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ON THE SIZE OF DIOPHANTINE M-TUPLES FOR LINEAR POLYNOMIALS

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Abstract. In this paper we prove that there does not exist a set with more than 16 nonzero polynomials in $\mathbb{K}[X]$, where $\mathbb{K}$ is any field of characteristic 0, such that the product of any two of them increased by a linear polynomial $n \in \mathbb{K}[X]$ is a square of a polynomial from $\mathbb{K}[X]$.

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1. INTRODUCTION

Diophantus of Alexandria [1] was the first who studied the problem of finding sets with the property that the product of any two of its distinct elements increased by 1 is a perfect square. Such a set consisting of m elements is therefore called a Diophantine m-tuple. Diophantus found the first Diophantine quadruple consisting of rational numbers $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$, while the first Diophantine quadruple of integers, the set $\{1, 3, 8, 120\}$, was found by Fermat. In the case of rational numbers no upper bound for the size of such sets is known. In integer case which is the most studied, Dujella [4] proved that there does not exist a Diophantine sextuple and there are only finitely many Diophantine quintuples. The folklore conjecture is that there does not exist a Diophantine quintuple over the integers.

Many generalizations of this problem were also considered, for example by adding a fixed integer n instead of 1, looking at kth powers instead of squares, or considering the problem over other domains than $\mathbb{Z}$ or $\mathbb{Q}$.

Definition 1. Let $m \geq 2$, $k \geq 2$ and let R be a commutative ring with 1. Let $n \in R$ be a nonzero element and let $\{a_1, \ldots, a_m\}$ be a set of m distinct nonzero elements from R such that $a_i a_j + n$ is a kth power of an element of R for $1 \leq i < j \leq m$. The set $\{a_1, \ldots, a_m\}$ is called a kth power Diophantine m-tuple with the property $D(n)$ or simply a kth power $D(n)$-m-tuple in R.

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The first such set, a second power $D(256)$-quadruple $\{1,33,68,105\}$, was found by Diophantus [1]. It is interesting to find upper bounds for the number of elements of such sets. Dujella [2,3] found such bounds for the integer case and for $k=2$. For other similar results see [5,6,9,12].

The first polynomial variant of the above problem was studied by Jones [13,14] for $R=\mathbb{Z}[X]$, $k=2$ and $n=1$. In this case, Dujella and Fuchs [6] proved that there does not exist a second power Diophantine quintuple. Dujella and Luca [11] considered the case $n=1$, $k\geq 3$ and $R=\mathbb{K}[X]$, where $\mathbb{K}$ is an algebraically closed field of characteristic 0. Using many results from [11], Dujella and Jurasić [9] proved that there does not exist a second power Diophantine 8-tuple in $\mathbb{K}[X]$ for $n=1$. There were also considered other variants of such a polynomial problem. Dujella and Fuchs, jointly with Tichy [7] and later with Walsh [8], considered the case $R=\mathbb{Z}[X]$, $k=2$ and n is a linear polynomial from $\mathbb{Z}[X]$. They proved that in this case $m\leq 12$. Jurasić [15] proved that $m\leq 98$ for n a quadratic polynomial in $\mathbb{Z}[X]$.

We will consider the case $R=\mathbb{K}[X]$, where $\mathbb{K}$ is any field of characteristic 0, $k=2$ and n is a linear polynomial from $\mathbb{K}[X]$. Without loss of generality we assume that $\mathbb{K}$ is algebraically closed. If we omit the condition that $\mathbb{K}$ is a field of characteristic 0, then we could not obtain some results where the factorisation of a polynomial is considered. For brevity, instead of second power $D(n)$-m-tuple in $\mathbb{K}[X]$, from now on we shall refer to a polynomial $D(n)$-m-tuple. Observe that, at most one polynomial a_i for $i\in\{1,\ldots,m\}$ in such a polynomial $D(n)$-m-tuple is constant. Otherwise, we would have two different constants a and b for which $ab+n=r^2$, where $r\in\mathbb{K}[X]$. This is not possible, because then $\deg(n)=1=2\deg(r)$.

An improper kth power $D(n)$-m-tuple in R is an m-tuple with the property from Definition 1, but with relaxed condition that its elements need not be distinct and need not be nonzero. For linear n, we cannot have 0 in an improper polynomial $D(n)$-m-tuple. Let us assume that there exists a non-constant polynomial a such that $a^2+n=r^2$, for some $r\in\mathbb{K}[X]$. Then $\deg(a)=\deg(r)\geq 1$ and $\deg(n)=\deg(r-a)+\deg(r+a)$. This is not possible if $\deg(a)\geq 2$ but, for example, we have $(X+3)^2-4X-8=(X+1)^2$. So, in an improper polynomial $D(n)$-m-tuple we can have infinitely many equal linear polynomials. In the rest of the paper we consider only proper $D(n)$-m-tuples, described in Definition 1. We have the following theorem.

Theorem 1. *There are at most* 16 *elements in a polynomial* $D(n)$*-*m*-tuple for a linear polynomial* n*, i.e.*

$$m\leq 16.$$

In order to prove Theorem 1, we follow the strategy used in [7,8] for linear n and in [15] for quadratic n. In those papers the ring $\mathbb{Z}[X]$ was considered, using the relation "$<$" between its elements. Instead of that, in $\mathbb{K}[X]$ we have to use the relation "$\leq$" between the degrees of its elements. The paper is organized as follows. In Section 2 we estimate the number of polynomials with given degree k in a polynomial $D(n)$-m-tuple and consider separate cases depending on k. In Section 3, we adapt the gap principle for the degrees of the elements of a polynomial $D(n)$-quadruple, proved in [7] for the ring $\mathbb{Z}[X]$, to $\mathbb{K}[X]$. Using the bounds from Section 2 and combining the gap principle with an upper bound for the degree of the largest element in a polynomial $D(n)$-quadruple, obtained in [8] and also valid in $\mathbb{K}[X]$, in Section 4 we give the proof of Theorem 1.

