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QUANTITATIVE ESTIMATES FOR SOME MODIFIED BERNSTEIN-STANCU OPERATORS

VOICHIŢA ADRIANA RADU

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Abstract. In the papers [11,12] starting with the Bernstein operators, some Stancu type operators are constructed

$$C_n : Y \to \Pi_n$$

$$(C_n f)(x) = \sum_{k=0}^{n} \frac{k!}{n^k} \binom{n}{k} \mathbf{m}_{k,n} \left[0, \frac{1}{n}, \dots, \frac{k}{n}; f\right] x^k,\ f \in C[0,1],$$

where Y is the linear space of all functions $[0,1] \to \mathbb{R}$ and the real numbers $(m_{k,n})_{k=0}^{\infty}$ are selected in order to preserve some important properties of Bernstein operators.

For $\mathbf{m}_{j,n} = \frac{(a_n)_j}{j!}$, $a_n \in (0,1]$ we obtained Bernstein-Stancu operators

$$\left(\overline{C}_n f\right)(x) = \sum_{k=0}^{n} \frac{(a_n)_k}{n^k} \binom{n}{k} \left[0, \frac{1}{n}, \dots, \frac{k}{n}; f\right] x^k.$$

The aim of this paper is to give some estimates for this operators using moduli of smoothness of first and second order.

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1. Introduction

First of all, we recall some notions and operators which will be used in the paper.

Let Π be the algebra of polynomials with real coefficients and Π_n be the linear space of all real polynomials of degree $\leq n$.

For $k \in \mathbb{N}$, $z \in \mathbb{C}$ let $(z)_0 = 1$ and $(z)_k = z(z+1)\dots(z+k-1)$.

For $n \in \mathbb{N}$, let $B_n : Y \to \Pi_n$ be the Bernstein operators, defined for any $f \in C[0,1]$ by

$$(B_n f)(x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right)$$

where

$$b_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$$

are the Bernstein fundamental polynomials.

For $g:[0,1]\to\mathbb{R}$ the Stancu operators $S_k^{<b>}:g\to S_k^{<b>}g,\ k\in\mathbb{N}$ are defined as $\left(S_0^{<b>}g\right)(x)=g(0)$ and for $k\in\{1,2,...\}$

$$\left(S_k^{<b>}g\right)(x)=\frac{1}{b_k}\sum_{j=0}^{k}\binom{k}{j}(bx)_j(b-bx)_{k-j}\cdot g\left(\frac{j}{k}\right),\ x\in[0,1]$$

where $b\in[0,1]$ is a real parameter (see [13, 24, 25]).

2. The construction of the modified Bernstein-Stancu operators

Approximation theory has been used in the theory of approximation of continuous functions by means of sequences of positive linear operators and still there remains a very active area of research. There are many approximating operators that their Korovkin type approximation properties and rates of convergence are investigated.

We list some of the mathematicians that relate their names to this fields of constructing and studding approximation properties of the linear and positive operators: A. Lupaş [20], O. Agratini [5], D. Bărbosu [8], [9], I. Gavrea, H.H. Gonska and D.P. Kacso [16], [17], O. Dogru [14], U. Abel, M. Ivan, R. Păltănea [1] and Y. Kageyama [18].

A new direction of generalization of the linear and positive operators are q-calculus as we can see in the pioneering works of A. Lupaş [19] and G.M. Philips [23]. Some of the most recent appearances in this direction are the papers of O. Agratini [6], P.N. Agrawal, Z. Finta and A. Sathish Kumar [15], [7], G. Nowak and V. Gupta [21], A.M. Acu, C.V. Muraru, D. Bărbosu and D.F. Sofonea [2], [4] and [3].

In [11] was constructed a new class of linear and positive operators starting with the derivatives of the Bernstein operators.

