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CLOSURES OF PROPER CLASSES

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Abstract. For an integral domain R we consider the closures $\widehat{M}$ ($\widehat{M}_r, r \in R$) of a submodule M of an R-module N consisting of elements n of N with $tn \in M$ ($r^m n \in M$) for some nonzero $t \in R$ ($m \in \mathbb{Z}^+$) and its connections with usual closure $\overline{M}$ of M in N. Using these closures we study the closures $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}_r$ of a proper class $\mathcal{P}$ of short exact sequences and give a decomposition for the class of quasi-splitting short exact sequences of abelian groups into the direct sum of "p-closures" of the class $\mathcal{S}plit$ of splitting short exact sequences and description of closures of some classes. In the general case of an arbitrary ring we generalize these closures of a proper class $\mathcal{P}$ by means of homomorphism classes $\mathcal{F}$ and $\mathcal{G}$ and prove that under some conditions this closure $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ is a proper classes.

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1. Introduction

Throughout R is an associative ring with identity and all modules are unital left R-modules; restrictions on the ring R concerned, if there are any, are mentioned at the beginning of each section.

In this paper, we study closures of proper classes and their relation to closures of splitting short exact sequences. Over an integral domain R, for a class $\mathcal{P}$ of short exact sequences of R-modules, we denote by $\hat{\mathcal{P}}$ the class of short exact sequences E such that $kE \in \mathcal{P}$ for some $0 \neq k \in R$, where k also denotes the multiplication homomorphism by k. Thus

$$\hat{\mathcal{P}} = \{E | kE \in \mathcal{P} \text{ for some } 0 \neq k \in R\}.$$

In case of abelian groups, the class $\hat{\mathcal{P}}$ firstly was studied in [14] for $\mathcal{P} = \mathcal{S}plit$, the class of splitting short exact sequences, and it was denoted by Text (since $\text{Ext}_{\mathcal{S}\hat{p}lit}(C, A) = T(\text{Ext}(C, A))$. In [8] $\hat{\mathcal{P}}$ was studied for $\mathcal{P} = \mathcal{P}ure$, the class of pure exact sequences and $\mathcal{P} = \mathcal{D}$, torsion splitting short exact sequences. For an arbitrary proper class $\mathcal{P}$ in the category of abelian groups the class $\hat{\mathcal{P}}$ was studied in [3].

For an integral domain R we consider the closures $\widehat{M}$ ($\widehat{M}_r, r \in R$) of a submodule M of an R-module N consisting of elements n of N with $tn \in M$ ($r^m n \in M$) for some nonzero $t \in R$ ($m \in \mathbb{Z}^+$). We study its connections with usual closure of M in N (see [7]) and show that a submodule M of a torsion-free module N over an integral domain have unique closure which coincides with $\widehat{M}$.

For an element r of an integral domain R we define *r-closures* of the class $\mathcal{P}$ of short exact sequences contained in $\hat{\mathcal{P}}$ and give a direct sum decomposition for the class $\mathcal{S}\hat{plit}$ of quasi-splitting short exact sequences of abelian groups into the direct sum of p-closures of the class $\mathcal{S}plit$ (for sum of proper classes see [1, 12]).

The classes $\mathcal{S}mall$, $\mathcal{S}$, $\mathcal{SB}$ and $\mathcal{WS}$ determined by small submodules, submodules that have supplements, submodules that have supplements with bounded intersection and weak supplement submodules respectively, are studied in [4]. Over a hereditary ring, it is shown in the same work that the least proper class $\overline{\mathcal{WS}}$ can be defined using the class $\mathcal{WS}$. In section 3, we give the relation between $\mathcal{SB}$, $\overline{\mathcal{WS}}$ and Ivanov Classes (see [3, 9]).

For investigation of $\hat{\mathcal{P}}$ over an integral domain R we have used the fact that multiplication by an element of R is an endomorphism of a module if R is commutative which need not be true for arbitrary associative ring with identity. For this reason, the last section is devoted to the study of closures of proper classes over an associative ring with identity. Instead of multiplication by an element of R we take classes $\mathcal{F}$ and $\mathcal{G}$ of homomorphisms and define closure $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ for a class $\mathcal{P}$ of short exact sequences with respect to these classes of homomorphisms of R-modules. We define a *compatible* pair $(\mathcal{F},\mathcal{G})$ for a class $\mathcal{P}$ and use its properties to prove that for a compatible pair $(\mathcal{F},\mathcal{G})$ for a proper class $\mathcal{P}$, the class $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ is proper. We give some restrictions on the compatible pair $(\mathcal{F},\mathcal{G})$ to ensure that $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ contains $\mathcal{P}$. Furthermore, under some conditions on the compatible pair $(\mathcal{F},\mathcal{G})$ for the proper class $\mathcal{P}$, we obtain a relation between the closure $\mathcal{S}\hat{plit}_{\mathcal{F}}^{\mathcal{G}}$ of splitting short exact sequences and the class $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ at the end of this section.

2. Preliminaries

Let R be an associative ring with identity and $\mathcal{P}$ be a class of short exact sequences of left R-modules and R-module homomorphisms. If a short exact sequence

$E: 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ belongs to $\mathcal{P}$, then f is said to be a $\mathcal{P}$-*monomorphism* and g is said to be a $\mathcal{P}$-*epimorphism*. E is determined by the monomorphism f and the epimorphism g uniquely up to isomorphism.

Definition 1. The class $\mathcal{P}$ is said to be *proper* (in the sense of Buchsbaum) if it satisfies the following conditions (see [5, 10, 13]):

P-1) If a short exact sequence E is in $\mathcal{P}$, then every short exact sequence isomorphic to E is contained in $\mathcal{P}$.

P-2) $\mathcal{P}$ contains all splitting short exact sequences.

P-3) The composition of two $\mathcal{P}$-monomorphisms is a $\mathcal{P}$-monomorphism if it is defined.

