ON A DIOPHANTINE EQUATION ON TRIANGULAR NUMBERS

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Abstract. A number $N$ is a triangular number if it can be written as $N = 0 + 1 + \ldots + n$ for some natural number $n$. We study the problem of finding all nonnegative integer solutions of the Diophantine equation $(0 + 1 + \ldots + X) + (0 + 1 + \ldots + Y) = (0 + 1 + \ldots + Z)$. Using this equation, some new and curious increasing integer sequences are built.

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1. INTRODUCTION

A triangular number is a number of the form $T_n = \sum_{k=0}^{n} k = n(n + 1)/2$, where $n$ is a natural number.

So the first few triangular numbers are $0, 1, 3, 6, 10, 15, 21, 28, \ldots$ (sequence A000217 in [6]). A well known fact about the triangular numbers is that $X$ is a triangular number if and only if $8X + 1$ is a perfect square. Triangular numbers can be thought of as the numbers of dots needed to make a triangle.

We are concerned by the Diophantine equation of the form

$$T_X + T_Y = T_Z \quad (1.1)$$

Papers [1–5] gave many interesting results concerning the problem of solvability of Diophantine equations related to triangular numbers. The aim of this paper is two-fold: on the one hand, it gives all solutions of the Diophantine equation (1.1), and on the other hand, it gives us a method of finding explicitly (and quickly) infinite families of solutions of (1.1) in where many new integer sequences are derived.

2. GENERAL SOLUTION

Let us start with some technical lemmas.

**Lemma 1.** Let $a, b, c, d \in \mathbb{N}$. Then $ab = cd$ if and only if there exist $p, q, m, n \in \mathbb{N}$ such that $a = mn$, $b = pq$, $c = mp$ and $d = nq$. 

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Proof. Let \( a = \prod_{i \in A \cap N} p_i^{\alpha_i} \) and \( b = \prod_{i \in B \cap N} q_i^{\beta_i} \) where \( p_i, q_i \) are primes, \( \alpha_i, \beta_i \in \mathbb{N} \) and \( A, B \) are finite sets. The fact \( cd = \prod_{i \in A \cap N} p_i^{\alpha_i} \prod_{i \in B \cap N} q_i^{\beta_i} \) implies that \( c = \prod_{i \in A' \subseteq A} p_i^{\alpha_i'} \prod_{i \in B' \subseteq B} q_i^{\beta_i'} \) and \( d = \prod_{i \in A'' \subseteq A} p_i^{\alpha_i''} \prod_{i \in B'' \subseteq B} q_i^{\beta_i''} \), where \( 0 \leq \alpha_i', \alpha_i'' \leq \alpha_i, 0 \leq \beta_i', \beta_i'' \leq \beta_i \) with \( \alpha_i = \alpha_i' + \alpha_i'' \) and \( \beta_i = \beta_i' + \beta_i'' \). If we put \( m = \prod_{i \in A \cap N} p_i^{\alpha_i}, p = \prod_{i \in A' \subseteq A} p_i^{\alpha_i'} \), \( n = \prod_{i \in B \cap N} q_i^{\beta_i}, q = \prod_{i \in B' \subseteq B} q_i^{\beta_i'} \) then we obtain, \( a = mn, b = pq, c = mp \) and \( d = nq \). \( \square \)

Lemma 2. Equation (1.1) is equivalent to \( X(X+1) = (Z-Y)(Z+Y+1) \).

Proof. Replacing \( T_X \) by \( \frac{X(X+1)}{2} \), \( T_Y \) by \( \frac{Y(Y+1)}{2} \) and \( T_Z \) by \( \frac{Z(Z+1)}{2} \) equation (1.1) becomes

\[
\frac{X(X+1)}{2} + \frac{Y(Y+1)}{2} = \frac{Z(Z+1)}{2}
\]

i.e.,

\[
X^2 + X + Y^2 + Y = Z^2 + Z.
\]

Thus

\[
X(X+1) = (Z-Y)(Z+Y+1).
\]

\( \square \)

Now, we are able to establish the following theorem.

Theorem 1. All nonnegative integer solutions of the equation \( T_X + T_Y = T_Z \) are

given by \( \left\{ \begin{array}{l}
X = mn \\
Y = \frac{1}{2}(nq - mp - 1) : m, n, p, q \in \mathbb{N} \\
Z = \frac{1}{2}(nq + mp - 1)
\end{array} \right. \)

where \( pq - mn = 1 \) and \( nq - mp - 1 \in 2\mathbb{N} \).

