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FC-RINGS

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Abstract. We investigate properties of FC-rings (i.e. rings R in which the centralizer $C_R(a)$ of any element $a \in R$ is of finite index in R) and, in particular, characterize left Artinian rings with a finite set of all derivations $\operatorname{Der} R$ (respectively inner derivations $\operatorname{IDer} R$). We show that if R is a Jacobson radical ring in which its adjoint group R° has a finite number of conjugacy classes, then

$$R = R_{p_1} \bigoplus \cdots \bigoplus R_{p_t} \bigoplus D$$

is a ring direct sum of Jacobson radical rings R_{p_i} and D, where the additive group D^+ is a torsion-free divisible group, the adjoint group D° is a group with a finite number of conjugacy classes, $R^+_{p_i}$ is a finite p_i-group ($i = 1, \ldots, t$) and $p_1, \ldots, p_t$ are pairwise distinct primes.

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1. INTRODUCTION

Let R be an associative ring, not necessarily with identity.

It is well-known that a group G is called *an FC-group* if the centralizer $C_G(g)$ of any element $g \in G$ is of finite index in G [19]. Naturally, if the centralizer

$$C_R(a) = \{r \in R \mid ar = ra\}$$

of any element $a \in R$ is of finite index in the additive group R^+ of R, then R is called *an FC-ring*. Every finite ring and every commutative ring are FC-rings. Since

$$\partial_a : R \ni r \mapsto ar - ra = [a, r] \in R$$

(so-called *an inner* derivation of R induced by a) is an endomorphism of the group R^+, the kernel

$$\ker \partial_a = \{r \in R \mid ra = ar\} = C_R(a)$$

is the centralizer of $a \in R$ and the quotient group $R^+/\ker \partial_a$ is isomorphic to the image $\operatorname{Im} \partial_a$, we deduce that R is an FC-ring if and only if $\operatorname{Im} \partial_a$ is finite for any $a \in R$. A map $\delta : R \to R$ is said to be *a derivation* of R if

$$\delta(a + b) = \delta(a) + \delta(b) \text{ and } \delta(ab) = \delta(a)b + a\delta(b)$$

for all $a,b \in R$. Clearly, the zero map $0_R : R \ni r \mapsto 0 \in R$ is a derivation of R.

An algebraic operation "$\circ$" determined by the rule

$$a \circ b = a + b + ab$$

for any $a,b \in R$ is associative with the neutral element $0 \in R$. The set of all invertible elements in R with respect to "$\circ$" is a group (which is called *the adjoint group* of R and denoted by R°).

Notations. For a group G, g^{-1} is the inverse of $g \in G$, $a^G := \{g^{-1}ag \mid g \in G\}$ is the conjugacy class of $a \in G$, G' is the commutator subgroup of G (i.e. a subgroup generated by all multiplicative commutators $g^{-1}h^{-1}gh$, where $g,h \in G$), $Z_0(G) := 1$ is a trivial subgroup of G, $Z(G) = Z_1(G) := \{z \in G \mid zg = gz \text{ for all } g \in G\}$ is the center of G, $Z_{\alpha+1}(G)/Z_\alpha(G) = Z(G/Z_\alpha(G))$ and $Z_\lambda(G) = \cup_{\beta<\lambda} Z_\beta(G)$, where α is an ordinal and λ a limit ordinal. Recall that a group G is called:

- *hypercentral* (respectively *nilpotent*) if $Z_\theta(G) = G$ for some ordinal (respectively non-negative integer) θ,
- *locally nilpotent* if every its finitely generated subgroup is nilpotent,
- *simple* in case $G \neq 1$ and $1, G$ are the only normal subgroups of G,
- *solvable* if $G^{(n)} = 1$ for some integer $n \geq 0$, where $G^{(0)} = G$ and $G^{(m+1)} = (G^{(m)})'$ for any integer $m \geq 1$,
- *locally solvable* if every its finitely generated subgroup is solvable.

Every nilpotent group is hypercentral. It is known that a group is hypercentral if and only if every its non-trivial homomorphic image has the non-trivial center. A group G with a finite number of conjugacy classes is called *a υ-group*. A subgroup A of an (additive) abelian group G is called *pure* if $A \cap nG = nA$ for any integer n. An (additive) abelian group G is *divisible* if, for every positive integer n and every $a \in G$, there exists $x \in G$ such that $nx = a$.