P-3') The composition of two $\mathcal{P}$-epimorphisms is a $\mathcal{P}$-epimorphism if it is defined.

P-4) If g and f are monomorphisms, and $g \circ f$ is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism.

P-4') If g and f are epimorphisms, and $g \circ f$ is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.

Another and frequently used way to check if a given class $\mathcal{P}$ is proper is given by R. J. Nunke. A class $\mathcal{P}$ of short exact sequences is a proper class if $\mathrm{Ext}^1_{\mathcal{P}}(C, A)$ is a subfunctor of $\mathrm{Ext}^1_R(C, A)$, $\mathrm{Ext}^1_{\mathcal{P}}(C, A)$ is a subgroup of $\mathrm{Ext}^1_R(C, A)$ for all R-modules A, C and the composition of two $\mathcal{P}$-monomorphisms (or $\mathcal{P}$-epimorphisms) is a $\mathcal{P}$-monomorphism (a $\mathcal{P}$-epimorphism respectively) (see Theorem 1.1 in [11]).

For any class $\mathcal{E}$ of short exact sequences the proper class generated by $\mathcal{E}$ is the least proper class that contains $\mathcal{E}$ and denoted by $\langle \mathcal{E} \rangle$. It is the intersection of all proper classes containing $\mathcal{E}$ (see [12]).

A module M is $\mathcal{P}$-coprojective ($\mathcal{P}$-coinjective) if every short exact sequence of the form $0 \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow 0$ (of the form $0 \longrightarrow M \longrightarrow B \longrightarrow C \longrightarrow 0$) is in $\mathcal{P}$. For classes $\mathcal{M}$ and $\mathcal{J}$ of modules, the smallest proper class $\overline{k}(\mathcal{M})$ (resp. $\underline{k}(\mathcal{J})$) for which all modules in $\mathcal{M}$ (resp. $\mathcal{J}$) are $\overline{k}(\mathcal{M})$-coprojective (resp. $\underline{k}(\mathcal{J})$-coinjective) is said to be coprojectively (resp. coinjectively) generated by $\mathcal{M}$ (resp. $\mathcal{J}$).

For classes $\mathcal{R}$ and $\mathcal{L}$ of short exact sequences, the class $\mathcal{R} + \mathcal{L}$ is defined as the least proper class that contains $\mathcal{R}$ and $\mathcal{L}$, i.e. $\mathcal{R} + \mathcal{L} = \langle \mathcal{R} \cup \mathcal{L} \rangle$, and it is called the *sum* of the classes $\mathcal{R}$ and $\mathcal{L}$ (see [12]).

We will use Ext instead of Ext^1 and for a homomorphism f, we will denote $\mathrm{Ext}^1(1, f)$ by f_* and $\mathrm{Ext}^1(f, 1)$ by f^*.

3. THE CLASSES $\hat{\mathcal{P}}$ AND $\hat{\mathcal{P}}_r$

Let R be an integral domain throughout this section unless otherwise stated. For a class $\mathcal{P}$ of short exact sequences of R-modules, we denote by $\hat{\mathcal{P}}$ the class of short exact sequences $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules such that $kE \in \mathcal{P}$ for some $0 \neq k \in R$ where k also denotes the multiplication homomorphism by $k \in R$. Thus

$$\hat{\mathcal{P}} = \{E \,|\, kE \in \mathcal{P} \text{ for some } 0 \neq k \in R\}.$$

For $E \in \mathcal{P}$, we have $1 \cdot E = E \in \mathcal{P}$, therefore $\mathcal{P} \subseteq \hat{\mathcal{P}}$ for every class $\mathcal{P}$ of short exact sequences.

In case of abelian groups, the class $\hat{\mathcal{P}}$ was studied in [14] and in [3] for $\mathcal{P} = \mathcal{S}plit$,the class of splitting short exact sequences. In [8] $\hat{\mathcal{P}}$ was studied for $\mathcal{P} =$

$\mathcal{P}ure$, the class of pure exact sequences and $\mathcal{P} = \mathcal{D}$, torsion splitting short exact sequences. In [3] $\hat{\mathcal{P}}$ was studied for $\mathcal{P} = \mathcal{S}plit$, where it was denoted by *Text* (since $\mathrm{Ext}_{\hat{\mathcal{S}plit}}(C, A) = T(\mathrm{Ext}(C, A))$, the torsion part of $\mathrm{Ext}(C, A)$), and for every proper class $\mathcal{P}$. The following result gives a general answer when R is an integral domain.

Theorem 1 ([2], Theorem 3.1)**.** *For every proper class $\mathcal{P}$ of short exact sequences of R-modules, the class $\hat{\mathcal{P}}$ is proper.*

We can also consider $\hat{M}$ for R-modules $M \leq N$ with the definition

$$\hat{M} = \{x \in N \mid kx \in M \text{ for some } 0 \neq k \in R\}.$$

Some results involving this definition are obtained in [12] for the case of abelian groups (see [12, Lemma 4.1]).

The submodule $\hat{M}$ of the R-module N can be used in finding closures. Let us remind the definition of the closure of an R-module.

Definition 2. (see [7]) A submodule X of an R-module Z is closed in Z if $X \leq Y \leq Z$ and $X \trianglelefteq Y$ implies $Y = X$. For R-modules $M \leq X \leq N$, X is called a closure of M in N if $M \trianglelefteq X$ and X is closed in N.

Closure of a module need not be unique in general as the following example shows.

Example 1. Let R be the ring of integers and $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. Consider the subgroups $N = (1+2\mathbb{Z}, 2+8\mathbb{Z})\mathbb{Z}$, $L = (0+2\mathbb{Z}, 1+8\mathbb{Z})\mathbb{Z}$ and $X = (0+2\mathbb{Z}, 4+8\mathbb{Z})\mathbb{Z}$ of M. Then N and L are both closures of the subgroup X in M.

