Abstract. In this paper, our approach allows to refine the results announced by Ebadian et al. [Results Math., 36 (2013), 409–423]. Namely, we reduce the distance between approximate and exact double derivations on Banach algebras and Lie \( C^* \)-algebras up to \( \frac{1}{2n} \) and \( \frac{1}{2n^2} \) for \( n \geq 2 \). Indeed, we give a correct utilization of fixed point theory in the sense of Diaz and Margolis [Bull. Amer. Math. Soc., 74 (1968), 305–309] concerning the stability of double derivations.

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1. Introduction

A classical question in the theory of functional equations is the following: when is it true that a mapping which approximately satisfies a functional equation \( \xi \) must be somehow near to an exact solution of \( \xi \)?

In 1940, Ulam [7] gave a wide ranging talk and discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms.

Let \((G_1,\cdot)\) be a group and let \((G_2,\ast)\) be a metric group with the metric \(d(\cdot,\cdot)\). Given \( \epsilon > 0 \), does there exist a \( \delta > 0 \), such that if a mapping \( h: G_1 \to G_2 \) satisfies the inequality \( d(h(x,y), h(x) \ast h(y)) < \delta \) for all \( x, y \in G_1 \), then there exists a homomorphism \( H: G_1 \to G_2 \) with \( d(h(x), H(x)) < \epsilon \) for all \( x \in G_1 \)?

Generally, the concept of stability for a functional equation comes up when the functional equation is replaced by an inequality which acts as a perturbation of that equation. The case of approximately additive functions was solved by D. Hyers [5] under certain assumptions. In 1950, Hyers’ Theorem was generalized by Aoki [1] for additive mappings and independently, in 1978, by Rassias [6] for linear mappings considering the Cauchy difference controlled by sum of powers of norms. For the history and various aspects of this theory we refer the reader to [3] and the references therein. Note that a functional equation \( \xi \) is stable if any function \( g \) satisfying the equation \( \xi \) approximately is near to true solution of \( \xi \).
Recently, Ebadian et al. [3] used the fixed point alternative method to establish the Hyers–Ulam stability of double derivations on Banach algebras and Lie $*$-double derivations on Lie $C^*$-algebras associated with the following additive functional equation

$$\sum_{k=2}^{n} \left( \sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \ldots \sum_{i_{n-k+1}=i_{n-k}+1}^{k+1} f\left( \sum_{i=1}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r} \right) \right) + f\left( \sum_{i=1}^{n} x_i \right) = 2^{n-1} f(x_1).$$

Throughout this paper following [3], we assume that $\mathcal{A}$ is a Banach algebra (Lie $C^*$-algebra). For given mapping $f : \mathcal{A} \to \mathcal{A}$, we define the difference operator $D_\mu f : \mathcal{A}^n \to \mathcal{A}$ by

$$D_\mu f(x_1, \ldots, x_n) := \sum_{k=2}^{n} \left( \sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \ldots \sum_{i_{n-k+1}=i_{n-k}+1}^{k+1} f\left( \mu x_i - \sum_{r=1}^{n-k+1} \mu x_{i_r} \right) \right) + f\left( \sum_{i=1}^{n} \mu x_i \right) - 2^{n-1} f(\mu x_1)$$

for all $\mu \in \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$.

In this paper, we improve main results of [3] and reduce the distance between approximate and exact double derivations on Banach algebras and Lie $C^*$-algebras up to $\frac{1}{2n-1}$ and $\frac{1}{2n-2}$ for $n \geq 2$.

In section 2, we discuss on main results of [3] and improve some theorems and corollaries including Theorem 2.3, 2.5 and Corollary 2.4, 2.6. In section 3, we also refine some results of [3] including Theorem 3.2, 3.4 and Corollary 3.3, 3.5. Indeed, we are going to weaken their assumptions and giving a correct utilization of fixed point theory in the sense of Diaz and Margolis [2].

2. Almost Double Derivation

Throughout this section, we assume that $\mathcal{A}$ is a Banach algebra, $f(0) = g(0) = h(0) = 0$, and for given mappings $f, g, h : \mathcal{A} \to \mathcal{A}$, we define

$$C_{f,g,h}(a,b) := f(ab) - f(a)b - af(b) - g(a)h(b) - h(a)g(b)$$

for all $a, b \in \mathcal{A}$.