For any ring R, $[x,r] = xr - rx$ is an additive commutator of $x, r \in R$, $C(R)$ is the commutator ideal of R that is the ideal generated by the commutator set $[R,R] = \{[x,r] \mid x,r \in R\}$, $J(R)$ is the Jacobson radical of R, R^+ is the additive group of R, $N(R)$ is the set of all nilpotent elements of R, $U(R)$ is the unit group of R with identity, $Z(R)$ is the center of R, $F(R) = \{a \in R \mid a \text{ is of finite order in } R^+\}$ is the torsion part of R, $\exp F(R)$ is the exponent of the group $F(R)^+$, $g^{(-1)}$ is the inverse of g in the adjoint group R°, $x^{(n)}$ is the nth power of x in the adjoint group R°, $\operatorname{ann} X = \{a \in R \mid aX = 0 = Xa\}$ is the annihilator of $X \subseteq R$. By $\operatorname{Der} R$ we denote the set of all derivations of R. A subring S is *of finite index* in R (i.e. $|R : S| < \infty$) if the additive subgroup S^+ has a finite index in R^+. Recall that a ring R is called:

- *Jacobson radical* if $R^\circ = R$,
- *nil* if every its element x is nilpotent, i.e. there exists an integer $n = n(x) > 0$ such that $x^n = 0$; if there exists an integer $n > 0$ such that $x^n = 0$ for any $x \in R$, then R is *nil of bounded index*,

- *nilpotent* in case there is an integer $m > 0$ such that $x_1 x_2 \cdots x_m = 0$ for any $x_1, x_2, \ldots, x_m \in R$,
- *locally nilpotent* if every its finitely generated subring is nilpotent,
- *reduced* if it is without nonzero nilpotent elements,
- *simple* in case $R^2 \neq 0$ and 0, R are the only ideals of R,
- *local* if it has identity and the quotient ring $R/J(R)$ is simple,
- *semiprime* if it has no nonzero nilpotent ideals,
- *left Artinian* in case for every descending chain

$$I_1 \geq I_2 \geq \cdots \geq I_n \geq \cdots$$

of left ideals I_j of R, there is an integer $n \geq 1$ with $I_{n+1} = I_n$ $(j = 1, 2, \ldots)$.

Any unexplained terminology is standard as in [3, 10] and [19].

The purpose of this paper is to study associative FC-rings R and some related topics. Obviously, commutative rings and, in particular, differentially trivial rings (i.e. $\mathrm{Der}\, R = \{0_R\}$) are FC-rings. Associative rings R with finite sets $\mathrm{Der}\, R$, $\mathrm{IDer}\, R$ are very related to FC-rings. Rings R with the center $Z(R)$ of finite index are FC-rings. In [21, Problem 84] F. Szász asked: "In which rings R the additive group $Z(R)^+$ of the center $Z(R)$ has a finite group-theoretic index with respect to R^+?" From Corollary 2 it follows that this problem is equivalent to study of rings with a finite set of all inner derivations $\mathrm{IDer}\, R$. Y. Hirano [12, Proposition 1] has proved that the condition $|R : Z(R)| < \infty$ implies that the commutator ideal $C(R)$ is finite. F. Szász [21, Problem 83] asked: "In which rings the commutator ideal is finite or can be finitely generated?" H. Bell [5] has proved that if I is a nonzero right ideal of finite index in a prime ring R and $[I, I]$ is finite, then R is either finite or commutative (see also [17, Corollary 1.2]). C. Lanski [16] has showed that if T is a finite higher commutator of R containing no nonzero nilpotent element, then T generates a finite ideal of R.

We study left Artinian rings with the finite set of all derivations $\mathrm{Der}\, R$ (respectively inner derivations $\mathrm{IDer}\, R$) and prove the following with similar flavour.

Theorem 1. *Let R be a left Artinian ring. Then* $\mathrm{IDer}\, R$ *(respectively* $\mathrm{Der}\, R$*) is finite if and only if $R = A \oplus F$ is a direct sum of a finite ideal F and a commutative (respectively differentially trivial) reduced ideal A.*

An FC-ring R has the adjoint FC-group. It is easy to see that a Jacobson radical ring R is an FC-ring if and only if its adjoint group R° is an FC-group. In [21, Problem 88] F. Szász asked:" Let

$$\hat{a} = \{(1-x)a(1-x)^{-1} \mid x \in R\}$$

in a Jacobson radical ring R. When is every class $\hat{a}$ finite, and when is a number of the classes $\hat{a}$ finite?" About Jacobson radical rings we prove the following

Proposition 1. *If R is a Jacobson radical ring with the adjoint FC-group R°, then:*

(1) *every nonzero homomorphic image B of R is commutative or* ann B *is nonzero,*
(2) R° *is a hypercentral group,*
(3) *if R^+ is torsion-free, then R is commutative,*
(4) *if R is nonzero, then the commutator ideal $C(R)$ is proper in R; if, moreover, R is non-commutative, then R^2 is proper in R.*

We give a partial answer on the second part of [21, Problem 88] in the following

Proposition 2. *Let R be a Jacobson radical ring. Then the adjoint group R° is a υ-group if and only if*

$$R = R_{p_1} \bigoplus \cdots \bigoplus R_{p_t} \bigoplus D$$

is a ring direct sum of Jacobson radical rings R_{p_i} and D, where D^+ is a torsion-free divisible group, D° is a υ-group, $R_{p_i}^+$ is a finite p_i-group ($i = 1, \ldots, t$) and $p_1, \ldots, p_t$ are pairwise distinct primes.