Proposition 1. *Let $M \leq N$ be R-modules, then $\hat{M}$ is the unique closure of $M + T(N)$ in N, where $T(N)$ is the torsion part of N.*

Proof. For $x \in \hat{M}$, $rx \in M \leq M + T(N)$ for some $0 \neq r \in R$, therefore $M + T(N) \trianglelefteq \hat{M}$. If $\hat{M} \trianglelefteq M'$ for some $M' \leq N$, then for $y \in M'$, $ky \in \hat{M}$ for some $0 \neq k \in R$ which implies $l(ky) \in M$ for some $0 \neq l \in R$. Since $(lk)y = l(ky) \in M$ for $0 \neq lk \in R$, $y \in \hat{M}$. Therefore, $\hat{M}$ is closed.

Let N' be another closure of $M + T(N)$ in N. If $z \in N'$, then $sz \in M + T(N)$ for some $0 \neq s \in R$ which shows that $N' \leq \hat{M}$. Since $M + T(N) \trianglelefteq N' \leq \hat{M}$ and N' is closed in N, $N' = \hat{M}$. □

Corollary 1. *Let N be a torsion-free R-module. Then for every submodule M of N, $\hat{M}$ is the unique closure of M in N.*

It is possible to define closures of the class $\mathcal{P}$ that happen to be in between $\mathcal{P}$ and $\hat{\mathcal{P}}$. For a class $\mathcal{P}$ of short exact sequences of R-modules and $0 \neq r \in R$, we denote by $\hat{\mathcal{P}}_r$ (r*-closure* of $\mathcal{P}$) the class of short exact sequences
$E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules such that $r^t E \in \mathcal{P}$ for some

for some nonnegative integer t, where r^t also denotes the multiplication homomorphism by $r^t \in R$. Thus

$$\hat{\mathcal{P}}_r = \{E \mid r^t E \in \mathcal{P} \text{ for some nonnegative integer } t\}.$$

For $E \in \mathcal{P}$, we have $r^0 \cdot E = 1 \cdot E = E \in \mathcal{P}$ and clearly $\hat{\mathcal{P}}_r \subseteq \hat{\mathcal{P}}$, therefore $\mathcal{P} \subseteq \hat{\mathcal{P}}_r \subseteq \hat{\mathcal{P}}$ for every class $\mathcal{P}$ of short exact sequences.

For a given proper class $\mathcal{P}$, the following result gives us proper classes that are contained in $\hat{\mathcal{P}}$. We omit its proof since it uses similar ideas used in the proof of [3, Theorem 1].

Proposition 2. *$\hat{\mathcal{P}}_r$ is a proper class for every proper class $\mathcal{P}$ and every $0 \neq r \in R$.*

Direct sum of two proper classes are defined in [1]. We say that the sum $\sum_{i \in I} \mathcal{L}_i$ of proper classes is direct if $\mathcal{L}_j \bigcap \sum_{\substack{i \neq j \\ i \in F}} \mathcal{L}_i = \mathcal{S}plit$ for every $j \in I$ and for every finite index set $F \subseteq I$.

Over the ring of integers, for every group A we have a decomposition of $T(A)$ into primary parts as $T(A) = \bigoplus_{p \text{ prime}} T_p(A)$. The following result holds when $R = \mathbb{Z}$, and it gives a direct sum decomposition for the class $\mathcal{S}plit$ of splitting short exact sequences in terms of proper classes.

Proposition 3. *Over the ring of integers, $\widehat{\mathcal{S}plit} = \bigoplus_p \widehat{\mathcal{S}plit}_p$, where p ranges over all prime numbers.*

A relation between quasi-splitting short exact sequences and $\hat{\mathcal{P}}$ for a proper class $\mathcal{P}$ is given by [2, Theorem 3.2]. A similar result for the closures we have defined also holds. Its proof uses similar ideas used in [2, Theorem 3.2], therefore it is omitted.

Theorem 2. *Let $\mathcal{P}$ be a proper class. Then*

$$\mathcal{P} + \widehat{\mathcal{S}plit}_r = \hat{\mathcal{P}}_r$$

for every $0 \neq r \in R$.

In order to show the relation of the classes defined so far with the proper classes related to supplements, we use Ivanov classes and give some results over the ring of integers (see [3, 9]).

Definition 3. For the classes $\mathcal{M}$ and $\mathcal{J}$ of modules, the class $i(\mathcal{M}, \mathcal{J})$ of short exact sequences is the least proper class for which every module from $\mathcal{M}$ is coprojective and every module from $\mathcal{J}$ is coinjective.

These classes are called Ivanov classes.

Theorem 3 ([3], Theorem 2)**.** *For all classes $\mathcal{M}$ and $\mathcal{J}$ of abelian groups*

$$\widehat{i(\mathcal{M},\mathcal{J})} = i(\mathcal{M} \cup \mathcal{B}, \mathcal{J} \cup \mathcal{B})$$

where $\mathcal{B}$ denotes the class of bounded abelian groups.

This result is also true for modules over an integral domains since its proof can easily be modified. It is also possible to use the proof of this result and obtain another closure of the class $i(\mathcal{M},\mathcal{J})$.

Theorem 4. *Let R be an integral domain, $\mathcal{M}$ and $\mathcal{J}$ classes of R-modules. Then for every $0 \neq r \in R$*

$$\widehat{i(\mathcal{M},\mathcal{J})}_r = i(\mathcal{M} \cup \mathcal{B}_r, \mathcal{J} \cup \mathcal{B}_r)$$

where $\mathcal{B}_r$ denotes the class of R-modules bounded by a power of r.