**Definition 1.** Let $\mathcal{A}$ be a Banach algebra and let $\delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ be $\mathbb{C}$-linear mappings. A $\mathbb{C}$-linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a $(\delta, \varepsilon)$-double derivation if

$$f(ab) = f(a)b + af(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b)$$
for all $a, b \in A$.

A fundamental result in fixed point theory is the following theorem. We recall the following theorem by Diaz and Margolis [2].

**Theorem 1.** Let $(X, d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all $n \geq 0$, or there exists a natural number $n_0$ such that

1. $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$.
2. The sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$.
3. $y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$.
4. $d(y, y^*) \leq \frac{1}{1 - L} d(y, Jy)$ for all $y \in Y$.

The following theorem is a refined version of [3, Theorem 2.3]:

**Theorem 2.** Let $f, g, h : A \to A$ be mappings for which there exist functions $\varphi : A^n \to [0, \infty)$ and $\psi : A^2 \to [0, \infty)$ such that

\[
\lim_{m \to \infty} 2^m \varphi(x_1, \ldots, x_n) = 0, \tag{2.1}
\]

\[
\lim_{m \to \infty} 4^m \psi\left(\frac{a}{2^m}, \frac{b}{2^m}\right) = 0, \tag{2.2}
\]

\[
||D_\mu f(x_1, \ldots, x_n)|| \leq \varphi(x_1, \ldots, x_n), \tag{2.3}
\]

\[
||C f, g, h(a, b)|| \leq \psi(a, b) \tag{2.4}
\]

for all $a, b, x_1, \ldots, x_2 \in A$ and all $\mu \in T^1$. If there exists a constant $0 < L < 1$ such that $\varphi(x_1, \ldots, x_n) \leq \frac{L}{2} \varphi(2x_1, \ldots, 2x_n)$ for all $x_1, \ldots, x_n \in A$, then there exist unique $C$-linear mappings $d, \delta, \epsilon : A \to A$ such that

\[
||f(x) - d(x)|| \leq \frac{L}{\alpha(1 - L)} \varphi(x, x, 0, \ldots, 0), \tag{2.5}
\]

\[
||g(x) - \delta(x)|| \leq \frac{L}{\alpha(1 - L)} \varphi(x, x, 0, \ldots, 0), \tag{2.6}
\]

\[
||h(x) - \epsilon(x)|| \leq \frac{L}{\alpha(1 - L)} \varphi(x, x, 0, \ldots, 0) \tag{2.7}
\]

for all $x \in A$, where $\alpha = 2^{n-1}$ and $n \geq 2$. Moreover, $d$ is a $(\delta, \epsilon)$-double derivation on $A$.

**Proof.** Put $\mu = 1, x_1 = x_2 = x$, and $x_3 = x_4 = \ldots = x_n = 0$ in (2.3) to reach

\[
||\frac{\alpha}{2} f(2x) - \alpha f(x)|| \leq \varphi(x, x, 0, \ldots, 0)
\]

for all $x \in A$ and so

\[
||2f\left(\frac{x}{2}\right) - f(x)|| \leq \frac{2}{\alpha} \varphi\left(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0\right) \leq \frac{L}{\alpha} \varphi(x, x, 0, \ldots, 0). \tag{2.8}
\]
Define $F := \{ f : \mathcal{A} \to \mathcal{A} \}$. The metric defined on $F$ by

$$
\rho(f, g) := \inf \{ c \in [0, \infty] : \| f(x) - g(x) \| \leq c \varphi(x, x, 0, \ldots, 0), \forall x \in \mathcal{A} \}.
$$