F. Szász [20] has investigated properties of infinite Jacobson radical rings R whose adjoint groups R° have only two conjugacy classes. If, moreover, $R^2 = R$, then it is a simple domain by Propositions 1 and 2 from [20]. We make this result more precise in the following

Corollary 1. *If R is a simple Jacobson radical ring, then the following hold:*
(1) *if R° is a υ-group, then either R is a domain or contains a nonzero nilpotent element,*
(2) *if the adjoint group R° has only two conjugacy classes, then R is a domain with the torsion-free divisible additive group R^+ and the simple adjoint group R°.*

2. Preliminaries

For a convenience of the reader and in order to have the paper more self-contained in this section we collect some results needed in the next.

Lemma 1 ([1], Lemma 2.4(1))**.** *If R is a nil-ring and p is prime, then the additive group R^+ is a p-group if and only if the adjoint group R° is a p-group.*

Lemma 2 (see [2], Corollary 1)**.** *Let G be a subgroup of the adjoint group of a radical ring. If G has finite exponent, then it is locally nilpotent.*

If $(R, +, \cdot)$ is an associative ring, then R is a Lie ring with respect to the addition "$+$" and the Lie multiplication "$[-,-]$" (denoted by R^L) defined by the rule $[a,b] = a \cdot b - b \cdot a$ for any $a, b \in R$. Then the center $Z(R)$ of R is an ideal of the Lie ring R^L.

Lemma 3. *If R is an associative ring, then there is a Lie ring isomorphism*

$$\mathrm{IDer}\, R \ni \partial_a \mapsto a + Z(R) \in R^L / Z(R).$$

Proof. Immediate. □

From this we have the following

Corollary 2. *Let R be a ring. Then the set* IDer R *is finite if and only if* $|R : Z(R)| < \infty$.

Lemma 4. *Let R be a ring, I an ideal and S a subring of R. If the set of all inner derivations* IDer R *is finite, then sets* IDer S *and* IDer(R/I) *are finite.*

Proof. Straightforward. □

Lemma 5 ([13], Theorem 1)**.** *Let S be a subring of a ring R. If S has a finite index in R, then there exists an ideal I of R contained in S such that R/I is a finite ring.*

If R has identity 1, then

$$R^{\circ} \ni a \mapsto 1 + a \in U(R)$$

is a group isomorphism of the adjoint group R° and the unit group $U(R)$ of R. As proved in [8], a division ring D with multiplicative FC-group $U(D)$ is commutative.

Lemma 6. *A local ring R is an FC-ring if and only if its adjoint group R° is an FC-group.*

Proof. The adjoint group of any FC-ring is an FC-group. A commutative ring is an FC-ring. Therefore we assume that a local ring R is not commutative and R° is an FC-group. Since R has a proper ideal of finite index in view of Lemma 5, we conclude that the quotient ring $R/J(R)$ is finite. Moreover, $J(R)^{\circ} \cong 1 + J(R)$ is a subgroup of the unit group $U(R)$,

$$|J(R) : C_{J(R)}(a)| = |1 + J(R) : C_{1+J(R)}(a)| < \infty$$

for any $a \in U(R)$ and therefore $|R : C_R(a)| < \infty$. Inasmuch $R = J(R) \cup U(R)$ and $J(R)$ is a Jacobson radical FC-ring, we deduce that R is an FC-ring. □

Lemma 7 ([19], Theorem 14.5.9)**.** *If G is an FC-group, then the commutator subgroup G' is torsion.*

Lemma 8 ([15], Theorem 2)**.** *Let R be a semiprime ring and $S = \{x \in R \mid x^2 = 0\}$. If the cardinality* card S *is finite, then $R = A \oplus F$ is a direct sum of ideals A and F, A is reduced and F is finite. In particular, R has only finitely many nilpotent elements.*

Lemma 9 (see [11, 18])**.** *Let A be an algebra over a field of characteristic zero. Suppose that there is a positive integer n such that $a^n = 0$ for all $a \in A$. Then there is an integer N such that $a_1 a_2 \cdots a_N = 0$ for all $a_1, a_2, \ldots, a_N \in A$.*

Theorem 1 of [23] implies the next

Lemma 10. *A finite Jacobson radical ring is nilpotent.*

Lemma 11 ([15], Lemma 3)**.** *If R is a finite ring which is not nilpotent, then R contains a nonzero idempotent.*

Lemma 12 ([19], 4.3.8)**.** *A pure subgroup A of finite index of an abelian group G is a direct summand.*

A ring R is *a subdirect product* of some rings S_i $(i \in I)$ if, for any $i \in I$, $S_i \cong R/K_i$, where K_i is an ideal of R and

$$\bigcap_{i \in I} K_i = 0.$$

Lemma 13 (see [4])**.** *A ring R is reduced if and only if it is a subdirect product of domains.*

Lemma 14 ([10], §1.4, Corollary 2)**.** *If R is a left Artinian ring and its Jacobson radical $J(R) = 0$ is zero, then R contains identity.*

If I is an ideal of a ring R, then we say that an idempotent $g + I$ of the quotient ring R/I *can be lifted (to e) modulo I* in case there is an idempotent $e \in R$ such that $g + I = e + I$.