Proof. Let $i(\mathcal{M} \cup \mathcal{B}_r, \mathcal{J} \cup \mathcal{B}_r) = \mathcal{L}$. It is clear that $i(\mathcal{M},\mathcal{J}) \subseteq \mathcal{L}$ and $\mathcal{L} \subseteq \widehat{i(\mathcal{M},\mathcal{J})}_r$ since every module from $\mathcal{B}_r$ is $\widehat{i(\mathcal{M},\mathcal{J})}_r$-coinjective and $\widehat{i(\mathcal{M},\mathcal{J})}_r$-coprojective. To show that $\widehat{i(\mathcal{M},\mathcal{J})}_r \subseteq \mathcal{L}$, let
$E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \ \in \widehat{i(\mathcal{M},\mathcal{J})}_r$. Then there is a nonnegative integer t such that $r^t E \in i(\mathcal{M},\mathcal{J})$. We can write the homomorphism $r^t : C \longrightarrow C$ as $r^t = \alpha \circ \beta$, where $\alpha : r^t C \longrightarrow C$ is the inclusion and $\beta : C \longrightarrow r^t C$ is the standard epimorphism. Applying these homomorphisms we obtain the following commutative diagrams with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
E' : 0 & \longrightarrow & A & \xrightarrow{\delta} & B' & \xrightarrow{\sigma} & r^t C & \longrightarrow & 0 \\
 & & \| & & \downarrow \theta & & \downarrow \alpha & & \\
E : 0 & \longrightarrow & A & \xrightarrow{\gamma} & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C/r^t C & = & C/r^t C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C/r^tC & = & C/r^tC & & \\
 & & & & \downarrow & & \downarrow & & \\
r^{t*}(E):0 & \longrightarrow & A & \longrightarrow & B'' & \xrightarrow{\xi} & C & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \beta} & & \\
E':0 & \longrightarrow & A & \xrightarrow{\delta} & B' & \xrightarrow{\sigma} & r^tC & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

In the second diagram, β is an $\mathcal{L}$-epimorphism since $C[r^t] \in \mathcal{B}_r$ and ξ is also an $\mathcal{L}$-epimorphism. Then $\sigma \circ \eta = \beta \circ \xi$ is an $\mathcal{L}$-epimorphism since $\mathcal{L}$ is a proper class and σ is an $\mathcal{L}$-epimorphism since $\mathrm{Ext}_{\mathcal{L}}$ is a subfunctor of Ext. Therefore, $E' \in \mathcal{L}$ and δ is an $\mathcal{L}$-monomorphism.

In the first diagram, θ is an $\mathcal{L}$-monomorphism since $C/r^tC \in \mathcal{B}_r$. Since $\mathcal{L}$ is a proper class, $\alpha = \theta \circ \sigma$ is an $\mathcal{L}$-monomorphism and $E \in \mathcal{L}$. □

Corollary 2. *Let R be a Noetherian integral domain of Krull dimension 1, and $0 \neq r \in R$. Then*

$$\hat{\mathcal{S}\mathcal{B}} = i(\mathcal{B}, \mathcal{B})$$

$$\hat{\mathcal{S}\mathcal{B}_r} = i(\mathcal{B}_r, \mathcal{B})$$

where $\mathcal{B}$ is the class of bounded R-modules and $\mathcal{B}_r$ is the class of modules bounded by a power of r.

Proof. By [6, Proposition 4.3], $\mathcal{S}\mathcal{B} = \underline{k}(\mathcal{B}) = i(\varnothing, \mathcal{B})$. Theorem 3 and Theorem 4 complete the proof. □

We need the following result to see another application of these closures.

Proposition 4 ([4], Proposition 4.13)**.** *Over a hereditary ring $\overline{\mathcal{W}\mathcal{S}} = \underline{k}(\mathcal{S}m)$, where $\mathcal{S}m$ is the class of all small modules.*

Corollary 3. *Let R be a Dedekind domain and $0 \neq r \in R$. Then*

$$\hat{\overline{\mathcal{W}\mathcal{S}}} = i(\mathcal{B}, \mathcal{S}m)$$

$$\hat{\overline{\mathcal{W}\mathcal{S}}}_r = i(\mathcal{B}_r, \mathcal{S}m)$$

where $\mathcal{B}$ is the class of bounded R-modules, $\mathcal{B}_r$ is the class of modules bounded by a power of r and $\mathcal{S}m$ is the class of small modules.

Proof. By Proposition 4, $\overline{\mathcal{WS}} = \underline{k}(\mathcal{S}m) = i(\varnothing, \mathcal{S}m)$. Since $\mathcal{B} \subseteq \mathcal{S}m$, Theorem 3 and Theorem 4 completes the proof. □

Until now, we have dealt with the classes that include the given class $\mathcal{P}$. In [2], the existence of a class that is contained in the given class $\mathcal{P}$ is given using the class $\hat{\mathcal{P}}$ over a principal ideal domain. It is also possible to obtain a similar result using the class $\hat{\mathcal{P}}_r$. Note also that by $r\mathcal{P}$ we mean the class $r\mathcal{P} = \{E : E = rE' \text{ for some } E' \in \mathcal{P}\}$ defined in [2].

Theorem 5 ([2], Theorem 4.4)**.** *Let R be a principal ideal domain. Then for every proper class $\mathcal{P}$ with $\hat{\mathcal{P}} = \mathcal{P}$ and for every $k \in R$, the class $k\mathcal{P}$ is proper.*

With a slight change on the condition given, we can obtain a result for the class $\hat{\mathcal{P}}_r$.

Theorem 6. *Let R be a principal ideal domain. Then for every $0 \neq r \in R$ and every proper class $\mathcal{P}$ with $\hat{\mathcal{P}}_r = \mathcal{P}$, the class $r^t\mathcal{P}$ is proper for every nonnegative integer t.*

4. The class $\hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$

In the previous section, the classes $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}_r$ for the given class $\mathcal{P}$ of short exact sequences are defined over an integral domain using the homomorphisms that are multiplication by elements of R since multiplication by elements of R give homomorphisms when R is an integral domain. We cannot use elements of R to obtain homomorphisms in order to find a closure for the class $\mathcal{P}$ when R is an associative ring with an identity element since the multiplication by an element of R need not give a homomorphism in this case. Therefore, we turn our attention to the classes of R-module homomorphisms. Throughout this section we will study modules over an associative ring R with an identity element.