is a generalized metric and $(F, \rho)$ is a generalized complete metric space. Consider the mapping $(Jf)(x) := 2f(\frac{x}{2})$ for all $f \in F$ and $x \in \mathcal{A}$. Use [4, Lemma 1.3] to see that $J$ is a strictly contractive mapping with the Lipschitz constant $L$. It follows from (2.8) that $\rho(Jf, f) \leq \frac{L}{2}$. Therefore according to Theorem 1, the sequence $\{J^m f\}$ converges to a fixed point $d$ such that $d(x) = \lim_{m \to \infty} 2^m f(\frac{x}{2^m})$ and $d(2x) = 2d(x)$. Note that $d$ is the unique fixed point of $J$ and

$$
\rho(d, f) \leq \frac{1}{1-L} \rho(Jf, f) \leq \frac{L}{\alpha(1-L)}.
$$

This means that inequality (2.5) holds for all $x \in \mathcal{A}$. The proof of the linearity of $d$ and also the rest of the proof is similar to that of [3, Theorem 2.3] and we omit it. $\square$

The importance of our result becomes clear when we take

$$
\varphi(x_1, \ldots, x_n) = \theta_1 \sum_{i=1}^n \| x_i \|^p, \quad \psi(a, b) = \theta_2(\| a \|^q + \| b \|^q).
$$

In this situation, by choosing $L = 2^{1-p}$, we can get strong and close approximations of the functions $f, g, h$ with linear mappings $d, \delta, \varepsilon$, where $d$ is a $(\delta, \varepsilon)$-double derivation on $\mathcal{A}$. Thus, we improve [3, Corollary 2.4] up to $\frac{1}{2n-1}$ as follows:

**Corollary 1.** Let $p, q, \theta_1, \theta_2$ be non-negative real numbers with $p, q > 1$. Suppose that $f, g, h : \mathcal{A} \to \mathcal{A}$ are mappings such that

$$
\| D_\mu f(x_1, \ldots, x_n) \| \leq \theta_1 \sum_{i=1}^n \| x_i \|^p,
$$

$$
\| C_{f, g, h}(a, b) \| \leq \theta_2(\| a \|^q + \| b \|^q)
$$

for all $a, b, x_1, \ldots, x_n \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Then there exist unique $\mathcal{C}$-linear mappings $d, \delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ such that

$$
\| f(x) - d(x) \| \leq \frac{2\theta_1}{2n-1(2p-1-1)} \| x \|^p,
$$

$$
\| g(x) - \delta(x) \| \leq \frac{2\theta_1}{2n-1(2p-1-1)} \| x \|^p,
$$

$$
\| h(x) - \varepsilon(x) \| \leq \frac{2\theta_1}{2n-1(2p-1-1)} \| x \|^p
$$

for all $x \in \mathcal{A}$. Moreover, $d$ is a $(\delta, \varepsilon)$-double derivation on $\mathcal{A}$.

In the following theorem we give an improved version of [3, Theorem 2.5]:
Theorem 3. Suppose that $f, g, h : \mathcal{A} \to \mathcal{A}$ are mappings satisfying (2.3) and (2.4) for which there exist functions $\varphi : \mathcal{A}^n \to [0, \infty)$ and $\psi : \mathbb{R}^2 \to [0, \infty)$ such that
\begin{align*}
\lim_{m \to \infty} \frac{1}{2m} \varphi(2^m x_1, \ldots, 2^m x_n) &= 0, \\
\lim_{m \to \infty} \frac{1}{2m} \psi(2^m a, 2^m b) &= 0
\end{align*}
for all $a, b, x_1, \ldots, x_2 \in \mathcal{A}$. If there exists a constant $0 < L < 1$ such that $\varphi(x_1, \ldots, x_n) \leq 2L \psi \left( \frac{x_1}{2}, \ldots, \frac{x_n}{2} \right)$ for all $x_1, \ldots, x_n \in \mathcal{A}$, then there exist unique $\mathbb{C}$-linear mappings $d, \delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ such that
\begin{align*}
||f(x) - d(x)|| &\leq \frac{L}{\beta(1-L)} \varphi \left( \frac{x}{2}, \ldots, 0; 0; \ldots, 0 \right), \\
||g(x) - \delta(x)|| &\leq \frac{L}{\beta(1-L)} \varphi \left( \frac{x}{2}, \ldots, 0; 0; \ldots, 0 \right), \\
||h(x) - \varepsilon(x)|| &\leq \frac{L}{\beta(1-L)} \varphi \left( \frac{x}{2}, \ldots, 0; 0; \ldots, 0 \right)
\end{align*}
for all $x \in \mathcal{A}$, where $\beta = \frac{q}{2}$ and $n \geq 2$. Moreover, $d$ is a $(\delta, \varepsilon)$-double derivation on $\mathcal{A}$.