For $j\in\{0,1,...,n\}$

$$\frac{1}{j!}\frac{d^j(B_nf)(x)}{dx^j}=\binom{n}{j}\frac{j!}{n^j}\sum_{k=0}^{n-j}b_{n-j,k}(x)\left[\frac{k}{n},\frac{k+1}{n},...,\frac{k+j}{n};f\right],$$

the following formula holds

$$(B_nf)(x)=\sum_{k=0}^{n}\frac{k!}{n^k}\binom{n}{k}\left[0,\frac{1}{n},...,\frac{k}{n};f\right]x^k. \tag{2.1}$$

Starting with (2.1), we investigated the modifications

$$C_n : Y \to \Pi_n$$

$$(C_n f)(x) = \sum_{k=0}^{n} \frac{k!}{n^k} \binom{n}{k} \mathbf{m}_{k,n} \left[0, \frac{1}{n}, \dots, \frac{k}{n}; f\right] x^k, \ f \in Y, \tag{2.2}$$

where the real numbers $\left(m_{k,n}\right)_{k=0}^{\infty}$ are selected in order to preserve some important properties of Bernstein operators.

Observe that from (2.2)

$$\begin{cases} C_n e_0 = \mathbf{m}_{0,n} \\ C_n e_1 = \mathbf{m}_{1,n} e_1 \\ C_n e_2 = \mathbf{m}_{2,n} e_2 + \frac{e_1}{n} \left(\mathbf{m}_{1,n} - \mathbf{m}_{2,n} e_1\right) \\ \left(C_n \Omega_{2,x}\right)(x) = \left(\mathbf{m}_{2,n} - 2\mathbf{m}_{1,n} + \mathbf{m}_{0,n}\right) x^2 + \frac{x}{n} \left(\mathbf{m}_{1,n} - \mathbf{m}_{2,n} x\right) \end{cases}$$

where $e_j(t) = t^j$ and $\Omega_{2,x} = (t-x)^2$.

In the following we shall consider that $\mathbf{m}_{0,n} = 1$ and $\lim_{n\to\infty} \mathbf{m}_{1,n} = 1$.

We also consider that

$$\mathbf{m}_{j,n} = \frac{(a_n)_j}{j!}, \ a_n \in (0,1]$$

Then the operator C_n from (2.2), denoted further by $\overline{C}_n$, becomes

$$\left(\overline{C}_n f\right)(x) = \sum_{k=0}^{n} \frac{(a_n)_k}{n^k} \binom{n}{k} \left[0, \frac{1}{n}, \dots, \frac{k}{n}; f\right] x^k, \ f \in Y. \tag{2.3}$$

$\overline{C}_n$ are called **Bernstein-Stancu operators**, when $a_n \in (0,1)$ (see [11], Definition 11).

3. A SUMMING UP OF THE APPROXIMATION PROPERTIES OF $\overline{C}_n$ OPERATOR

First, in [12], it is demonstrated that $\overline{C}_n$ is a linear and positive operator that transform any polynomial of degree $s \le n$ into a polynomial of degree s and preserves the convexity of order j, if $j,n \in \mathbb{N}^*, 0 \le j \le n-2$.

Also, the operator $\overline{C}_n$ from (2.3) may be written in the Bernstein basis in the form ([11], Theorem 10)

$$(\overline{C}_n f)(x) = \sum_{k=0}^{n} b_{n,k}(x) \overline{C}_{k,n}[f]$$

with

$$\overline{C}_{k,n}[f] = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} f\left(\frac{j}{n}\right) (a_n)_j (1-a_n)_{k-j}.$$

In the same time $\overline{C}_n$ can be written using Stancu functionals.