Definition 4. Let $\mathcal{F}$ and $\mathcal{G}$ be families of homomorphisms of R-modules. We say that $\mathcal{F}$ is closed under pushout diagrams if $f : A \longrightarrow B$, $f \in \mathcal{F}$ and $\alpha : A \longrightarrow A'$ is a homomorphism, then in the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ A' & \xrightarrow[f']{} & P \end{array}$$

we have $f' \in \mathcal{F}$. We say that $\mathcal{G}$ is closed under pullback diagrams if $g : C \longrightarrow D$, $g \in \mathcal{G}$ and $\beta : D' \longrightarrow D$, then in the pullback diagram

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ \uparrow & & \uparrow \beta \\ P' & \xrightarrow[g']{} & D' \end{array}$$

we have $g' \in \mathcal{G}$.

Lemma 1. *Let $\mathcal{F}$ and $\mathcal{G}$ be families of homomorphisms of R-modules. Let $\mathcal{F}$ be closed under pushout diagrams and $\mathcal{G}$ closed under pullback diagrams. Then the following hold.*

(i) *If $f : A \longrightarrow A_1$ belongs to $\mathcal{F}$, then the homomorphisms $f \oplus 1_A : A \oplus A \longrightarrow A_1 \oplus A$ and $1_A \oplus f : A \oplus A \longrightarrow A \oplus A_1$ are also in $\mathcal{F}$.*

(ii) *If $f : A \longrightarrow B$ belongs to $\mathcal{F}$, then the inclusion $i : \mathrm{Im}\, f \longrightarrow B$ (or the monomorphism $\bar{i} : A/\operatorname{Ker} f \longrightarrow B$ induced by f) is also in $\mathcal{F}$.*

(iii) *If $g : C_1 \longrightarrow C$ belongs to $\mathcal{G}$, then the homomorphisms $g \oplus 1_C : C_1 \oplus C \longrightarrow C \oplus C$ and $1_C \oplus g : C \oplus C_1 \longrightarrow C \oplus C$ are also in $\mathcal{G}$.*

(iv) *If $g : C \longrightarrow D$ belong to $\mathcal{G}$, then the canonical epimorphism $\pi : C \longrightarrow C/\operatorname{Ker} g$ induced by g (or the epimorphism $\bar{\pi} : C \longrightarrow \mathrm{Im}\, g$ induced by g) is also in $\mathcal{G}$.*

Let R be an associative ring with identity. Let $\mathcal{F}$ and $\mathcal{G}$ be families of homomorphisms of R-modules, and $\mathcal{P}$ a class of short exact sequences. We say that the pair $(\mathcal{F}, \mathcal{G})$ is "*compatible*" for the class $\mathcal{P}$ if for every short exact sequence E, there is $f \in \mathcal{F}$ such that $f_*(E) \in \mathcal{P}$ if and only if there is $g \in \mathcal{G}$ such that $g^*(E) \in \mathcal{P}$ with one (or both) of the following conditions satisfied:

(i) $\mathcal{F}$ is closed under compositions and pushout diagrams,

(ii) $\mathcal{G}$ is closed under compositions and pullback diagrams.

For a class $\mathcal{P}$ of short exact sequences and a "compatible" pair $(\mathcal{F}, \mathcal{G})$ for $\mathcal{P}$, we define the class $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ as

$$\begin{aligned} \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}} &= \{E \mid f_*(E) \in \mathcal{P} \text{ for some } f \in \mathcal{F}\} \\ &= \{E \mid g^*(E) \in \mathcal{P} \text{ for some } g \in \mathcal{G}\}. \end{aligned}$$

Theorem 7. *For every proper class $\mathcal{P}$ of short exact sequences and every compatible pair $(\mathcal{F}, \mathcal{G})$ for $\mathcal{P}$, the class $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ is proper.*

Proof. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ with $f_{1*}(E), g_1^*(E) \in \mathcal{P}$ for $f_1 : A \longrightarrow A_1$, $f_1 \in \mathcal{F}$ and $g_1 : C_1 \longrightarrow C$, $g_1 \in \mathcal{G}$.

If $f : A \longrightarrow A'$, then $g_1^*(f_*(E)) = (g_1^* \circ f_*)(E) = (f_* \circ g_1^*)(E) = f_*(g_1^*(E)) \in \mathcal{P}$ since $\mathrm{Ext}_{\mathcal{P}}$ is a subfunctor of Ext. Then $f_*(E) \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ since $g_1 \in \mathcal{G}$.

If $g : C' \longrightarrow C$, then $f_{1*}(g^*(E)) = (f_{1*} \circ g^*)(E) = (g^* \circ f_{1*})(E) = g^*(f_{1*}(E)) \in \mathcal{P}$ since $\mathrm{Ext}_{\mathcal{P}}$ is a subfunctor of Ext. Then $g^*(E) \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ since $f_1 \in \mathcal{F}$.

These arguments show that $\mathrm{Ext}_{\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}}$ is a subfunctor of Ext.

At this point, we separate the proof into two parts and give the proof when $\mathcal{F}$ is closed under compositions and pushout diagrams. Proof for the other case can be done using similar arguments and duality.