Proof. According to Lemma 2, (1.1) is equivalent to \( X(X+1) = (Z-Y)(Z+Y+1) \), thus thanks to Lemma 1, one has

\[
X = mn \\
X + 1 = pq \\
Z - Y = mp \\
Z + Y + 1 = nq
\]

Therefore we get

\[
X = mn \\
Y = \frac{1}{2}(nq - mp - 1)
\]
\[ Z = \frac{1}{2}(nq + mp - 1) \text{ where } m, n, p, q \in \mathbb{N} \]
such that \( pq - mn = 1 \) and \( nq - mp - 1 \in 2\mathbb{N} \).

\begin{proof}
Example 1. \( p = 4, q = 7, m = 3 \) and \( n = 9 \). Clearly, \( X = 27, Y = 25 \) and \( Z = 37 \).
\end{proof}

3. Solutions Families

First of all, note that it is not easy to find integers \( m, n, p, q \) such that \( pq - mn = 1 \) and \( nq - mp - 1 \in 2\mathbb{N} \).

The usage of matrices enables us to obtain a large number of solutions of (1.1). In this section, we will construct several infinite many families solutions of our triangular equation.

By completing squares in (2.1), we obtain
\[
\left( X + \frac{1}{2} \right)^2 + \left( Y + \frac{1}{2} \right)^2 = \left( Z + \frac{1}{2} \right)^2 + \frac{1}{4}.
\]
Multiplying both sides by 4, we get
\[
(2X + 1)^2 + (2Y + 1)^2 = (2Z + 1)^2 + 1.
\]
By putting, \( x = 2X + 1, y = 2Y + 1 \) and \( z = 2Z + 1 \), we obtain
\[
x^2 + y^2 = 1 + z^2. \tag{3.1}
\]
Since the proof of the following proposition can be seen easily we omit it.

**Proposition 1.** The two families\[
\begin{pmatrix}
1 \\
a \\
a
\end{pmatrix},
\begin{pmatrix}
a \\
1 \\
a
\end{pmatrix}
\]
are solutions of (3.1) for all integer \( a \).

**Theorem 2.** There exist some \( 3 \times 3 \) matrices \( M \), with elements in \( \{1, 2, 3\} \), such that if
\[
\begin{pmatrix}
x_0 \\
y_0 \\
z_0
\end{pmatrix}
\]
is a solution of (3.1) then
\[
\begin{pmatrix}
x_0 \\
y_0 \\
z_0
\end{pmatrix}
\]
is also a solution.

**Proof.** Let \( M = \begin{pmatrix} A & B & C \\ E & F & G \\ L & M & N \end{pmatrix} \) and let \( \begin{pmatrix} x_0 \\
y_0 \\
z_0 \end{pmatrix} \) a solution of (3.1).

\[
\begin{pmatrix} A & B & C \\ E & F & G \\ L & M & N \end{pmatrix} \begin{pmatrix} x_0 \\
y_0 \\
z_0 \end{pmatrix} = \begin{pmatrix} Ax_0 + By_0 + Cz_0 \\
Fy_0 + Gz_0 + Ex_0 \\
Lx_0 + My_0 + Nz_0 \end{pmatrix}.
\]

It is clear that
\[
\begin{pmatrix} A & B & C \\ E & F & G \\ L & M & N \end{pmatrix} \begin{pmatrix} x_0 \\
y_0 \\
z_0 \end{pmatrix} = \begin{pmatrix} Ax_0 + By_0 + Cz_0 \\
Fy_0 + Gz_0 + Ex_0 \\
Lx_0 + My_0 + Nz_0 \end{pmatrix}.
\]

Now, if the following system holds
\[
A^2 - L^2 + E^2 = 1 \tag{3.2}
B^2 + F^2 - M^2 = 1
C^2 + G^2 - N^2 = -1
\]
\[ AB - LM + EF = 0 \]
\[ AC - LN + EG = 0 \]
\[ BC - MN + FG = 0 \]

then
\[ (Ax_0 + By_0 + Cz_0)^2 + (Fy_0 + Gz_0 + Ex_0)^2 - (Lx_0 + My_0 + Nz_0)^2 = 1. \]

It follows from this, \( \begin{pmatrix} Ax_0 + By_0 + Cz_0 \\ Fy_0 + Gz_0 + Ex_0 \\ Lx_0 + My_0 + Nz_0 \end{pmatrix} \) is a solution of (3.1).

We will explore, in five steps, all solutions of the system (S) are in the space \{1, 2, 3\}.

**Case 1:** \( A = L = E = 1 \).

\[ B^2 + F^2 - M^2 = 1 \]
\[ C^2 + G^2 - N^2 = -1 \]
\[ B + F = M \]
\[ C + G = N \]
\[ BC - MN + FG = 0. \]

From (3.4), we obtain \( F = 1 \) or \( B = 1 \), then (3.3) implies \( B = M \) or \( F = M \) then \( F = 0 \) or \( B = 0 \) which is impossible.

**Case 2:** \( A = 1, L = E = 2 \).

\[ B^2 + F^2 - M^2 = 1 \]
\[ C^2 + G^2 - N^2 = -1 \]
\[ B - 2M + 2F = 0 \]
\[ C - 2N + 2G = 0 \]
\[ BC - MN + FG = 0. \]

Equations (3.5) and (3.6) imply that \( B \) and \( C \) are even.