Observe that

$$\overline{C}_0 f = \overline{C}_{0,0}[f] := f(0)$$

and

$$\left(S_k^{<1>} g\right)(a_n) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(a_n)_j(1-a_n)_{k-j}\, g\left(\frac{j}{k}\cdot\frac{k}{n}\right),\ k \geq 1.$$

Therefore,

$$\overline{C}_{k,n}[f] = \left(S_k^{<1>} g_{n,k}^{<f>}\right)(a_n)$$

with

$$g_{n,k}^{<f>}(t) = f\left(t\frac{k}{n}\right),\ k \geq 1.$$

In order to provide the convergence theorem, the following identities hold true ([11], Lemma 8):

$$\begin{cases} \left(\overline{C_n}e_0\right)(x) = 1 \\ \left(\overline{C_n}e_1\right)(x) = a_n x = x - (1-a_n)x \\ \left(\overline{C_n}e_2\right)(x) = x^2 + \dfrac{x(1-x)}{n}a_n + \dfrac{1-a_n}{2}\left(\frac{a_n}{n} - (2+a_n)\right)x^2. \end{cases}$$

Also, if $f \in Y$, $\Omega_{2,x} = (t-x)^2$ and $a_n \in (0,1]$, then

$$\left(\overline{C_n}\Omega_{2,x}\right)(x) = \frac{x(1-x)}{n}a_n + x^2(1-a_n)\left(\frac{2-a_n}{2} + \frac{a_n}{2n}\right).$$

Moreover

$$\left|\left(\overline{C_n}\Omega_{2,x}\right)(x)\right| \leq \frac{a_n}{4n} + (1-a_n),\ \ \forall x \in [0,1]. \tag{3.1}$$

Applying the Bohman - Korovkin theorem and the above assertions ([12], Theorem 5) we can see that the sequence $\{\overline{C}_n f\}_{n\geq 1}$ converges to f, uniformly on $[0,1]$ for any $f \in Y$.

The asymptotic behavior of the sequence $\left(\overline{C}_n\right)_{n=1}^{\infty}$ on a certain subspaces of $C[-1,1]$ is given in the following proposition ([11], Theorem 15) and it was demonstrated applying a version of a general proposition given by R. G. Mamedov:

Theorem 1 ([11], Theorem 15)**.** *Suppose* $x_0 \in [0,1]$ *and* $f''(x_0)$ *exists. If* $a_n \in (0,1)$, $\lim\limits_{n\to\infty} a_n = 1$ *and exists* $L := \lim\limits_{n\to\infty} n(1-a_n)$, *then*

$$\lim_{n\to\infty} n\left[f(x_0) - \left(\overline{C}_n f\right)(x_0)\right] = -\frac{x(1-x)}{2}f''(x_0) + \left[x_0 f'(x_0) - \frac{x_0^2}{4}f''(x_0)\right]L.$$

In order to obtain an overview of the approximation properties of this ope- rator, we add here some more properties.

Lemma 1. *If $f \in C[0,1]$, then $||\overline{C}_n f|| \leq ||f||$, where $||\cdot||$ is the uniform norm on $C[0,1]$.*

Proof. Using the fact that the operator is linear and positive we have the identity

$$||\overline{C}_n|| = ||\overline{C}_n e_0||.$$

And from the above property $(\overline{C_n} e_0)(x) = e_0$ then $||\overline{C}_n|| = 1$. □

Lemma 2. *The operator $\overline{C}_n f$ can be represented by the following expansion*

$$\left(\overline{C}_n f\right)(x) = f(0) + \sum_{k=1}^{n} \frac{(a_n)_k x^k}{k!} \binom{n}{k} \left(\Delta_{n^{-1}}^k f\right)(0),$$

where $\Delta_{n^{-1}}^k f(0)$ is the finite difference of order k, with the step n^{-1} and the starting point 0 of the function f, that is

$$\Delta_{n^{-1}}^k f(0) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f\left(\frac{k-j}{n}\right).$$

Proof. By making use of the following relation between divided differences and finite differences

$$[x_0, x_0+h, ..., x_0+kh; f] = \frac{1}{k! \cdot h^k} \cdot \Delta_h^k f(x_0)$$

we obtain

$$\left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] = \frac{n^k}{k!} \Delta_{n^{-1}}^k f(0).$$

Replacing it in (2.3) we are led to the desired formula. □

Remark 1. If we set $x = 0$ in the expansion formula from above, then we find

$$(\overline{C}_n f)(0) = f(0).$$

Naturally follows the fact that the polynomial (2.3) is interpolating at the end 0 of the interval $[0,1]$ i.e. $(\overline{C}_n f)(0) = f(0)$.