Lemma 15 ([3], Proposition 27.1)**.** *If I is a nil-ideal of a ring R, then idempotents lift modulo I.*

Lemma 16 ([9], Theorem 18.13)**.** *A left Artinian ring is left Noetherian (i.e. every its ideal is finitely generated).*

A commutative ring V is called *a v-ring* if it is complete (in the $J(V)$-adic topology), discrete, unramified valuation ring of characteristic 0 with the quotient ring $V/J(V)$ of prime characteristic p [6, p.79]. Then $J(V) = pV$, V/pV is a field, $p^k V/p^{k+1}$ and V/pV are isomorphic as (V/pV)-linear spaces.

Lemma 17. *Let R be a complete (in the $J(R)$-adic topology) local Noetherian commutative ring of prime power characteristic p^n. Then the following hold:*

(1) (see [6, Theorem 9]) *if $n = 1$, then there exists a subfield C of R such that $R = J(R) + C$ is a group direct sum,*
(2) (see [6, Theorem 11]) *if $n \geq 2$, then there exists a subring C of R such that $R = J(R) + C$ is a group sum, where $C \cong V/p^n V$ for some v-ring V and $J(R) \cap V = pV$.*

Any local Artinian ring R is a complete local Noetherian ring.

3. Properties of FC-rings

Lemma 18. *If R is an infinite ring with a finite set of all inner derivations* IDer R, *then the following hold:*

(1) *the centralizer $C_R(a)$ is a subring of finite index in R for any $a \in R$ (i.e. R is an FC-ring),*
(2) *the adjoint group R° is an FC-group,*
(3) *if R is a simple ring, then R is a field,*
(4) *R contains a central ideal I of finite index such that $I \cdot C(R) = 0$,*
(5) *the commutator ideal $C(R)$ is finite.*

Proof. (1) Since the set $\{\partial_r(a) \mid r \in R\}$ is finite, the index $|R : C_R(a)|$ is finite.

(2) It follows from the part (1).

(3) It holds from Corollary 2 in view of Lemma 5.

(4) Corollary 2 and Lemma 5 imply that R contains an ideal I of finite index such that $I \leq Z(R)$. Moreover, for any $r, t \in R$ and $i \in I$ we obtain that

$$(rt)i = r(ti) = (ti)r = t(ir) = t(ri) = (tr)i,$$

and so $(rt - tr)i = 0$. As a consequence, $I \cdot C(R) = 0$.

(5) By Corollary 2, the center $Z(R)$ is of finite index in R and so, by the part (4), the annihilator $\operatorname{ann} \partial_x(y)$ has a finite index in R. Then the commutator ideal

$$C(R) = \sum_{x,y \in R \setminus Z(R)} R\partial_x(y)R$$

is finite.

□

Proof of Proposition 1. (1) Assume that B is a non-commutative homorphic image of R and $a \in B \setminus Z(B)$. Then $C_B(a)$ is of finite index in B and, by Lemma 5, the centralizer $C_B(a)$ contains a proper ideal I of B such that $|B : I| < \infty$. If $i \in I$ and $r \in B$, then

$$ari = ria = rai, \; iar = air = ira$$

and so

$$[a, r]i = 0 = i[a, r].$$

This gives that $[B, B] \subseteq \operatorname{ann} I$. Since B/I is a finite radical ring, it is nilpotent by Lemma 10. We have that $B^n \leq I$ for some positive integer n and $\operatorname{ann} I \subseteq \operatorname{ann}(B^n)$. From this it holds that $\operatorname{ann} B \neq 0$. Then every non-trivial homomorphic image of R is commutative or has a non-trivial annihilator.

(2) Since $\operatorname{ann} R \subseteq Z(R)$, we deduce that every proper quotient group of R° has the non-trivial center. This means that the adjoint group R° is hypercentral [19, Exercises 12.2.2].

(3) By Lemma 7, every torsion-free FC-group is abelian.

(4) The commutator ideal of a commutative ring R is zero (and so it is proper in $R \neq 0$). Assume that R is non-commutative. As in the part (1), we can prove that R contains a proper ideal of finite index and therefore R^2 is proper in R in view of Lemma 10. Obviously that $C(R) \subseteq R^2$. Hence $C(R)$ is proper in a non-commutative ring R.