2. Sets with polynomials of equal degree

Let L_k be the number of polynomials of degree k from $\mathbb{K}[X]$ in a polynomial $D(n)$-m-tuple for linear n. The first step which leads us to the proof of Theorem 1 is to estimate the numbers L_k for $k \geq 0$. We already proved that

$$L_0 \leq 1.$$

The following lemma, which is [7, Lemma 1], plays the key role in our proofs. It is proved for polynomials with integer coefficients, but the proof is obtained using only algebraic manipulations so it holds in $\mathbb{K}[X]$ also.

Lemma 1. *Let $\{a,b,c\}$ be a polynomial $D(n)$-triple in $\mathbb{K}[X]$ and let*

$$ab+n=r^2,\ \ ac+n=s^2,\ \ bc+n=t^2, \tag{2.1}$$

for some $r,s,t \in \mathbb{K}[X]$. Then there exist polynomials $e,u,v,w \in \mathbb{K}[X]$ such that

$$ae+n^2=u^2,\ \ be+n^2=v^2,\ \ ce+n^2=w^2. \tag{2.2}$$

More precisely,

$$e=n(a+b+c)+2abc-2rst, \tag{2.3}$$

$$u=at-rs,\ \ v=bs-rt,\ \ w=cr-st \tag{2.4}$$

and

$$c=a+b+\frac{e}{n}+\frac{2}{n^2}(abe+ruv). \tag{2.5}$$

We also define

$$\overline{e}=n(a+b+c)+2abc+2rst \tag{2.6}$$

and we get

$$e\cdot\overline{e}=n^2(c-a-b-2r)(c-a-b+2r). \tag{2.7}$$

Also, from (2.3), using (2.1) and (2.4), we get

$$e=n(a+b-c)+2rw, \tag{2.8}$$

$e=n(a-b+c)+2sv$ and $e=n(-a+b+c)+2tu$.

Let $\deg(n)=1$ and $\deg(a)=\deg(b)=\deg(c)=k\geq 1$. From (2.1) we obtain $\deg(r)=\deg(s)=\deg(t)=k\geq 1$. Let A,B,C,R,S,T be the leading coefficients

of the polynomials a,b,c,r,s,t, respectively. Then, from (2.1), we have $AB = R^2$, $AC = S^2$ and $BC = T^2$ so $ABC = \pm RST$. Let us consider both cases.
1.) If $ABC = RST$, from (2.6) we have $\deg(\overline{e}) = 3k$ and then, from (2.7),

$$\deg(e) \leq 2-k. \tag{2.9}$$

2.) If $ABC = -RST$, from (2.3) we conclude

$$\deg(e) = 3k, \tag{2.10}$$

and then, from (2.7), $\deg(\overline{e}) \leq 2-k$.

In order to bound the number L_k for $k \geq 1$, we are interested to find the number of possible c-s, for fixed a and b, such that (2.1) holds. The first step is finding all possible e-s from Lemma 1. The trivial situation in **1.)** is $e = 0$. Then, from (2.7), we obtain

$$c_{\pm} = a+b \pm 2r. \tag{2.11}$$

Analogue situation in **2.)** is $\overline{e} = 0$. Beside those trivial situations we suppose that $\deg(e) \geq 0$ and $\deg(\overline{e}) \geq 0$.

Polynomial $D(n)$-triples can be classified as regular or irregular, depending on whether they satisfy the condition given in the next definition (see e.g. [10]).

Definition 2. A $D(n)$-triple $\{a,b,c\}$ is called regular if it satisfies the condition

$$(c-b-a)^2 = 4(ab+n). \tag{2.12}$$

Observe that any permutation of a,b,c leaves equation (2.12) invariant. Also, from (2.12), using (2.1), we get (2.11) and we have

$$ac_{\pm} + n = (a \pm r)^2, \quad bc_{\pm} + n = (b \pm r)^2. \tag{2.13}$$

Moreover, a semi-regular $D(n)$-quadruple is one which contains a regular triple, and a twice semi-regular $D(n)$-quadruple is one that contains two regular triples.

Let us consider a polynomial $D(n)$-triple $\{a,b,c\}$ in the two following lemmas. We will use the divisibility of polynomials in $\mathbb{K}[X]$. A polynomial p divides a polynomial q if there exists a polynomial m from $\mathbb{K}[X]$ such that $q = mp$. This situation will be denoted by $p|q$.

Lemma 2. *At most one element of a polynomial $D(n)$-triple $\{a,b,c\}$ is divisible by n.*

Proof. Without loss of generality, let a and b be divisible by n. Then, from (2.1), it follows that $n|r$. Hence, $n^2|n$, a contradiction. □

In the following lemma, we adapt the important result from [8] for $\mathbb{K}[X]$.

Lemma 3. *Let $\{a,b\}$ be a polynomial $D(n)$-pair with $\deg(a) = \deg(b) = k \geq 1$ and such that $ab+n = r^2$, where $r \in \mathbb{K}[X]$. Let $ae+n^2 = u^2$, $be+n^2 = v^2$, where $u,v,e \in \mathbb{K}[X]$ and $\deg(e) \geq 0$. Then for each such e there is at most one $c \in \mathbb{K}[X]$, with $\deg(c) = k$, such that $\{a,b,c\}$ is a polynomial $D(n)$-triple.*

Proof. Suppose that $\{a,b\}$ is a polynomial $D(n)$-pair such that $ab+n=r^2$, where $r\in\mathbb{K}[X]$. From $ae+n^2=u^2$ and $be+n^2=v^2$, where $u,v\in\mathbb{K}[X]$, and from the fact that $\mathbb{K}$ is an algebraically closed field, we conclude that there are at most two u-s and at most two v-s. Namely, we have $\pm u$ and $\pm v$. Then, from (2.5), we obtain two possible c-s:

$$c_{\pm}=a+b+\frac{e}{n}+\frac{2}{n^2}(abe\pm ruv). \tag{2.14}$$

From this, we get

$$c_{+}\cdot c_{-}=b^2+a(a-2b)+\frac{e^2}{n^2}-\frac{2ae}{n}-\frac{2be}{n}-4n. \tag{2.15}$$