Proof. It follows from (2.8) that
\begin{align*}
||\frac{1}{2} f(2x) - f(x)|| &\leq \frac{1}{\alpha} \varphi(x, 0; 0; \ldots, 0) \leq \frac{2L}{\alpha} \varphi \left( \frac{x}{2}, \ldots, 0; 0; \ldots, 0 \right)
\end{align*}
for all $x \in \mathcal{A}$. Consider the generalized complete metric $(\mathcal{F}, \rho)$ with the generalized metric $\rho$ defined by
\[\rho(f, g) := \inf\{c \in [0, \infty] : ||f(x) - g(x)|| \leq c \varphi \left( \frac{x}{2}, \ldots, 0; 0; \ldots, 0 \right), \forall x \in \mathcal{A}\}.
\]
Define the mapping $(Jf)(x) := \frac{1}{2} f(2x)$ for all $f \in \mathcal{F}$ and $x \in \mathcal{A}$. Apply [4, Lemma 1.2]) to find that $J$ is a strictly contractive mapping with the Lipschitz constant $L$. It follows from (2.14) that $\rho(Jf, f) \leq \frac{2L}{\alpha}$. Applying Theorem 1, we get the sequence $\{J^m f\}$ converges to a unique fixed point $d$ of $J$ such that
\[\rho(d, f) \leq \frac{1}{1-\frac{2L}{\alpha}} \rho(Jf, f) \leq \frac{2L}{\alpha(1-L)} = \frac{L}{\beta(1-L)},
\]
i. e., inequality (2.11) holds for all $x \in \mathcal{A}$. The rest of the proof is similar to that of [3, Theorem 2.3].

As we mentioned in Corollary 1, the importance of Theorem 3 becomes also clear when we put $L = 2^{p-1}$ and
\[\varphi(x_1, \ldots, x_n) = \theta_1 + \theta_2 \sum_{i=1}^n ||x_i||^p, \ \psi(a, b) = \theta_1 + \theta_2(||a||^q + ||b||^q).
\]
However, we can improve [3, Corollary 2.6] up to $\frac{1}{2^n-2}$ as follows:

**Corollary 2.** Let $p, q, \theta_1, \theta_2$ be non-negative real numbers with $p, q \in (0, 1)$. Suppose that $f, g, h : A \rightarrow A$ are mappings such that

$$||D_\mu f(x_1, \ldots, x_n)|| \leq \theta_1 + \theta_2 \sum_{i=1}^{n} ||x_i||^p,$$

$$||C_{f,g,h}(a, b)|| \leq \theta_1 + \theta_2(||a||^q + ||b||^q)$$

for all $a, b, x_1, \ldots, x_n \in A$ and all $\mu \in \mathbb{T}^1$. Then there exist unique $\mathbb{C}$-linear mappings $d, \delta, \varepsilon : A \rightarrow A$ such that

$$||f(x) - d(x)|| \leq \frac{1}{2^n-2} \left( \frac{\theta_1}{2^{1-p} - 1} + \frac{\theta_2}{1 - 2^{p-1}} ||x||^p \right),$$

$$||g(x) - \delta(x)|| \leq \frac{1}{2^n-2} \left( \frac{\theta_1}{2^{1-p} - 1} + \frac{\theta_2}{1 - 2^{p-1}} ||x||^p \right),$$

$$||h(x) - \varepsilon(x)|| \leq \frac{1}{2^n-2} \left( \frac{\theta_1}{2^{1-p} - 1} + \frac{\theta_2}{1 - 2^{p-1}} ||x||^p \right)$$

for all $x \in A$. Moreover, $d$ is a $(\delta, \varepsilon)$-double derivation on $A$.