Then from
\[ M^2 - F^2 = 3 \]
\[ N^2 - G^2 = 5 \]
\[ M - F = 1 \]
\[ N - G = 1 \]
\[ MN - FG = 4. \]

Equation (3.7) gives \( M = 2, F = 1 \) and (3.8) gives \( N = 3, G = 2 \).
then \( M_1 = \begin{pmatrix} A & B & C \\ E & F & G \\ L & M & N \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} \).

**Case 3**: \( A = 1, L = E = 3 \).

\[
\begin{align*}
B^2 + F^2 - M^2 &= 1 \\
C^2 + G^2 - N^2 &= -1 \\
B - 3M + 3F &= 0 \\
C - 3N + 3G &= 0
\end{align*}
\]

Equations (3.9) and (3.10) give us \( B = C = 3 \), then
\[
\begin{align*}
F^2 - M^2 &= -8 \\
G^2 - N^2 &= -10 \\
-M + F &= -1 \\
-N + G &= -1 \\
-MN + FG &= -9.
\end{align*}
\]

It is easy to see that equation (3.11) is impossible in \( \{1, 2, 3\}^2 \)

**Case 4**: \( A = 2 \).

Equation 3.2 implies that \( L = 2, E = 1 \).

So, we get
\[
\begin{align*}
B^2 + F^2 - M^2 &= 1 \\
C^2 + G^2 - N^2 &= -1 \\
2B - 2M + F &= 0 \\
2C - 2N + G &= 0
\end{align*}
\]

Equations (3.12) and (3.13) give us \( F = G = 2 \), thus
\[
\begin{align*}
M^2 - B^2 &= 3 \\
N^2 - C^2 &= 5 \\
M - B &= 1 \\
N - C &= 1 \\
BC - MN + 4 &= 0
\end{align*}
\]

Equation (3.14) gives us \( M = 2, B = 1 \) and (3.15) gives \( N = 3, C = 2 \).
We obtain \( \mathfrak{M}_2 = \begin{pmatrix} A & B & C \\ E & F & G \\ L & M & N \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix} \).

**Case 5 :** \( A = 3 \).

Equation (1.6) in (5) implies that : \( L = 3, E = 1 \). Now

\[
\begin{align*}
B^2 + F^2 - M^2 &= 1, \\
C^2 + G^2 - N^2 &= -1, \\
3B - 3M + F &= 0, \\
3C - 3N + G &= 0, \\
BC - MN + FG &= 0.
\end{align*}
\]

Equations (3.16) and (3.17) give us \( F = G = 3 \), thus

\[
\begin{align*}
M^2 - B^2 &= 8, \\
N^2 - C^2 &= 10, \\
B - M + 1 &= 0, \\
C - N + 1 &= 0, \\
BC - MN + 9 &= 0.
\end{align*}
\]

It is easy to see that equation (3.18) is impossible in \( \{1, 2, 3\}^2 \).

\( \square \)

**Corollary 1.** The equation (1.1) admits infinite many solutions.

**Proof.** Let \( \mathfrak{M} = \mathfrak{M}_1 \) and \( \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) a solution of (3.1) then \( \mathfrak{M}^n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) is a solution of (3.1) for all integer \( n \geq 0 \).

\( \square \)

4. NEW INTEGER SEQUENCES

Using the on-line Encyclopedia of integer sequences (see [6]), we can easily check that the following two integer sequences formed by solutions \( X \) of triangular equation (1.1) : \( 1, 5, 35, 203, 1179, 6929, 40391, 235415, 1372105, 7997213 \ldots \) and \( 0, 8, 54, 322, 1884, 10988, 64050, 373318, 2175864, 12681872, \ldots \) are new.

4.1. Constructions

**1:** We start with \( \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \), a solution of (3.1), thanks to theorem 2, by multiplying recursively by \( \mathfrak{M}_1 \), we obtain
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| 0 | @ |
| 71 | 69 |
| 99 | 577 |
| 2744211 | 2744209 |
| 3880899 | 22619537 |

are also solutions of (3.1).

We get, 
\[ x = 3, 11, 71, 407, 2379, 13859, 80783, 470831, 2744211, 15994427, \ldots \]

But 
\[ x = 2X + 1, \]

so

\[ X = 1, 5, 35, 203, 1179, 6929, 40391, 235415, 1372105, 7997213, \ldots \]
is a new sequence of odd integers formed by solutions \( X \) of our triangular equation (1.1).

2: We start with \( \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix} \) a solution of (3.1), thanks to theorem 2, by multiply-

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are also solutions of (3.1).

We get, 
\[ x = 1, 17, 109, 645, 3769, 21977, 128101, 746637, 4351729, 25363745, \ldots \]

But 
\[ x = 2X + 1, \]

then

\[ X = 0, 8, 54, 322, 1884, 10988, 64050, 373318, 2175864, 12681872, \ldots \]
is a new sequence of even integers formed by solutions \( X \) of our triangular equation (1.1).

REFERENCES

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