□

4. Rings with a finite set of derivations

Lemma 19. *If e is an idempotent of a commutative ring R, then $d(e) = 0$ for any $d \in \operatorname{Der} R$.*

Proof. In fact, $d(e) = d(e^2) = d(e)e + ed(e)$ implies that $ed(e) = ed(e)e + ed(e)$ and so $d(e)e = ed(e) = e^2 d(e) = ed(e)e = 0$. Hence $d(e) = 0$. □

Lemma 20. *Let R be a ring. If the set $\operatorname{Der} R$ (respectively $\operatorname{IDer} R$) is finite, then $\operatorname{Der} R = \{0_R\}$ (respectively R is commutative) or $\delta(R) \subseteq F(R)$ for any $\delta \in \operatorname{Der} R$ (respectively $\delta \in \operatorname{IDer} R$).*

Proof. Assume that $\delta(a) \neq 0$ for some $\delta \in \operatorname{Der} R$ and $a \in R$. Since the set

$$\{n\delta(a) \mid n \text{ is an integer}\}$$

is finite, the torsion part $F(R) \neq 0$ is nonzero and $\delta(a) \in F(R)$. □

Lemma 21. *Any finite semiprime commutative ring R is differentially trivial.*

Proof. By Lemma 14, R contains identity 1 and, by the Artin-Wedderburn structure theorem, $R = R_1 \oplus \cdots \oplus R_m$ is a ring direct sum of finite fields R_i $(i = 1, \ldots, m)$. Since $1 = f_1 + \cdots + f_m$, where f_i is identity of R_i and $R_i = f_i R$, we obtain that

$$d(R_i) = d(f_i)R + f_i d(R) = f_i d(R) \subseteq f_i R = R_i$$

for any $d \in \operatorname{Der} R$ by Lemma 19. Every finite field is differentially trivial and we conclude that $d(R) = 0$. Hence R is differentially trivial. □

Proposition 3. *Let R be a reduced ring (respectively a ring with the torsion-free additive group R^+). Then the following hold:*

(1) *the set of all inner derivations $\operatorname{IDer} R$ is finite if and only if R is commutative,*
(2) *the set of all derivations $\operatorname{Der} R$ is finite if and only if R is differentially trivial.*

Proof. If the additive group R^+ is torsion-free and $\operatorname{IDer} R$ (respectively $\operatorname{Der} R$) is finite, the assertion follows in view of Lemma 20. Therefore we suppose that R is reduced.

(1) Assume that the set $\operatorname{IDer} R$ is finite. By Lemmas 13 and 20, R is a subdirect product of domains D with finite sets $\operatorname{IDer} D$ of inner derivations. If D is finite, then it is a field. If D is infinite, then it does not contain a proper ideal of finite index and therefore it is commutative in view of Lemma 18(4). This implies that R is commutative.

The converse is clear.

(2) If $\operatorname{Der} R$ is finite, then R is commutative in view of the part (1). Assume that $d(a) \neq 0$ for some $d \in \operatorname{Der} R$ and $a \in R$. The rule $rd : R \ni x \mapsto rd(x) \in R$ determines a derivation rd of R and so $Rd \subseteq \operatorname{Der} R$. Then the set $Rd(a)$ is a finite

ring and Lemma 11 implies that there exists a nonzero idempotent $e \in Rd(a)$ such that

$$Re \leq Rd(a)$$

and $e = td(a)$ for some $t \in R$. By Lemma 21, Re is differentially trivial. This gives that

$$0 = d(ae) = d(a)e = d(a)td(a).$$

Then $d(a)t$ is a nilpotent element and therefore $e = 0$, a contradiction. Hence $\operatorname{Der} R = \{0_R\}$.

The converse is clear. □

Corollary 3. *Let R be a semiprime ring. Then* $\operatorname{IDer} R$ *(respectively* $\operatorname{Der} R$*) is finite if and only if $R = A \oplus F$ is a direct sum of a finite ideal F and a commutative (respectively differentially trivial) reduced ideal A (in particular, $A = 0$).*

Proof. Assume that R is infinite and $\operatorname{IDer} R$ (respectively $\operatorname{Der} R$) is finite. If $a \in Z(R) \cap N(R)$, then aR is a nilpotent ideal of R and therefore $|N(R)| \leq |R : Z(R)|$. Since the index $|R : Z(R)|$ is finite by Corollary 2, we deduce that

$$R = A \bigoplus F$$

is a direct sum of a finite ideal F and a reduced ideal A by Lemma 8 (in particular, $A = 0$). If $A \neq 0$, then A is a commutative (respectively differentially trivial) ring by Proposition 3.

The converse is clear. □

Corollary 4. *A semiprime Jacobson radical FC-ring is commutative.*

Proof. In view of Lemma 10, R does not contain a nonzero finite ideal and so $C(R) = 0$ by Lemma 18(5). Hence R is commutative. □

Lemma 22. *If R is a commutative Artinian ring such that $|R : J(R)| < \infty$, then it is finite.*