### 3. Almost Lie $*$-double derivation

A unital $C^*$-algebra $A$, endowed with the Lie product $[x, y] = xy - yx$ on $A$, is called a Lie $C^*$-algebra. In this section, we assume that $A$ is a Lie $C^*$-algebra and $U(A) = \{u \in A : uu^* = u^*u = e\}$. For given mappings $f, g, h : A \rightarrow A$, we let $f(0) = g(0) = h(0) = 0$ and define

$$J_{f,g,h}(a, b) := f([a, b]) - [f(a), b] - [a, f(b)] - [g(a), h(b)] - [h(a), g(b)]$$

for all $a, b \in A$.

**Definition 2.** Let $A$ be a Lie $C^*$-algebra and let $\delta, \varepsilon : A \rightarrow A$ be $\mathbb{C}$-linear mappings. A $\mathbb{C}$-linear mapping $f : A \rightarrow A$ is called a Lie $(\delta, \varepsilon)$-double derivation if

$$f([a, b]) = [f(a), b] + [a, f(b)] + [\delta(a), \varepsilon(b)] + [\epsilon(a), \delta(b)]$$

for all $a, b \in A$.

The presented results in this section are refinements of [3, Theorem 3.2, 3.4] and [3, Corollary 3.3, 3.5]:

**Theorem 4.** Let $f, g, h : A \rightarrow A$ be mappings for which there exist functions $\varphi : \mathbb{A}^n \rightarrow [0, \infty)$ and $\psi : \mathbb{A}^2 \rightarrow [0, \infty)$ such that

$$\lim_{m \rightarrow \infty} 2^m \varphi(x_1^{2^m}, \ldots, x_n^{2^m}) = 0,$$

$$\lim_{m \rightarrow \infty} 4^m \psi\left(\frac{a}{2^m}, \frac{b}{2^m}\right) = 0,$$

$$||D_\mu f(x_1, \ldots, x_n)|| \leq \varphi(x_1, \ldots, x_n),$$

for all $x_1, \ldots, x_n \in A$, then there exist unique $\mathbb{C}$-linear mappings $d, \delta, \varepsilon : A \rightarrow A$ such that

$$||f(x) - d(x)|| \leq \frac{1}{2^n-2} \left( \frac{\theta_1}{2^{1-p} - 1} + \frac{\theta_2}{1 - 2^{p-1}} ||x||^p \right),$$

$$||g(x) - \delta(x)|| \leq \frac{1}{2^n-2} \left( \frac{\theta_1}{2^{1-p} - 1} + \frac{\theta_2}{1 - 2^{p-1}} ||x||^p \right),$$

$$||h(x) - \varepsilon(x)|| \leq \frac{1}{2^n-2} \left( \frac{\theta_1}{2^{1-p} - 1} + \frac{\theta_2}{1 - 2^{p-1}} ||x||^p \right)$$

for all $x \in A$. Moreover, $d$ is a $(\delta, \varepsilon)$-double derivation on $A$. 

\begin{align}
|f(a, b)| & \leq \psi(a, b) \\
\max\{f(u_k^*) - f(u_{2k})^* - g(u_k^*) - g(u_{2k})^* + h(u_k^*) - h(u_{2k})^*\} & \leq \varphi(u_{2k}^*, \ldots, u_{2k}^*) 
\end{align}

for all \(a, b, x_1, \ldots, x_2 \in \mathcal{A}, k = 0, 1, 2, \ldots, u \in U(A),\) and \(\mu \in \mathbb{T}^1.\) If there exists a constant \(0 < L < 1\) such that \(\varphi(x_1, \ldots, x_n) \leq \frac{L}{2} \varphi(2x_1, \ldots, 2x_n)\) for all \(x_1, \ldots, x_n \in \mathcal{A},\) then there exist unique \(\mathbb{C}\)-linear mappings \(d, \delta, \varepsilon : \mathcal{A} \to \mathcal{A}\) such that

\[
\max\{|f(x) - d(x)|, |g(x) - \delta(x)|, |h(x) - \varepsilon(x)|\} \leq \frac{L}{\alpha(1 - L)} \varphi(x, 0, \ldots, 0)
\]

for all \(x \in \mathcal{A}.\) Moreover, \(d\) is a Lie \(*-(\delta, \varepsilon)\)-double derivation on \(\mathcal{A}.\)

\begin{proof}
Using the same methods as in the proof of [3, Theorem 2.3, 3.2], we can obtain the desired results. \(\square\)
\end{proof}