If $c_{+}\cdot c_{-}\in\mathbb{K}[X]$, then $\frac{e(e-2n(a+b))}{n^2}\in\mathbb{K}[X]$. From this, we conclude that $n|e$. Then, from (2.2), we get that $n|u$, $n|v$, and further $n^2|ae$ and $n^2|be$. If $n^2\nmid e$, then $n|a$ and $n|b$, which is in contradiction with Lemma 2. Hence, $n^2|e$. In that case (2.9) is not possible and we must have (2.10). Then, from (2.15), we obtain that

$$\deg(c_{+})+\deg(c_{-})=\deg\left(\frac{e^2}{n^2}\right)=6k-2.$$

Finally, we conclude that one of the polynomials $c_{\pm}$ has a degree $5k-2$ if the other of them has a degree k. ☐

Remark 1. From the proof of Lemma 3 we have that if $n|e$, then $n^2|e$. A polynomial $D(16X+9)$-triple

$$\{X,16X+8,36X+20\}, \tag{2.16}$$

from [7], for which $e=33X+18$ is an example for which (2.9) holds. But in this case $c_{+}\cdot c_{-}\notin\mathbb{K}[X]$.

2.1. *Linear polynomials*

Let us prove the following proposition.

Proposition 1. $L_1\leq 7$.

For the proof of Proposition 1 we use the results from the previous lemmas and the results stated below. Let us fix $a,b\in\mathbb{K}[X]$ such that $\deg(a)=\deg(b)=1$. We are looking at the extensions of $\{a,b\}$ to a polynomial $D(n)$-triple $\{a,b,c\}$ with $\deg(c)=1$ and then at the corresponding $e\in\mathbb{K}[X]$ defined by (2.3). From (2.9), we have the first possibility that $\deg(e)\leq 1$ and from (2.10) we have the second possibility that $\deg(e)=3$. Therefore, we have to consider the possibilities that $\deg(e)\in\{0,1,3\}$.

Lemma 4. *For a fixed polynomial $D(n)$-pair $\{a,b\}$ with $\deg(a)=\deg(b)=1$ there is at most one c with $\deg(c)=1$ such that $\{a,b,c\}$ is a polynomial $D(n)$-triple and such that the corresponding e, defined by (2.3), is from $\mathbb{K}\backslash\{0\}$.*

Proof. By Lemma 1, there is $u \in \mathbb{K}[X]$ such that $ae+n^2=u^2$ and $\deg(u)=1$. Since $a=A(X-\phi)$, where $\phi \in \mathbb{K}$ and $A \in \mathbb{K}\backslash\{0\}$, we assume without loss of generality that $u-n=\varepsilon_1(X-\phi)$, $u+n=\varepsilon_2$, where $\varepsilon_1, \varepsilon_2 \in \mathbb{K}\backslash\{0\}$ and $\varepsilon_1\varepsilon_2=Ae$. This implies that

$$2n=-\varepsilon_1 X+\varepsilon_2+\varepsilon_1\phi. \tag{2.17}$$

Assume that, for fixed a and b, two distinct e-s exist, namely e and f, where $f \in \mathbb{K}\backslash\{0\}$ such that Lemma 1 holds. Therefore, $af+n^2=u_1^2$, where $u_1 \in \mathbb{K}[X]$ and $\deg(u_1)=1$. We conclude that

$$\begin{aligned} u_1-n&=\varphi_1(X-\phi),\\ u_1+n&=\varphi_2, \end{aligned} \tag{2.18}$$

or

$$\begin{aligned} u_1-n&=\varphi_1,\\ u_1+n&=\varphi_2(X-\phi), \end{aligned} \tag{2.19}$$

where $\varphi_1\varphi_2=Af$ and $\varphi_1,\varphi_2 \in \mathbb{K}\backslash\{0\}$. Let us first consider the case (2.18). We get $2n=-\varphi_1 X+\varphi_2+\varphi_1\phi$. Hence, from (2.17), $-\varepsilon_1=-\varphi_1$ and $\varepsilon_2+\varepsilon_1\phi=\varphi_2+\varphi_1\phi$. So, we have $\varepsilon_1=\varphi_1$ and $\varepsilon_2-\varphi_2=\phi(-\varepsilon_1+\varphi_1)$, from which it follows that and $\varepsilon_2=\varphi_2$. Therefore, $e=f$.

Assume now that (2.19) holds. Then, $2n=\varphi_2 X-\varphi_2\phi-\varphi_1$. Therefore, $-\varepsilon_1=\varphi_2$ and $\varepsilon_2=-\varphi_1$. This yields $e=f$. Hence, for fixed a and b, there is at most one $e \in \mathbb{K}\backslash\{0\}$. For that e, from Lemma 3, there is at most one $c \in \mathbb{K}[X]$, with $\deg(c)=1$, such that $\{a,b,c\}$ is a polynomial $D(n)$-triple. □

Remark 2. Using the proof of Lemma 4, we get a polynomial $D(-2X+\frac{1}{2})$-triple

$$\left\{X+\frac{\sqrt{2}-10}{49}, 8X+\frac{64\sqrt{2}+46}{49}, X(9-4\sqrt{2})\right\},$$

for which $e=\frac{36+16\sqrt{2}}{49}$. Hence, the situation described in Lemma 4 is possible.

Lemma 5. *For a fixed polynomial $D(n)$-pair $\{a,b\}$ with $\deg(a)=\deg(b)=1$ there are at most two c-s with $\deg(c)=1$ such that $\{a,b,c\}$ is a polynomial $D(n)$-triple and such that the corresponding e, defined by (2.3), is from $\mathbb{K}[X]$ and $\deg(e)=1$.*

Proof. From (2.2), we have

$$c=\frac{(w-n)(w+n)}{e}. \tag{2.20}$$

Therefore, $w \pm n=\lambda e$ where $\lambda \in \mathbb{K}\backslash\{0\}$, for at least one of the signs $\pm$. Inserting that into (2.20), we get

$$c=\lambda(\lambda e \mp 2n). \tag{2.21}$$

Then, from (2.8), it follows that $e(1-2r\lambda) = n(a+b-c \mp 2r)$. From Remark 1, we know that if $n|e$ then $n^2|e$, so $n \nmid e$. Therefore,

$$1-2r\lambda = \lambda_1 n, \tag{2.22}$$

$$\lambda_1 e = a+b-c \mp 2r, \tag{2.23}$$

where $\lambda, \lambda_1 \in \mathbb{K}\backslash\{0\}$.