4. Estimates for $\overline{C}_n$ in terms of moduli of smoothness

The main tools to estimating the degree of approximation by positive linear functionals and operators are the moduli of smoothness of first and second order, given by (see, for example, [10])

$$\omega(f;\delta) := \sup\{|f(x) - f(t)| : x, t \in I, |x-t| \leq \delta\}$$

$$\omega_2(f;\delta) := \sup\left\{\left|f(x) - 2f\left(\frac{x+t}{2}\right) + f(t)\right| : x, t \in I, |x-t| \leq 2\delta\right\}$$

for $f \in C(I)$ and $\delta \geq 0$.

For $f \in C[a,b]$, a useful modification represents the least concave majorant of $\omega(f;\cdot)$ given by

$$\tilde{\omega}(f;\epsilon) = \begin{cases} \sup\limits_{0\leq x\leq \epsilon \leq y \leq b-a,\ x\neq y} \frac{(\epsilon-x)\omega(f,y)+(y-\epsilon)\omega(f,x)}{y-x} & \text{if} \quad 0\leq \epsilon \leq b-a \\ \tilde{\omega}(f;\epsilon) = \omega(f;b-a) & \text{if} \quad \epsilon > b-a \end{cases}$$

First, using the result obtained by O. Shisha and B. Mond in 1968, we recall from [12] the estimation for $\overline{C_n}$ using the modulus of continuity.

Theorem 2 ([12], Theorem 9)**.** *If $a_n \in (0,1)$ such that $\lim\limits_{n\to\infty} a_n = 1$ and $L := \lim\limits_{n\to\infty} n(1-a_n)$ exists, then for any $f \in Y$, $x \in [0,1]$ and $n \geq 1$, the Bernstein-Stancu operators (2.3) verify*

$$\left|\left(\overline{C_n} f\right)(x) - f(x)\right| \leq \left(1 + \frac{1}{\delta}\sqrt{\frac{a_n}{4n} + (1-a_n)}\right)\omega(f;\delta).$$

Corollary 1 ([12], Corollary 10)**.** *If $a_n \in (0,1)$ such that $\lim\limits_{n\to\infty} a_n = 1$ and $L := \lim\limits_{n\to\infty} n(1-a_n)$ exists, then for any $f \in Y$, $x \in [0,1]$ and $n \geq 1$, the Bernstein-Stancu operators (2.3) verify*

$$\left|\left(\overline{C_n} f\right)(x) - f(x)\right| \leq \left(1 + \sqrt{\frac{a_n}{4} + n(1-a_n)}\right)\omega(f;\frac{1}{\sqrt{n}}).$$

Corollary 2 ([12], Corollary 11)**.** *If $a_n \in (0,1)$ such that $\lim\limits_{n\to\infty} a_n = 1$ and $L := \lim\limits_{n\to\infty} n(1-a_n)$ exists, then for any $f \in Y$, $x \in [0,1]$, $n \geq 1$ and*

$$\delta = \sqrt{\frac{a_n}{4n} + (1-a_n)}$$

then Bernstein-Stancu operators verify

$$\left|\left(\overline{C_n} f\right)(x) - f(x)\right| \leq 2\omega(f;\delta).$$

Theorem 3 ([12], Theorem 12)**.** *If $a_n \in (0,1)$ such that $\lim\limits_{n\to\infty} a_n = 1$ and $L := \lim\limits_{n\to\infty} n(1-a_n)$ exists, then for any $f \in Y$, $x \in [0,1]$ and $n \geq 1$, the Bernstein-Stancu operators (2.3) verify*

$$\left|\left(\overline{C_n} f\right)(x) - f(x)\right| \leq |a_n x - x| \cdot \left|f'(x)\right| + 2\delta\omega(f';\delta),$$

where

$$\delta = \sqrt{\frac{a_n}{4n} + (1-a_n)}.$$

The main results of this paper are direct estimates via ω, $\tilde{\omega}$ and ω_2.