Proof. Since R is a ring direct sum of local Artinian rings of prime power characteristics by the Artin-Wedderburn structure theorem, we may assume that R is local Artinian of characteristic p^n for some prime p and an integer $n \geq 1$. By Lemma 17, $R = J(R) + C$ is a group sum, where either C is a field (and consequently it is finite) or $C \cong V/p^n V$ for some v-ring V and $n \geq 2$. Since $R/J(R) \cong V/pV$ is a field, $p^k V/p^{k+1} V$ and V/pV are isomorphic as (V/pV)-linear spaces $(k = 1, \ldots, n-1)$, we deduce that C is finite. In view of Lemma 16,

$$J(R) = \sum_{s=1}^{t} j_s R$$

for some integer $t \geq 1$ and elements $j_1, \ldots, j_s \in R$. Then

$$J(R) = \sum_{s=1}^{t} (j_s J(R) + j_s C)$$
$$= \sum_{s=1}^{t} (j_s \sum_{l=1}^{t} (j_l J(R) + j_l C) + j_s C)$$
$$\ldots = \sum_{u=1}^{w} g_w C$$

for some integer $w \geq 1$ and $g_1, \ldots, g_w \in J(R)$. Thus R is finite. □

As usual, if e is an idempotent od a ring R (not necessary with 1), then we write

$$eR(1-e) := \{er - ere \mid r \in R\}, \ (1-e)Re := \{re - ere \mid r \in R\}$$

and

$$(1-e)R(1-e) := \{r - er - re + ere \mid r \in R\}.$$

Proof of Theorem 1. $a)$ Suppose that the set $\operatorname{IDer} R$ is finite and R is infinite. By Lemma 18, the commutator ideal $C(R)$ is finite. The quotient ring

$$R/J(R) = F_1 \bigoplus A_1$$

is semisimple and so, by the Artin-Wedderburn structure theorem, it is a direct sum of ideals F_1 and A_1, where F_1 is finite. If $A_1 = 0$, then $R/C(R)$ (and consequently R) is finite by Lemma 22, a contradiction with the assumption. Thus $A_1 \neq 0$ and we may assume that A_1 does not contain a nonzero finite ideal and so, by the Artin-Wedderburn structure theorem, A_1 is a ring direct sum of finitely many infinite semisimple Artinian rings (which are fields). An idempotent that is identity of A_1 can be lifted to an idempotent $e \in R$ by Lemma 15. We denote eRe by A and $A \cap \operatorname{ann} C(R)$ by C_1. Then $|A : C_1| < \infty$ in view of Lemma 18(4) and $J(A) = A \cap J(R)$. Since A_1 does not contain a proper ideal of finite index and the quotient ring

$$A/J(A) \cong (A + J(R))/J(R) \cong A_1$$

is a ring direct sum of finitely many infinite fields, we deduce that

$$A/J(A) = (C_1 + J(A))/J(A) \text{ and } A = J(A) + C_1.$$

The Jacobson radical $J(A)$ does not contain e and, as a consequence, $A = C_1 \leq \operatorname{ann} C(R)$. Since $e \in A$, we have that $eC(R) = C(R)e = 0$ and hence

$$rea - erea = (re - er)ea = 0 \text{ and } aer - aere = ae(er - re) = 0$$

for any $a \in A$ and $r \in R$. If

$$I = (1-e)R(1-e) + eR(1-e) + (1-e)Re + C(R),$$

then I is an ideal of R and $R = I + A$. Inasmuch $IA = 0 = AI$, we conclude that A is an ideal of R. Moreover, if $z \in I \cap A$, then

$$exe = z = (u - eu - ue + eue) + (ev - eve) + (we - ewe) + c$$

for some $x, u, v, w \in R$, $c \in C(R)$ and $exe = e^2xe^2 = eze = ece = 0$ what forces that $I \cap A = 0$. Hence

$$R = I \bigoplus A \tag{4.1}$$

is a ring direct sum, where A is commutative.

b) Now assume that the set $\operatorname{Der} R$ is finite. Then the set $\operatorname{IDer} R$ is also finite and (4.1) it follows. Consequently, $\operatorname{Der} A$ is finite. If $\delta \in \operatorname{Der} A$, then the rule

$$b\delta : A \ni r \mapsto b\delta(r) \in A$$

determines a derivation $b\delta$ of A for any $b \in A$. Since sets $A\delta$ and $A\delta(a)$ are finite, the index $|A : \operatorname{ann}\delta(a)| < \infty$ for any $a \in A$. But $A/J(A)$ is a ring direct sum of finitely many infinite fields and so A does not contain a proper ideal of finite index. This yields that $\delta = 0_A$ is zero and consequently A is a differentially trivial ring.

The converse is clear.

□

5. On Jacobson radical rings with two conjugacy classes

It is well known that any linear group (over a field) and any SI-group (i.e. group which possess a normal series with abelian factors) with a finite number of conjugacy classes are finite (see e.g. [19] or [14]). A result of Cohn [7] says that there exists a radical ring whose adjoint group has only two conjugacy classes and it is a torsion-free simple group.

Lemma 23. *If R is a Jacobson radical ring with the adjoint $\circ$-group R°, then:*

(1) *R has only the finite number of two-sided ideals,*
(2) *R has a simple homomorphic image,*
(3) *the center $Z(R)$ is finite.*

Proof. (1) If I is an ideal of R, then I° is a normal subgroup of R°. Since every normal subgroup is a set-theoretic sum of some conjugacy classes, the assertion holds.