**Corollary 3.** Let \(p, q, \theta_1, \theta_2\) be non-negative real numbers with \(p, q > 1.\) Suppose that \(f, g, h : \mathcal{A} \to \mathcal{A}\) are mappings such that

\[
|D \mu f(x_1, \ldots, x_n)| \leq \theta_1 \sum_{i=1}^{n} |x_i|^p,
\]
\[
|f(g(u, b))| \leq \theta_2(1 + |b|^q)
\]
\[
\max\{f(u_{2k}^*) - f(u_{2k})^* - g(u_k^*) - g(u_{2k})^* + h(u_k^*) - h(u_{2k})^*\} \leq \frac{\theta_1 + \theta_2}{2^k}
\]

for all \(a, b, x_1, \ldots, x_n \in \mathcal{A}, k = 0, 1, 2, \ldots, u \in U(A),\) and \(\mu \in \mathbb{T}^1.\) There exist unique \(\mathbb{C}\)-linear mappings \(d, \delta, \varepsilon : \mathcal{A} \to \mathcal{A}\) such that

\[
\max\{|f(x) - d(x)|, |g(x) - \delta(x)|, |h(x) - \varepsilon(x)|\} \leq \frac{2\theta_1}{2^{n-1}(2^p - 1 - 1)} |x|^p
\]

for all \(x \in \mathcal{A}.\) Moreover, \(d\) is a \((\delta, \varepsilon)\)-double derivation on \(\mathcal{A}.\)

\begin{proof}
The results follows from above theorem by taking \(L = 2^{1-p}\) and

\[
\varphi(x_1, \ldots, x_n) = \theta_1 \sum_{i=1}^{n} |x_i|^p, \ \psi(a, b) = \theta_2(1 + |b|^q).
\]
\(\square\)

**Theorem 5.** Suppose that \(f, g, h : \mathcal{A} \to \mathcal{A}\) are mappings satisfying (3.3) and (3.4) for which there exist functions \(\varphi : \mathcal{A}^n \to [0, \infty)\) and \(\psi : \mathcal{A}^2 \to [0, \infty)\) such that

\[
\lim_{m \to \infty} \frac{1}{2^m} \varphi(2^m x_1, \ldots, 2^m x_n) = 0,
\]
\[
\lim_{m \to \infty} \frac{1}{4^m} \psi(2^m a, 2^m b) = 0
\]

\[
\max\{f(2^k u^*) - f(2^k u)^* - g(2^k u^*) - g(2^k u)^* + h(2^k u^*) - h(2^k u)^*\} \leq \varphi(2^k u, \ldots, 2^k u)
\]
for all \( a, b, x_1, \ldots, x_2 \in \mathcal{A} \), \( k = 0, 1, 2, \ldots, u \in U(A) \), and \( \mu \in \mathbb{T}^1 \). If there exists a constant \( 0 < L < 1 \) such that \( \varphi(x_1, \ldots, x_n) \leq 2L\varphi(\frac{x_1}{2}, \ldots, \frac{x_n}{2}) \) for all \( x_1, \ldots, x_n \in \mathcal{A} \), then there exist unique \( \mathbb{C} \)-linear mappings \( d, \delta, \epsilon : \mathcal{A} \to \mathcal{A} \) such that
\[
\max\{||f(x) - d(x)||, ||g(x) - \delta(x)||, ||h(x) - \epsilon(x)||\} \leq \frac{L}{\beta(1 - L)} \varphi\left(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0\right)
\]
for all \( x \in \mathcal{A} \), where \( \beta = \frac{a}{2} \) and \( n \geq 2 \). Moreover, \( d \) is a Lie \( \ast \)-\((\delta, \epsilon)\)-double derivation on \( \mathcal{A} \).