Let $\mathcal{F}$ be closed under compositions and pushout diagrams. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, $E' : 0 \longrightarrow A \longrightarrow B' \longrightarrow C \longrightarrow 0$ $\in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ with $f_{1*}(E) \in \mathcal{P}$ for $f_1 : A \longrightarrow A_1$, $f_1 \in \mathcal{F}$ and $f_{2*}(E') \in \mathcal{P}$ for $f_2 : A \longrightarrow A_2$, $f_2 \in \mathcal{F}$, say $f_{1*}(E) : 0 \longrightarrow A_1 \longrightarrow B_1 \longrightarrow C \longrightarrow 0$ and $f_{2*}(E') : 0 \longrightarrow A_2 \longrightarrow B_2 \longrightarrow C \longrightarrow 0$. We have the commutative diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A \oplus A & \longrightarrow & B \oplus B' & \longrightarrow & C \oplus C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1 \oplus 1_A} & & \downarrow & & \| & & \\
0 & \longrightarrow & A_1 \oplus A & \longrightarrow & B_1 \oplus B' & \longrightarrow & C \oplus C & \longrightarrow & 0
\end{array}$$

and

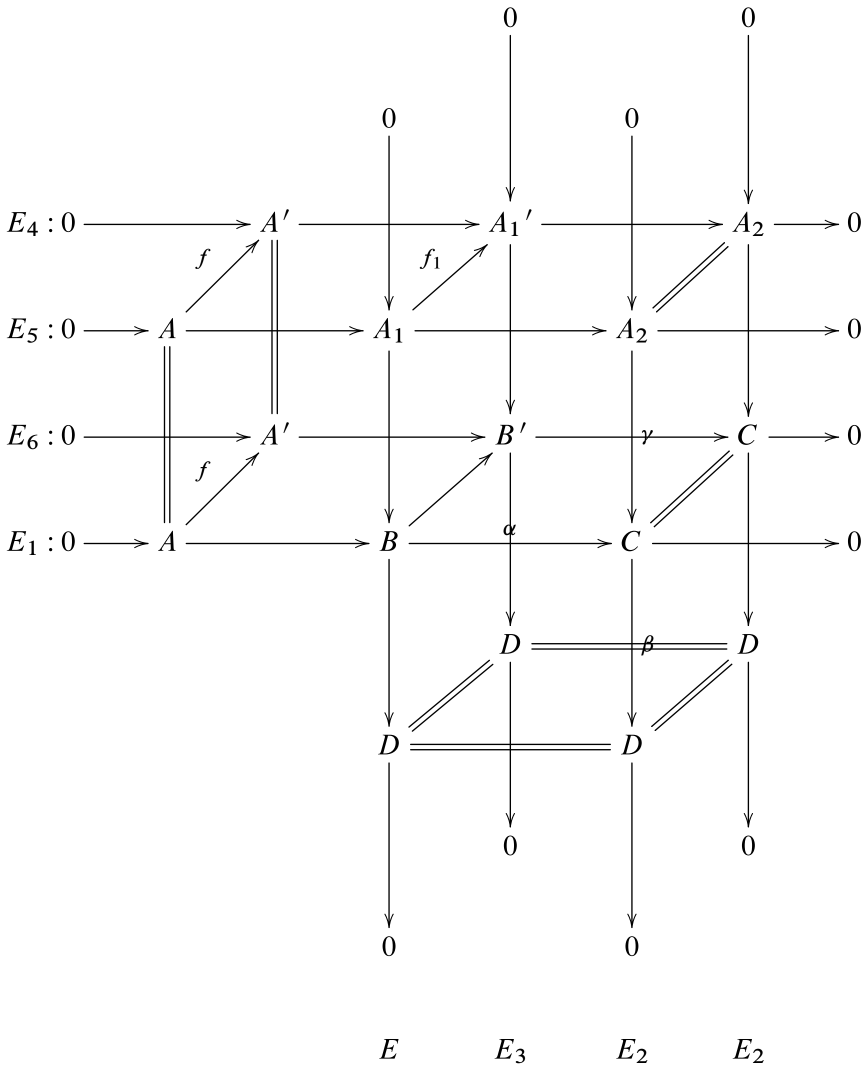
$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 \oplus A & \longrightarrow & B_1 \oplus B' & \longrightarrow & C \oplus C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_{A_1} \oplus f_2} & & \downarrow & & \| & & \\
0 & \longrightarrow & A_1 \oplus A_2 & \longrightarrow & B_1 \oplus B_2 & \longrightarrow & C \oplus C & \longrightarrow & 0
\end{array}$$

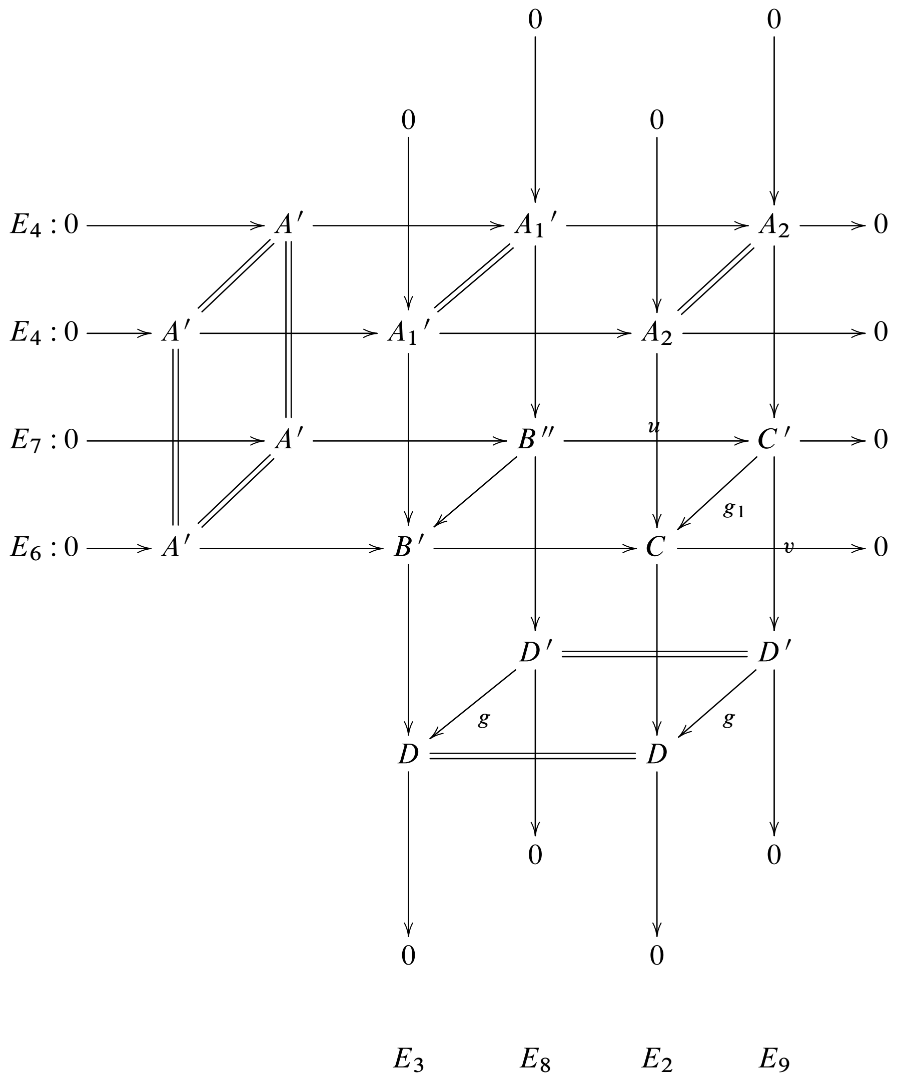
with exact rows.