Suppose that for fixed a and b another e exists. We call it f, and we suppose that for such $f \in \mathbb{K}[X]$, where deg(f) $= 1$, Lemma 1 holds. For a polynomial $D(n)$-triple $\{a,b,c'\}$, where deg(c') $= 1$, by Lemma 1 there is $w' \in \mathbb{K}[X]$ such that $c'f + n^2 = (w')^2$. Analogously as for e, it holds

$$1-2r\xi = \xi_1 n, \tag{2.24}$$

where $\xi_1 f = a+b-c' \mp 2r$, $w' \pm n = \xi f$ (for at least one of the signs $\pm$) and $\xi, \xi_1 \in \mathbb{K}\backslash\{0\}$.

From (2.22) and (2.24), we get $-2r(\lambda-\xi) = n(\lambda_1 - \xi_1)$. If $n|r$, we obtain a contradiction with (2.22). Therefore, $\lambda = \xi$ and $\lambda_1 = \xi_1$.

By inserting (2.21) into (2.23), we obtain

$$e = \frac{1}{\lambda^2+\lambda_1}(a+b \mp 2r \pm 2n\lambda). \tag{2.25}$$

Analogously, we get

$$f = \frac{1}{\xi^2+\xi_1}(a+b \mp 2r \pm 2n\xi). \tag{2.26}$$

Comparing (2.26) and (2.25), we conclude that there is at most one such $f \neq e$, namely one obtained for different combination of signs in (2.25) and (2.26).

Suppose that there is a third e, namely $h \in \mathbb{K}[X]$, where deg(h) $= 1$, for which Lemma 1 holds. Analogously we conclude that $h = e$ or $h = f$. For each of polynomials e and f, by Lemma 3, we have at most one different linear polynomial c, namely c and c', such that $\{a,b,c\}$ and $\{a,b,c'\}$ are polynomial $D(n)$-triples. □

Remark 3. The situation from the proof of Lemma 5 is possible. Examples for that are the polynomial $D(16X+9)$-triples (2.16) and $\{X, 16X+8, 100X+44\}$, for which $f = 273X+126$. In this case $\lambda = \frac{2}{3}$ and $\lambda_1 = -\frac{1}{3}$. Also, by iserting X^2 instead of X in those examples, we get examples obtained in [15].

Let us now consider the last possibility, that (2.10) holds, i.e. deg($\overline{e}$) ≤ 1. Since we assumed that deg($\overline{e}$) ≥ 0, from (2.7) we conclude that $n|e$.

Lemma 6. *For a fixed polynomial $D(n)$-pair $\{a,b\}$ with* deg(a) $=$ deg(b) $= 1$ *there does not exist c with* deg(c) $= 1$ *such that $\{a,b,c\}$ is a polynomial $D(n)$-triple and such that the corresponding e, defined by (2.3), is from* $\mathbb{K}[X]$ *and* deg(e) $= 3$.

Proof. Assume that such an e exists. Since $n|e$ then $n^2|e$, by Remark 1. If we, for fixed a and b, have such a triple $\{a,b,c\}$, then $ce+n^2=w^2$, for $w\in\mathbb{K}[X]$. We conclude that $\deg(w)=2$ and $n|w$. Therefore, from the last equation, we get $ce_1+1=w_1^2$, where $e_1,w_1\in\mathbb{K}[X]$ and $\deg(e_1)=\deg(w_1)=1$. We actually have two possibilities for the polynomial w_1, namely $\pm w_1$.

By dividing (2.8) by n, we obtain

$$c=a+b-ne_1\pm 2rw_1, \tag{2.27}$$

which is actually $c_\pm$ given with (2.14), and one of the polynomials $c_\pm$ must have a degree equal to 2. Since $\deg(c_\pm)+\deg(c_\mp)=4$, we conclude that neither one of the polynomials c obtained in this way has degree equal to 1. □

Now we can estimate the number L_1.

Proof of Proposition 1. Let $a,b\in\mathbb{K}[X]$ be linear polynomials such that $ab+n=r^2$, with $r\in\mathbb{K}[X]$. We want to find the number of possible $D(n)$-triples $\{a,b,c\}$, where $c\in\mathbb{K}[X]$ is also a linear polynomial.

For $e=0$, from (2.11), we get as candidates for c at most two polynomials

$$c_{1,2}=c_\pm=a+b\pm 2r.$$

By Lemma 4, there can exists another c, which is one of the polynomials

$$c_3=a+b+\frac{e_3}{n}+\frac{2}{n^2}(abe_3\pm ru_3v_3),$$

with $e_3\in\mathbb{K}\backslash\{0\}$, for which $ae_3+n^2=u_3^2$, $be_3+n^2=v_3^2$, where $u_3,v_3\in\mathbb{K}[X]$. By Lemma 5, as a possible c we obtain at most one of the polynomials

$$c_i=a+b+\frac{e_i}{n}+\frac{2}{n^2}(abe_i\pm ru_iv_i)$$

for each $i=4,5$, with linear polynomials $e_i\in\mathbb{K}[X]$ and with $u_i,v_i\in\mathbb{K}[X]$ such that $ae_i+n^2=u_i^2$, $be_i+n^2=v_i^2$.