(2) It follows from the part (1).

(3) If $a \in Z(R)$, then the conjugacy class a^G contains only one a. □

Lemma 24. *Let R be a Jacobson radical ring and $R^2 \neq R$. If the adjoint group R° has only two conjugacy classes, then R is a zero-ring that contains only two elements.*

Proof. Assume $G = R^\circ$ has only exactly two conjugacy classes. Now, $\{0\}$ is a conjugacy class by itself, and the other class is a^G for some $0 \neq a \in G$. Then $(R^2)^\circ$

is a normal subgroup of G, because R^2 is an ideal of R, thus $(R^2)^\circ$ is the union of conjugacy classes. As $0 \in R^2$, we have that $(R^2)^\circ$ is either $\{0\}$ or G. Since the ring R is Jacobson radical, we obtain that R^2 is either $\{0\}$ or R. From $R^2 \neq R$ it follows that $R^2 = \{0\}$, i.e. R is a zero-ring. Hence "$\circ$" is simply the addition, and then a number of conjugacy classes is card R from the commutativity of the addition "$+$". □

Lemma 25 ([22], Corollary, p. 332)**.** *Let G be a locally solvable group. If G has a finite number of conjugacy classes, then it is finite.*

Corollary 5. *Any Jacobson radical ring R with the torsion adjoint υ-group R° is finite.*

Proof. Since elements of the same conjugacy class have the same order, R° is of finite exponent. Then, by Lemma 2, the adjoint group R° is locally nilpotent. By Lemma 25, R is finite. □

Proof of Proposition 2. a) By Corollary 5, the torsion part $F(R)$ is finite and so D° is a υ-group.

b) Since $F(R)$ is an ideal of R and R° is a υ-group, we deduce that the exponent $\exp F(R)$ is finite. Then $F(R)$ is a ring direct sum of finitely many p-components for pairwise distinct primes p.

c) Suppose that the additive group R^+ is torsion-free. Assume that qR is proper in R for some prime q. Since the set

$$\{nR \mid n \text{ is a positive integer}\}$$

is finite by Lemma 23(1), we conclude that

$$q^l R = q^k R$$

for some positive integers l, k, where $l > k$. If $a \in R \setminus qR$, then $q^k a = q^l b$ for some $b \in R$ and

$$q^k(a - q^{l-k}b) = 0.$$

This yields $a = q^{l-k}b \in qR$, a contradiction. Hence R^+ is divisible.

d) Let R be any Jacobson radical ring with the adjoint υ-group R°. Then $R^+/F(R)$ is a divisible group and, by Lemma 12,

$$R^+ = F(R) \bigoplus D$$

is a group direct sum, where D is a divisible group. If $c, d \in D$, then $cd = f + h$ for some $f \in F(R)$ and $h \in D$. Since $c = nc_1$ and $h = nh_1$ for some $c_1, h_1 \in D$, where n is a positive integer, we conclude that

$$f \in \bigcap_{n=1}^{\infty} nR.$$

Therefore $f = 0$ and D is a ring. Moreover $F(R)D = 0 = DF(R)$. Then $R = F(R) \oplus D$ is a ring direct sum and so $F(R)$ is a ring with a finite number of conjugacy classes. □

6. Some corollaries

If B is the two-element zero ring, then B is Jacobson radical and its adjoint group B° has only two conjugacy classes. But B is not simple because $B^2 = 0$.

Proof of Corollary 1. (1) Assume that R is not a domain and does not contain a nonzero nilpotent element. If there exists a nonzero element $a \in R$ with a nonzero right annihilator $A = \operatorname{ann}_r a$, then $A \neq R$, $(Aa)^2 = 0$ and, as a consequence, $Aa = 0$. This means that A is contained in the two-sided annihilator $\operatorname{ann} a$. Since R does not contain a nonzero nilpotent element, $\operatorname{ann} a$ is a nonzero proper ideal of R, a contradiction.

(2) In view of the part (1), assume that R contains a nonzero nilpotent element $x \in R$ such that $x^2 = 0$. Then any $0 \neq y \in R$ is contained in the class x^{R° and so there exists $a \in R$ such that

$$y = a^{(-1)} \circ x \circ a.$$

Since

$$\begin{aligned}
y^2 &= (x + xa + a^{(-1)}x + a^{(-1)}xa)(x + xa + a^{(-1)}x + a^{(-1)}xa)\\
&= (xa^{(-1)}xa + xaxa + xaa^{(-1)}xa) + (xax + xa^{(-1)}x + xaa^{(-1)}x)\\
&\quad + (a^{(-1)}xa^{(-1)}x + a^{(-1)}xax + a^{(-1)}xaa^{(-1)}x)\\
&\quad + (a^{(-1)}xaxa + a^{(-1)}xa^{(-1)}xa + a^{(-1)}xaa^{(-1)}xa)\\
&= x(a^{(-1)} \circ a)xa + x(a^{(-1)} \circ a)x + a^{(-1)}x(a^{(-1)} \circ a)x\\
&\quad + a^{(-1)}x(a^{(-1)} \circ a)xa\\
&= 0,
\end{aligned}$$

we obtain that R is a nil-ring of bounded index. If $F(R) \neq 0$, then the p-part

$$F_p(R) = \{a \in R \mid a \text{ is of finite order } p^n \text{ for some non-negative integer } n\}$$

is a nonzero ideal of R for some prime p and therefore $R = F_p(R)$. Hence R^+ is a p-group. If $F(R) = 0$, then R^+ is torsion-free.

