setminus N_A$. Then $(c + 1)(n + n') = 0$, thus $c + 1 \in H_A$. Let $m - m_1 - m_2 - m$ be a 3-cycle in $TT_c(M, N_A)$. Then $n_1 = cm + m_1, n_2 = cm + m_2, m_1 + cm_2 \in N_A$. Thus $0 - n_1 - n_2 - 0$ is a 3-cycle in $TT_c(N_A, N_A)$; therefore $gr(TT_c(N_A, N_A)) = 3$.

(3) Since $N_A$ is not a submodule of $M$, there are distinct elements $n, n' \in N_A$ such that $n + n' \in M \setminus N_A$. Then $0 - (cn) - n + n' - (cn') - 0$ is a 4-cycle in $TT_c(N_A, N_A)$.

(4) This follows by parts (1), (2) and (3) above.

Example 3. (a) Let $M = \mathbb{Z}[X]$ be an $\mathbb{Z}[X]$-module. Then $N_A = \mathbb{Z}[X] \setminus \mathbb{Z}^*$ is a union of prime submodule of $M$, that is not a submodule of $M$. Thus $TT_c(N_A, N_A)$ is connected with $diam(TT_c(N_A, N_A)) = 2$ by Theorem

Since $c + 1 \notin (0 :_R M)$, there is a $m \in M$ such that $0 \neq (c + 1)m \in \bigcap \lambda N_\lambda$. Thus $gr(TT_c(M \setminus N_A, N_A)) = 3$ by Theorem 15(1).
Moreover, by Theorem 12 and 14, $T \Gamma_c(M, N_A)$ is connected with $\text{diam}(T \Gamma_c(M, N_A)) = 2$ since $z = X + z - X$ for $X, z - X \in N_A$ for every $z \in \mathbb{Z}^*$. However, $T \Gamma_c(M \setminus N_A, N_A)$ is not connected since there is no path from 1 to 2 in $T \Gamma_c(M \setminus N_A, N_A)$. Thus the converse of Theorem 11(3) need not hold.

(b) Let $k \in \mathbb{N}$, $M = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $R = \mathbb{Z}_{2k}$, $N_1 = \mathbb{Z}_2 \times 0 \times 0$, $N_2 = 0 \times \mathbb{Z}_2 \times 0$, $N_3 = 0 \times 0 \times \mathbb{Z}_2$ and $N_A = \bigcup_{1 \leq j \leq 3} N_j$ then by Theorem 12 and 14, $T \Gamma_c(M, N_A)$ is connected with $\text{diam}(T \Gamma_c(M, N_A)) = 3$, note that $(1, 1, 0)$ is the sum of two elements of $N_A$ but $(1, 1, 1)$ is the sum of three element of $N_A$, and $T \Gamma_c(R, H_A)$ is disconnected so the converse of Theorem 13 is fails, also $gr(T \Gamma_c(N_A, N_A)) = \infty$ and $gr(T \Gamma_c(M, N_A)) = 4$ by Theorem 16.

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