By Lemma 1, $f_1 \oplus 1_A$ and $1_{A_1} \oplus f_2$ are in $\mathcal{F}$. Since $\mathcal{F}$ is closed under compositions, we have $(1_{A_1} \oplus f_2) \circ (f_1 \oplus 1_A) \in \mathcal{F}$. Since $\mathcal{P}$ is a proper class, $[(1_{A_1} \oplus f_2) \circ (f_1 \oplus 1_A)]_*(E \oplus E') \in \mathcal{P}$ with $(1_{A_1} \oplus f_2) \circ (f_1 \oplus 1_A) \in \mathcal{F}$. Therefore, $(E \oplus E') \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$.

These arguments show that $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ is an e-functor. Using [11, Theorem 1.1], in order to show that $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ is a proper class, it is enough to show that the composition of two $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$-epimorphisms is a $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$-epimorphism.

Let $E_1 : 0 \longrightarrow A \longrightarrow B \xrightarrow{\alpha} C \longrightarrow 0$ and $E_2 : 0 \longrightarrow A_2 \xrightarrow{\gamma} C \xrightarrow{\beta} D \longrightarrow 0 \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$. Then there are homomorphisms $f : A \longrightarrow A'$, $f \in \mathcal{F}$ and $g : D' \longrightarrow D$, $g \in \mathcal{G}$ such that $f_*(E_1) \in \mathcal{P}$ and $g^*(E_2) \in \mathcal{P}$. We have the following commutative diagrams with exact rows and columns:


$E_4 : 0$
$E_5 : 0$
$E_6 : 0$
$E_1 : 0$
A'
A
$A_1{}'$
A_1
A_2
B'
B
C
D
f
f_1
α
β
γ
E
E_3
E_2
E_2



In the first diagram, $E_6 = f_*(E_1) \in \mathcal{P}$, $E_4 = f_*(E_5) = f_*(\gamma^*(E_1)) = \gamma^*(f_*(E_1))$ since $\text{Ext}_{\mathcal{P}}$ is a subfunctor of Ext, and $f_1 \in \mathcal{F}$ since f_1 is constructed using pushout.

In the second diagram, $E_9 = g^*(E_2)$, $E_7 = g_1^*(E_6) \in \mathcal{P}$. Then $u : B'' \longrightarrow C'$ and $v : C' \longrightarrow D'$ are $\mathcal{P}$-epimorphisms. Since $\mathcal{P}$ is a proper class, $v \circ u$ is a $\mathcal{P}$-epimorphism and $E_8 \in \mathcal{P}$. $g^*(E_3) = E_8$ and $g \in \mathcal{G}$ implies $E_3 \in \hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$, so there is $f_2 : {A_1}' \longrightarrow {A_1}''$, $f_2 \in \mathcal{F}$ for some R-module ${A_1}''$ such that $f_{2*}(E_3) \in \mathcal{P}$.

Then $(f_2 \circ f_1)_*(E) = f_{2*}(f_{1*}(E)) = f_{2*}(E_3) \in \mathcal{P}$ and $(f_2 \circ f_1) \in \mathcal{F}$ since $f_1, f_2 \in \mathcal{F}$, and $\mathcal{F}$ is closed under compositions. Hence $E \in \hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$. $\square$

Remark 1. For the compatible pair $(\mathcal{F}, \mathcal{G})$ for the proper class $\mathcal{P}$, if there is a homomorphism $f \in \mathcal{F}$ from every R-module A or equivalently there is a homomorphism $g \in \mathcal{G}$ to every R-module C, then $\mathcal{P} \subseteq \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$.

In case the condition $\mathcal{P} \subseteq \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ is satisfied, a similar result obtained for the classes $\mathcal{P}$ and $\hat{\mathcal{P}}_r$ in the previous section also holds for the class $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ under some extra conditions on the classes $\mathcal{F}$ and $\mathcal{G}$.