Since there are no other possibilities for e, we conclude that a polynomial $D(n)$-tuple which contains polynomials a and b, and consists only of linear polynomials, has at most seven elements, namely, $\{a,b,c_1,c_2,c_3,c_4,c_5\}$. □

Remark 4. The example $a=X-\frac{7}{12}$, $b=4X$, $n=X+\frac{1}{9}$, $r=2X-\frac{1}{3}$, $c_1=c_+=9X-\frac{5}{4}$ and $c_2=c_-=X+\frac{1}{12}$ shows that a twice semi-regular $D(n)$-quadruple of the form $\{a,b,c_1,c_2\}$ is possible in the case of linear polynomials so we can not exclude this possibility in the proof of Proposition 1.

2.2. *Quadratic polynomials*

We intend to prove the following proposition.

Proposition 2. $L_2\le 4$.

The proof of Proposition 2 is based on the constructions from Lemmas 1- 3 and from the following lemmas which deal with a polynomial $D(n)$-triple $\{a,b,c\}$, where $\deg(a)=\deg(b)=\deg(c)=2$. First, we are looking for the possible e-s for fixed a and b. We have the following possible cases: $\deg(e)\leq 0$, which comes from (2.9), and $\deg(e)=6$, from (2.10). Therefore, we consider the possibilities that $\deg(e)=0$ and $\deg(e)=6$.

Lemma 7. *For a fixed polynomial $D(n)$-pair $\{a,b\}$ with $\deg(a)=\deg(b)=2$ there is at most one c with $\deg(c)=2$ such that $\{a,b,c\}$ is a polynomial $D(n)$-triple and such that the corresponding e, defined by (2.3), is from $\mathbb{K}\backslash\{0\}$.*

Proof. By Lemma 1, there is $u\in\mathbb{K}[X]$ such that $ae+n^2=u^2$ and $\deg(u)\leq 1$. Since $a=A(X-\phi_1)(X-\phi_2)$, where $\phi_1,\phi_2\in\mathbb{K}$ and $A\in\mathbb{K}\backslash\{0\}$, we assume that

$$\begin{aligned} u-n&=\varepsilon_1(X-\phi_1), \\ u+n&=\varepsilon_2(X-\phi_2), \end{aligned} \tag{2.28}$$

where $\varepsilon_1,\varepsilon_2\in\mathbb{K}\backslash\{0\}$ and $\varepsilon_1\varepsilon_2=Ae$. From that we conclude

$$2n=X(\varepsilon_2-\varepsilon_1)+\varepsilon_1\phi_1-\varepsilon_2\phi_2. \tag{2.29}$$

Let, for fixed a and b, two distinct e-s from $\mathbb{K}\backslash\{0\}$ exist, i.e. there is also $f\in\mathbb{K}\backslash\{0\}$ for which $af+n^2=u_1^2$, where $u_1\in\mathbb{K}[X]$ and $\deg(u_1)\leq 1$. We have

$$\begin{aligned} u_1-n&=\varphi_1(X-\phi_1), \\ u_1+n&=\varphi_2(X-\phi_2), \end{aligned} \tag{2.30}$$

or

$$\begin{aligned} u_1-n&=\varphi_1(X-\phi_2), \\ u_1+n&=\varphi_2(X-\phi_1), \end{aligned} \tag{2.31}$$

where $\varphi_1\varphi_2=Af$ and $\varphi_1,\varphi_2\in\mathbb{K}\backslash\{0\}$.

From (2.30), we get $2n=X(\varphi_2-\varphi_1)+\varphi_1\phi_1-\varphi_2\phi_2$. By comparing that with (2.29), we obtain $\phi_1(\varepsilon_1-\varphi_1)=\phi_2(\varepsilon_2-\varphi_2)=\phi_2(\varepsilon_1-\varphi_1)$, from which it follows that $\phi_1=\phi_2$ or $\varepsilon_1=\varphi_1$. If $\phi_1=\phi_2$ then, from (2.28), we conclude that $n^2|a$. Further, from (2.1), we get that $n|r$, and then, from (2.3), it follows that $n|e$, which is not possible. If $\varepsilon_1=\varphi_1$, then $\varepsilon_2=\varphi_2$, so $e=f$.

Assume now that (2.31) holds. Then, $2n=(\varphi_2-\varphi_1)X-\varphi_2\phi_1+\varphi_1\phi_2$. By comparing that with (2.29), we obtain $\phi_1(\varepsilon_1+\varphi_2)=\phi_2(\varepsilon_2+\varphi_1)=\phi_2(\varepsilon_1+\varphi_2)$, from which $\phi_1=\phi_2$ or $\varepsilon_1=-\varphi_2$. For $\phi_1=\phi_2$ we obtain a contradiction, as in the previous case. If $\varepsilon_1=-\varphi_2$, then $\varepsilon_2=-\varphi_1$, so $e=f$. Hence, for fixed a and b, there is at most one $e\in\mathbb{K}\backslash\{0\}$ for which Lemma 1 holds. For that e, from Lemma 3, there is at most one $c\in\mathbb{K}[X]$ with $\deg(c)=2$, such that $\{a,b,c\}$ is a polynomial $D(n)$-triple. □

We are left with the last possibility, that (2.10) holds, i.e. $\deg(\overline{e}) \le 0$. Since we assumed that $\deg(\overline{e}) \ge 0$, from (2.7) we conclude that $n^2|e$.

Lemma 8. *For a fixed polynomial $D(n)$-pair $\{a,b\}$ with $\deg(a)=\deg(b)=2$ there does not exist c with $\deg(c)=2$ such that $\{a,b,c\}$ is a polynomial $D(n)$-triple and such that the corresponding e, defined by (2.3), is from $\mathbb{K}[X]$ and $\deg(e)=6$.*

Proof. Assume, on the contrary that such an e exists. Then $ce+n^2=w^2$ for some $w\in\mathbb{K}[X]$. We conclude that $\deg(w)=4$ and $n|w$. By dividing the last equation by n^2 we get $ce_1+1=w_1^2$, where $e_1,w_1\in\mathbb{K}[X]$, $\deg(e_1)=4$ and $\deg(w_1)=3$. The polynomial w_1 is only determined up to the sign.