Let $K = [a,b]$ and $K' \subseteq K$ be also compact and let $L : C(K) \to C(K')$ be a positive linear operator. The following result was obtain by H.H. Gonska in [17].

Theorem 4. *For the linear and positive operator L that reproduce the constant functions, the following inequality holds:*

$$|L(f;x) - f(x)| \leq \max\left\{1, \frac{1}{h} \cdot L(|e_1 - x|;x)\right\} \cdot \tilde{\omega}(f;h),$$

for all $f \in C(K)$, $x \in K'$ and $h > 0$.

In our case, we have

Theorem 5. *If $a_n \in (0,1)$ such that $\lim_{n\to\infty} a_n = 1$ and $L := \lim_{n\to\infty} n(1-a_n)$ exists, then for any $f \in Y$, $x \in [0,1]$ and $n \geq 1$, for Bernstein-Stancu operators (2.3), the following inequality holds:*

$$\left|\left(\overline{C_n} f\right)(x) - f(x)\right| \leq \tilde{\omega}\left(f; \sqrt{\frac{a_n}{4n} + (1-a_n)}\right).$$

Proof. Using the Gonska's result and the inequality (3.1) we obtain

$$\left|\left(\overline{C_n} f\right)(x) - f(x)\right| \leq \max\left\{1, \frac{1}{h} \cdot \sqrt{\frac{a_n}{4n} + (1-a_n)}\right\} \cdot \tilde{\omega}(f;h),$$

and putting $h = \sqrt{\dfrac{a_n}{4n} + (1-a_n)}$ leads to the desired result. □

Due to the fact that ω_2 annihilates linear functions, it is advantageous to measure the degree of approximation by means of this modulus of smoothness.

Further, we recall the following results given by R. Păltănea in [22].

Theorem 6. *For any $f \in C(K)$ all $x \in C(K')$ and $0 < h < \frac{1}{2}$length(K) we have*

$$|L(f;x) - f(x)| \leq |L(e_0;x) - 1| \cdot |f(x)| + |L(e_1 - x;x)| \cdot \frac{1}{h}\omega_1(f,h)$$
$$+ \left(L(e_0;x) + \frac{1}{2} \cdot \frac{1}{h^2} L((e_1 - x)^2;x)\right)\omega_2(f,h).$$

Thus we can state

Theorem 7. *If $a_n \in (0,1)$ such that $\lim_{n\to\infty} a_n = 1$ and $L := \lim_{n\to\infty} n(1-a_n)$ exists, then for any $f \in Y$, $x \in [0,1]$, $n \geq 1$ and $0 < h \leq \dfrac{1}{2}$ we have:*

$$\left|\left(\overline{C_n} f\right)(x) - f(x)\right| \leq \frac{1}{h}(1-a_n)\omega(f,h) + \left(1 + \frac{1}{2} \cdot \frac{1}{h^2}\left(\frac{a_n}{4n} + (1-a_n)\right)\right)\omega_2(f,h)$$

Proof. The assertion follows from the Păltănea's theorem using inequality (3.1). □

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Author's address

Voichiţa Adriana Radu
Babes-Bolyai University, FSEGA, Department of Statistics-Forecasts-Mathematics, Str. Teodor Mihali, No.58-60, RO-400591 Cluj Napoca, Romania
E-mail address: `voichita.radu@econ.ubbcluj.ro`, `voichita.radu@gmail.com`