Theorem 8. *Let $\mathcal{P}$ be a proper class, $(\mathcal{F}, \mathcal{G})$ a compatible pair for $\mathcal{P}$, and one of the classes $\mathcal{F}$ and $\mathcal{G}$ contain identity homomorphisms. Moreover, let the inclusions $i_f : \operatorname{Ker} f \longrightarrow A$ belong to $\mathcal{F}$ for all $f : A \longrightarrow A' \in \mathcal{F}$ for R-modules A, A' if $\mathcal{F}$ is closed under pushout diagrams, and the epimorphisms $\pi_g : C \longrightarrow C/\operatorname{Im} g$ belong to $\mathcal{G}$ for all $g : C' \longrightarrow C \in \mathcal{G}$ for R-modules C, C' if $\mathcal{G}$ is closed under pullback diagrams. Then we have*

$$\mathcal{P} + \widehat{\mathcal{S}plit}_{\mathcal{F}}^{\mathcal{G}} = \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}.$$

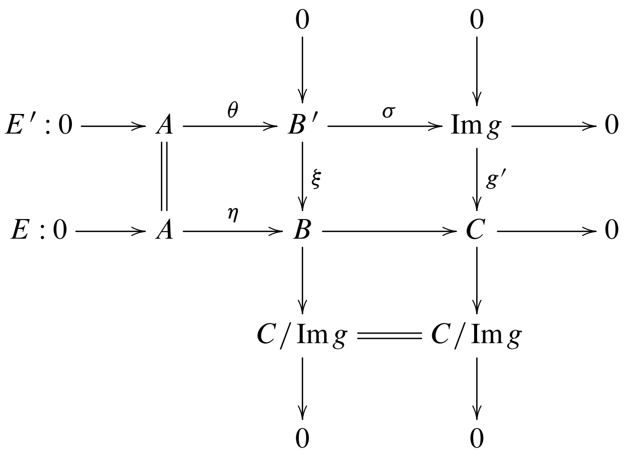
Proof. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \in \mathcal{P}$. If $\mathcal{F}$ contains identity homomorphisms, then $E = 1 \cdot E = 1_{A*}(E) \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ since $E \in \mathcal{P}$. If $\mathcal{G}$ contains identity homomorphisms, then $E = 1 \cdot E = 1_C^*(E) \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ since $E \in \mathcal{P}$. Since $\mathcal{S}plit \subseteq \mathcal{P}$ for every proper class $\mathcal{P}$, $\widehat{\mathcal{S}plit}_{\mathcal{F}}^{\mathcal{G}} \subseteq \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$. Combining these inclusions we obtain $\mathcal{P} + \widehat{\mathcal{S}plit}_{\mathcal{F}}^{\mathcal{G}} \subseteq \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$.

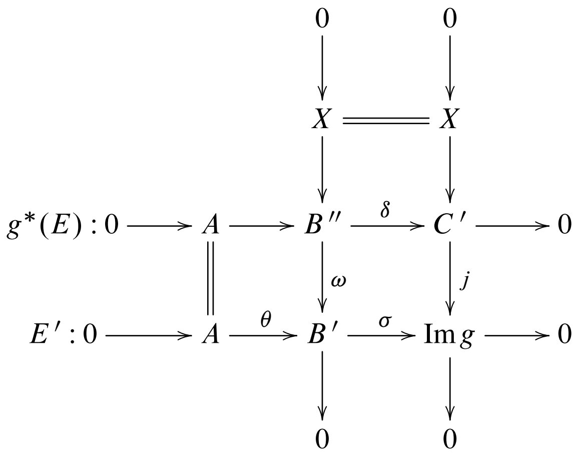
Let us write $\mathcal{P} + \widehat{\mathcal{S}plit}_{\mathcal{F}}^{\mathcal{G}} = \mathcal{L}$. We will show that $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}} \subseteq \mathcal{L}$.

At this point, we separate the proof into two parts and give the proof when $\mathcal{G}$ is closed under compositions and pullback diagrams. Proof for the other case can be done using similar arguments and duality.

Let $\mathcal{G}$ be closed under compositions and pullback diagrams. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$. Then there is a homomorphism $g : C' \longrightarrow C$, $g \in \mathcal{G}$ for some R-module C' such that $g^*(E) \in \mathcal{P}$. We can write the homomorphism g as $g = g' \circ j$, where $g' : \operatorname{Im} g \longrightarrow C$ is the inclusion and $j : C' \longrightarrow \operatorname{Im} g$ is the epimorphism induced by g.

Using the homomorphisms j and g', we obtain the following commutative diagrams with exact rows and columns:





E_2

We have $g'^{*}(E') = g^{*}(E) \in \mathcal{P}$, then δ is a $\mathcal{P}$-epimorphism. Since $\mathcal{P} \subseteq \mathcal{L}$, δ is an $\mathcal{L}$-epimorphism. Since $j \in \mathcal{G}$ by Lemma 1 and $j^{*}(E_2) \in \mathcal{S}plit$ by [10, Ch. 3, Proposition 1.7], j is a $\mathcal{S}\hat{p}lit_{\mathcal{F}}^{\mathcal{G}}$-epimorphism. Since $\mathcal{S}\hat{p}lit_{\mathcal{F}}^{\mathcal{G}} \subseteq \mathcal{L}$, j is an $\mathcal{L}$-epimorphism. Then $\sigma \circ \omega = j \circ \delta$ is an $\mathcal{L}$-epimorphism since $\mathcal{L}$ is a proper class by definition, and σ is an $\mathcal{L}$-epimorphism since $\mathrm{Ext}_{\mathcal{L}}$ is a subfunctor of Ext. Therefore θ is an $\mathcal{L}$-monomorphism.

Since $\pi_g : C \longrightarrow C/\mathrm{Im}\, g$ and $g : C' \longrightarrow C$ belong to $\mathcal{G}$, $\pi_g \circ g$ is in $\mathcal{G}$ since $\mathcal{G}$ is closed under compositions. Therefore, $C/\mathrm{Im}\, g$ is $\mathcal{S}\hat{p}lit_{\mathcal{F}}^{\mathcal{G}}$-coprojective and ξ is a

$\mathcal{S}\hat{plit}_{\mathcal{F}}^{\mathcal{G}}$-monomorphism, and ξ is an $\mathcal{L}$-monomorphism since $\mathcal{S}\hat{plit}_{\mathcal{F}}^{\mathcal{G}} \subseteq \mathcal{L}$. Then $\eta = \xi \circ \theta$ is an $\mathcal{L}$-monomorphism since $\mathcal{L}$ is a proper class by definition. Hence $E \in \mathcal{L}$ and $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}} \subseteq \mathcal{L}$. □

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