Analogously as in the proof of Lemma 6, we obtain (2.27), from which we conclude that one of the polynomials $c_\pm$ has a degree equal to 5. Since $\deg(c_\pm)+\deg(c_\mp)=10$, we conclude that neither one of polynomials $\pm w_1 c$ obtained this way has a degree equal to 2. □

Before we estimate the number L_2 we will exclude one possibility, which exists in the case of linear polynomials.

Lemma 9. *Let $a,b\in\mathbb{K}[X]$ such that $\deg(a)=\deg(b)=2$ and $ab+n=r^2$. Then the set of the form $\{a,b,a+b+2r,a+b-2r\}$, which contains only quadratic polynomials, is not a polynomial $D(n)$-quadruple.*

Proof. Assume that such a set is a polynomial $D(n)$-quadruple. Let us consider the triples $\{a,a+b+2r,a+b-2r\}$ and $\{b,a+b+2r,a+b-2r\}$. If the first triple is regular, then, from (2.12), we have $(-4r-a)^2=4(a(a+b+2r)+n)$. From that, $-4r-a=\pm 2(a+r)$ so $r|a$. Then, from (2.1), it follows that $r|n$, which is not possible, because $\deg(r)=2$ and $\deg(n)=1$. The second case is analogous. Therefore, neither one of those triples is regular.

By Lemma 7, for the pair $\{a+b+2r,a+b-2r\}$, there is at most one $e\in\mathbb{K}\backslash\{0\}$ such that Lemma 1 holds. Since no other e-s exist for that pair, it holds $a=b$ which is not possible. □

Remark 5. The example $a=X^2+X+\frac{17}{36}$, $b=X^2+2X-\frac{1}{9}$, $n=-\frac{2}{3}X+\frac{1}{18}$, $r=X^2+\frac{3}{2}X+\frac{1}{18}$, $a+b+2r=4X^2+6X+\frac{17}{36}$ and $a+b-2r=\frac{1}{4}$ shows that if we omit the condition in Lemma 9 that all polynomials in a set of the form $\{a,b,a+b+2r,a+b-2r\}$ are quadratic, then such a $D(n)$-quadruple can exist.

Now we determine the upper bound for L_2.

Proof of Proposition 2. Let $a,b\in\mathbb{K}[X]$ be quadratic polynomials such that $ab+n=r^2$, with $r\in\mathbb{K}[X]$. We look for the number of possible $D(n)$-triples $\{a,b,c\}$, where $c\in\mathbb{K}[X]$ and c is also a quadratic polynomial.

By Lemma 7, there is at most one possibility for c, namely c_1, for which $e\in\mathbb{K}\backslash\{0\}$. For $e=0$ we obtain $c_{2,3}=a+b\pm 2r$. In Lemma 8, we excluded the last option which comes from Lemma 1, those that $\deg(e) = 6$. Hence, the largest set which we can obtain is of the form $\{a, b, c_1, c_2, c_3\}$.

By Lemma 9, the set $\{a, b, c_2, c_3\}$ is not a polynomial $D(n)$-quadruple. Therefore, the pair $\{a, b\}$ can be extended with at most 2 quadratic polynomials (c_1 and one of the polynomials $c_{2,3}$), so $L_2 \leq 4$. □

2.3. *Polynomials of degree $k \geq 3$*

Now we are looking for the upper bound for the number of polynomials of degree $k \geq 3$ in a polynomial $D(n)$-tuple.

Proposition 3. *$L_k \leq 3$ for $k \geq 3$.*

Let $a, b \in \mathbb{K}[X]$ be polynomials of degree $k \geq 3$ such that $ab + n = r^2$, with $r \in \mathbb{K}[X]$. We look for the number of possible c-s such that $\{a, b, c\}$, where $c \in \mathbb{K}[X]$ and $\deg(c) = k$ is a polynomial $D(n)$-triple. First, we look for possible e-s such that Lemma 1 holds. From (2.9), $\deg(e) \leq -1$, and from (2.10), $\deg(\overline{e}) \leq -1$. Therefore, $e = 0$ or $\overline{e} = 0$. From (2.7), we obtain $c_{1,2} = a + b \pm 2r$. Hence, we have at most four elements $\{a, b, c_1, c_2\}$ in such a polynomial $D(n)$-tuple, but we also have the following lemma.

Lemma 10. *Let $a, b \in \mathbb{K}[X]$ such that $\deg(a) = \deg(b) = k \geq 3$ and $ab + n = r^2$. Then the set of the form $\{a, b, a+b+2r, a+b-2r\}$, which contains only polynomials of degree $k \geq 3$, is not a polynomial $D(n)$-quadruple.*

Proof. The first part of the proof is analogue as the proof of Lemma 9, except that here $\deg(r) = k \geq 3$. Since in the case of polynomials of degree $k \geq 3$ the only possible triples are regular ones, we proved the lemma. □

Therefore, the pair $\{a, b\}$ can be extended with at most one polynomial of degree k, where $k \geq 3$, (namely, one of $c_{1,2}$), so we proved Proposition 3.

Example for this case is $D(X)$-triple $\{X^3 - 1, X^3 + 2X^2 + X - 1, 4X^3 + 4X^2 + X - 4\}$, from [8].

3. Gap principle

We will prove a gap principle for the degrees of the elements in a polynomial $D(n)$-quadruple. This result will be used in the proof of Theorem 1, together with the bounds from Section 2 and with the upper bound for the degree of the element in a polynomial $D(n)$-quadruple [8, Lemma 1], given in the following lemma. This bound was obtained for polynomials with integer coefficients but it also holds in $\mathbb{K}[X]$ with slightly different assumption on the degrees of polynomials in quadruple.

Lemma 11. *Let $\{a, b, c, d\}$, where $\deg(a) \leq \deg(b) \leq \deg(c) \leq \deg(d)$, be a polynomial $D(n)$-quadruple with $n \in \mathbb{K}[X]$. Then*

$$\deg(d) \leq 7\deg(a) + 11\deg(b) + 15\deg(c) + 14\deg(n) - 4.$$

The proof of this lemma is based on the theory of function fields, precisely it is obtained by using Mason's inequality [16].