**Proof.** The proof is similar to that of [3, Theorem 2.5, 3.2].

**Corollary 4.** Let \( p, q, \theta_1, \theta_2 \) be non-negative real numbers with \( p, q \in (0, 1) \). Suppose that \( f, g, h : \mathcal{A} \to \mathcal{A} \) are mappings such that
\[
||D_\mu f(x_1, \ldots, x_n)|| \leq \theta_1 + \theta_2 \sum_{i=1}^{n} ||x_i||^p,
\]
\[
||Jf, g, h(u, b)|| \leq \theta_1 + \theta_2(1 + ||b||^q)
\]
\[
\max\{||f(2^k u^*) - f(2^k u)^*||, ||g(2^k u^*) - g(2^k u)^*||, ||h(2^k u^*) - h(2^k u)^*||\} \leq \frac{\theta_1 + \theta_2}{2^kp}
\]
for all \( a, b, x_1, \ldots, x_n \in \mathcal{A}, k = 0, 1, 2, \ldots, u \in U(A) \), and \( \mu \in \mathbb{T}^1 \). Then there exist unique \( \mathbb{C} \)-linear mappings \( d, \delta, \epsilon : \mathcal{A} \to \mathcal{A} \) such that
\[
\max\{||f(x) - d(x)||, ||g(x) - \delta(x)||, ||h(x) - \epsilon(x)||\} \leq \frac{\theta_1 + \theta_2}{2n - 2^{1-p} - 1}||x||^p + \frac{\theta_1}{1 - 2^{1-p} - 1}||x||^p
\]
for all \( x \in \mathcal{A} \). Moreover, \( d \) is a Lie \( \ast \)-\((\delta, \epsilon)\)-double derivation on \( \mathcal{A} \).

**Proof.** Apply above theorem by putting \( L = 2^{p-1} \) and
\[
\varphi(x_1, \ldots, x_n) = \theta_1 + \theta_2 \sum_{i=1}^{n} ||x_i||^p, \quad \psi(a, b) = \theta_1 + \theta_2(1 + ||b||^q).
\]

4. CONCLUSION

Our results can give the results proved by Ebadian et al. [3]. For instance, under the hypotheses of Theorem 2 we can conclude [3, Theorem 2.3], but not vice versa. In other words, if there exists a constant \( 0 < L < 1 \) such that \( \varphi(x_1, \ldots, x_n) \leq L\varphi(2x_1, \ldots, 2x_n) \) for all \( x_1, \ldots, x_n \in \mathcal{A} \), then \( \varphi(x_1, \ldots, x_n) \leq \frac{L}{2}L\varphi(2x_1, \ldots, 2x_n) \), i.e., all of the hypotheses of [3, Theorem 2.3] hold. On the other hand, Theorem 2 says that there exist unique \( \mathbb{C} \)-linear mappings \( d, \delta, \epsilon : \mathcal{A} \to \mathcal{A} \) such that
\[
||f(x) - d(x)|| \leq \frac{L}{\alpha(1 - L)} \varphi(x, x, 0, \ldots, 0),
\]
\[ ||g(x) - \delta(x)|| \leq \frac{L}{\alpha(1-L)} \varphi(x,x,0,...,0), \]
\[ ||h(x) - \epsilon(x)|| \leq \frac{L}{\alpha(1-L)} \varphi(x,x,0,...,0) \]
for all \( x \in \mathcal{A} \), where \( \alpha = 2^{n-1} \) and \( n \geq 2 \). Since \( \frac{L}{\alpha(1-L)} \leq \frac{L}{1-L} \), we have
\[ ||f(x) - d(x)|| \leq \frac{L}{1-L} \varphi(x,x,0,...,0), \]
\[ ||g(x) - \delta(x)|| \leq \frac{L}{1-L} \varphi(x,x,0,...,0), \]
\[ ||h(x) - \epsilon(x)|| \leq \frac{L}{1-L} \varphi(x,x,0,...,0), \]
which coincide with the results of [3, Theorem 2.3]. The same arguments can be applied for Theorem 2.5, 3.2, 3.4 and Corollary 2.4, 2.6, 3.3, 3.5 of [3].

References


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