Now we will adjust the result from [7, Lemma 3], very similar as in the classical case for integers, to achieve the needed gap principle. We cannot use the gap principle from [7] and [8], because we do not have the relation " $<$ " between elements of $\mathbb{K}[X]$.

Lemma 12. *Let $\{a,b,c,d\}$, where $3 \leq \deg(a) \leq \deg(b) \leq \deg(c) \leq \deg(d)$, be a polynomial $D(n)$-quadruple for linear $n \in \mathbb{K}[X]$. Then*

$$\deg(d) \geq \deg(b) + \deg(c) - 2.$$

Proof. Applying Lemma 1 to the polynomial $D(n)$-triple $\{a,c,d\}$, we conclude that there exist $e, \overline{e} \in \mathbb{K}[X]$ such that, by (2.7), we have

$$e \cdot \overline{e} = n^2(d-a-c-2s)(d-a-c+2s). \tag{3.1}$$

From (3.1), for $e = 0$ or $\overline{e} = 0$, we get

$$d = a + c \pm 2s. \tag{3.2}$$

In this case $\deg(d) \leq \deg(c)$, so $\deg(d) = \deg(c)$.

Let $e, \overline{e} \in \mathbb{K}[X]$ be nonzero polynomials. Since $\deg(s) = \frac{\deg(a)+\deg(c)}{2} \leq \deg(c)$, from (3.1) we obtain

$$\deg(e) + \deg(\overline{e}) \leq 2 + 2\deg(d). \tag{3.3}$$

By (2.3) and (2.6), we conclude that the degree of one of the polynomials e and $\overline{e}$ is equal to $\deg(a) + \deg(c) + \deg(d)$ and the degree of the other one is ≥ 0. Hence, from (3.3) it follows that

$$\deg(d) \geq \deg(a) + \deg(c) - 2. \tag{3.4}$$

Analogously, applying Lemma 1 to the polynomial $D(n)$-triple $\{b,c,d\}$, we obtain that either

$$d = b + c \pm 2t \tag{3.5}$$

or

$$\deg(d) \geq \deg(b) + \deg(c) - 2. \tag{3.6}$$

Assume that (3.5) holds. As for (3.2), we conclude that $\deg(d) = \deg(c)$. Therefore, (3.4) cannot hold at the same time. Otherwise, we would obtain $\deg(a) \leq 2$, a contradiction. Assume that (3.2) also holds. From (3.2) and (3.5) we obtain $a = b \pm 2s \pm 2t$. Therefore, if $\deg(a) < \deg(c)$ then $\deg(s) > \deg(a)$ and the polynomial on the right hand side of the previous equation has degree $> \deg(a)$ unless $\deg(a) = \deg(b)$ and $S = \pm T$. We conclude that

$$\deg(a) = \deg(b) = \deg(c) = \deg(d) \tag{3.7}$$

or

$$\deg(a) = \deg(b) < \deg(c) = \deg(d) \text{ and } S = \pm T. \tag{3.8}$$

If (3.7) holds, then we have a polynomial $D(n)$-quadruple whose elements have degrees equal to k, where $k \geq 3$. This is not possible, according to Proposition 3. Therefore, it holds (3.8). Since $\{a,c,d\}$ and $\{b,c,d\}$ are both regular triples (i.e. $\{a,b,c,d\}$ is a twice semi-regular $D(n)$-quadruple), by (2.13) we have $cd+n=(c\pm s)^2$ and $cd+n=(c\pm t)^2$, where the signs $\pm$ in last two equations are the same as signs in (3.2) and (3.5), respectively. If $c\pm s=c\pm t$, then we obtain $s^2=t^2$ so $a=b$, which is not possible. If $c\pm s=-c\mp t$, then $\pm s=-2c\mp t$. Since (3.8) holds, the degree of the polynomial on the left hand side of the previous equation is $<\deg(c)$ and the polynomial on the right hand side of that equation has a degee equal to $\deg(c)$. This is not possible.

Assume that (3.6) holds. Then (3.2) can not hold, because otherwise we would have $\deg(b)\leq 2$, a contradiction. But, from (3.6) we obtain (3.4), which shows that this situation is indeed possible. □

4. Proof of Theorem 1

Let $S=\{a_1,a_2,...,a_m\}$, where $\deg(a_1)\leq\deg(a_2)\leq...\leq\deg(a_m)$, be a polynomial $D(n)$-m-tuple with $n\in\mathbb{K}[X]$ a linear polynomial. Since the product of each two elements from S increased by n is a square of a polynomial in $\mathbb{K}[X]$, it follows that if S contains a polynomial of degree ≥ 1, then it contains only polynomials of even or only polynomials of odd degree. We proved that in S we have at most 1 nonzero constant. By Proposition 1, in S there are at most 7 linear polynomials. By Proposition 2, the number of quadratic polynomials in S is at most 4, and, by Proposition 3, in S there are at most 3 polynomials of degree k for every $k\geq 3$.

Assume that in S there is a polynomial of degree ≥ 1. Let us first consider the case where the degrees of all polynomials in S are odd. We will combine the gaps between the degrees of the elements in S with the upper bound on the degree of the element of a $D(n)$-quadruple. We assume that there are smallest possible gaps between the degrees of the elements in S. Then, there are possible the following bounds for the degrees:

$$\deg(a_1)\geq 1,\ \deg(a_2)\geq 1,\ ...,\ \deg(a_7)\geq 1,$$
$$\deg(a_8)\geq 3,\ \deg(a_9)\geq 3,\ \deg(a_{10})\geq 3.$$

Applying Lemma 12 to the polynomial $D(n)$-quadruple $\{a_8,a_9,a_{10},a_{11}\}$ gives $\deg(a_{11})\geq 4$ and, since this degree is odd, we conclude that

$$\deg(a_{11})\geq 5.$$

If we continue in analogue way, we obtain

$$\begin{array}{llll}\deg(a_{12})\geq 7, & \deg(a_{13})\geq 11, & \deg(a_{14})\geq 17, & \deg(a_{15})\geq 27,\\ \deg(a_{16})\geq 43, & \deg(a_{17})\geq 69, & \deg(a_{18})\geq 111, & \deg(a_{19})\geq 179,\ldots.\end{array}$$

We will separate the cases depending on the number of linear polynomials in S. Assume first that in S we have at least three (and ≤ 7) linear polynomials. For the quadruple $\{a_1, a_2, a_3, a_m\} \subseteq S$, where $\deg(a_1) = \deg(a_2) = \deg(a_3) = 1$ and $\deg(a_m) \geq 1$, applying Lemma 11, we get

$$\deg(a_m) \leq 7+11+15+14-4 = 43.$$

Hence, in this case

$$m \leq 16.$$

Assume next that in S we have two linear polynomials. Now we observe the quadruple $\{a_1, a_2, a_3, a_m\} \subseteq S$, where $\deg(a_1) = \deg(a_2) = 1$ and $\deg(a_3) = \mathrm{A}$, for $A \geq 3$ an odd positive integer. As before, we have

$$\deg(a_m) \leq 7+11+15A+14-4 = 15A+28$$

and

$$\begin{array}{lll} \deg(a_4) \geq \mathrm{A}, & \deg(a_5) \geq \mathrm{A}, & \deg(a_6) \geq 2A-2, \ldots, \\ \deg(a_{10}) \geq 13A-24, & \deg(a_{11}) \geq 21A-40, & \deg(a_{12}) \geq 34A-66, \\ \deg(a_{13}) \geq 55A-108, & \deg(a_{14}) \geq 89A-176, & \deg(a_{15}) \geq 144A-286, \ldots, \end{array}$$

so we obtain $m \leq 13$.

Let in S we have one linear polynomial, i.e. $\deg(a_1) = 1, \deg(a_2) = A$, $\deg(a_3) = B$, where $3 \leq A \leq B$ and A, B are odd positive integers. We obtain

$$\deg(a_m) \leq 7+11A+15B+14-4 \leq 26B+17.$$

Further,

$$\begin{array}{lll} \deg(a_4) \geq B, & \deg(a_5) \geq 2B-2, & \deg(a_6) \geq 3B-4, \ldots, \\ \deg(a_{12}) \geq 55B-108, & \deg(a_{13}) \geq 89B-176, & \deg(a_{14}) \geq 144B-286, \ldots, \end{array}$$

so $m \leq 13$.

Finally, suppose that $\deg(a_1) = A, \deg(a_2) = B, \deg(a_3) = C$ where $3 \leq A \leq B \leq C$ and A, B, C are odd positive integers. We get

$$\deg(a_m) \leq 7A+11B+15C+14-4 \leq 33C+10$$

and

$$\begin{array}{ll} \deg(a_4) \geq B+C-2, & \deg(a_5) \geq B+2C-4, \ldots, \\ \deg(a_{12}) \geq 34B+55C-176, & \deg(a_{13}) \geq 55B+89C-286, \ldots, \end{array}$$

then $m \leq 12$. Hence, if S contains only polynomials of odd degree then $m \leq 16$.

Let all polynomials in S have even degree. Now we have

$$\deg(a_1) \geq 0,$$

$$\deg(a_2) \geq 2,\ \deg(a_3) \geq 2,\ \deg(a_4) \geq 2,\ \deg(a_5) \geq 2 \text{ and}$$

$$\deg(a_6) \geq 4,\ \deg(a_7) \geq 4,\ \deg(a_8) \geq 4.$$

Applying Lemma 12 to the polynomial $D(n)$-quadruple $\{a_6, a_7, a_8, a_9\}$ we get

$$\deg(a_9) \geq 6.$$

Analogously, it follows

$$\begin{array}{llll} \deg(a_{10}) \geq 8, & \deg(a_{11}) \geq 12, & \deg(a_{12}) \geq 18, & \deg(a_{13}) \geq 28, \\ \deg(a_{14}) \geq 44, & \deg(a_{15}) \geq 70, & \deg(a_{16}) \geq 112, & \deg(a_{17}) \geq 180, \dots . \end{array}$$

Assume first that $\deg(a_1) = 0, \deg(a_2) = A, \deg(a_3) = B$ where $2 \leq A \leq B$ and A, B are even positive integers. If we apply Lemma 11 to a polynomial $D(n)$-quadruple $\{a_1, a_2, a_3, a_m\}$, it follows that

$$\deg(a_m) \leq 0 + 11A + 15B + 14 - 4 \leq 26B + 10.$$

If $A = B = 2$, then $m \leq 14$. If $B \geq 4$, then

$$\begin{array}{lll} \deg(a_4) \geq B, & \deg(a_5) \geq B, & \deg(a_6) \geq 2B - 2, \dots, \\ \deg(a_{12}) \geq 34B - 66, & \deg(a_{13}) \geq 55B - 108, & \deg(a_{14}) \geq 89B - 176, \dots, \end{array}$$

so we obtain that $m \leq 13$.

Suppose finally that $\deg(a_1) = A, \deg(a_2) = B, \deg(a_3) = C$, where $2 \leq A \leq B \leq C$ and A, B, C are even positive integers. We have

$$\deg(a_m) \leq 7A + 11B + 15C + 14 - 4 \leq 33C + 10.$$

If $A = B = C = 2$ and

$$\begin{array}{lll} \deg(a_4) \geq C, & \deg(a_5) = D, & \deg(a_6) \geq D, \\ \deg(a_7) \geq D, & \deg(a_8) \geq 2D - 2, & \deg(a_9) \geq 3D - 4, \dots, \\ \deg(a_{14}) \geq 34D - 66, & \deg(a_{15}) \geq 55D - 108, & \deg(a_{16}) \geq 89D - 176, \dots, \end{array}$$

where $D \geq 4$ and D is even positive integer, then $m \leq 15$. Also, $m \leq 13$ if $C \geq 4$. We conclude that the set S has at most 15 polynomials of even degree. Therefore,

$$L \leq 16.$$

□

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