a) If R^+ is a p-group, then, by Lemma 1, R° is a p-group and, by Corollary 5, R is finite. Then R is a nilpotent ring, a contradiction with the simplicity of R.

b) Let R^+ be a torsion-free group. By Proposition 2, R is a $\mathbb{Q}$-algebra and, by Lemma 9, it is nilpotent, a contradiction.

Hence R is reduced and, by the part (1), R is a domain. As a consequence, R° is a simple group. By Proposition 2, R^+ is torsion-free divisible. □

Corollary 6. *If R is an infinite nil-ring, then $R^2 \neq R$ or R° is not a υ-group.*

Proof. By contrary. If R° is a υ-group and $R^2 = R$, then, in view of Lemma 23(2), there exists an ideal I of R with the simple homomorphic image R/I. If $(R/I)^{+}$ is a torsion group, then $(R/I)^{\circ}$ is torsion by Lemma 1 and so R/I is a finite nilpotent ring, which leads to a contradiction. If $(R/I)^{+}$ is a torsion-free group, then, as in the proof of Corollary 1, R/I is a nil $\mathbb{Q}$-algebra of bounded index and it is nilpotent by Lemma 9, a contradiction with $R^2 = R$. □

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REFERENCES

[1] B. Amberg and O. Dickenschied, "On the adjoint group of a radical ring," *Can. Math. Bul.*, vol. 38, no. 3, pp. 262–270, 1995, doi: http://dx.doi.org/10.4153/CMB-1995-039-2.

[2] B. Amberg, O. Dickenschied, and Y. P. Sysak, "Subgroups of the adjoint group of a radical ring," *Canad. J. Math.*, vol. 50, no. 1, pp. 3–15, 1998, doi: http://dx.doi.org/10.4153/CJM-1998-001-9.

[3] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, 2nd ed., ser. Graduate Text in Mathematics. New York: Springer, 2013, vol. 13, doi: 10.1007/978-1-4612-4418-9.

[4] V. A. Andrunakievič and J. M. Rjabuhin, "Rings without nilpotent elements, and completely prime ideals (russian)," *Dokl. Akad. Nauk SSSR*, vol. 180, pp. 9–11, 1968, doi: S0161171200004245.

[5] H. E. Bell, "A note on centralizers," *Int. J. Math. Math. Sci.*, vol. 24, no. 1, pp. 55–57, 2000, doi: S0161171200004245.

[6] I. S. Cohen, "On the structure and ideal theory of complete local rings," *Trans. Amer. Math. Soc.*, vol. 59, pp. 54–106, 1946, doi: 10.1137/050641867.

[7] P. M. Cohn, "The embedding of radical ring in simple radical ring," *Bull. London Math. Soc.*, vol. 3, pp. 185–188 (Corr. ibid 4 (1972), 54 and ibid 5(1973), 322), 1971, doi: 10.1112/blms/5.3.322-s and 10.1112/blms/4.1.54-s.

[8] F. D'Alessandro, "Note on the multiplicative group of a division ring," *Internat. J. Algebra Comput.*, vol. 7, no. 1, pp. 51–53, 1997, doi: 10.1142/S0218196797000058.

[9] C. Faith, *Algebra II. Ring Theory*, ser. Grundlehren der mathematischen Wissenschaften. Berlin New York London: Springer, 1976, vol. 191, doi: 10.1007/978-1-4614-6946-9.

[10] I. N. Herstein, *Noncommutative rings*. New York: The Mathematical Association of America, J. Wiley and Sons, 1969. doi: https://doi.org/10.5948/UPO9781614440154.

[11] G. Higman, "On a conjecture of nagata," *Proc. Cambridge Phil. Soc.*, vol. 52, no. 1, pp. 1–4, 1956, doi: https://doi.org/10.1017/S0305004100030899.

[12] Y. Hirano, "On a problem of szász," *Bull. Austral. Math. Soc.*, vol. 40, no. 3, pp. 363–364, 1989, doi: https://doi.org/10.1017/S000497270001738X.

[13] Y. Hirano, "On extensions of rings with finite additive index," *Math. J. Okayama Univ.*, vol. 32, pp. 93–95, 1990, doi: http://www.math.okayama-u.ac.jp/mjou/mjou-32.html.

[14] A. G. Kuroš, *The theory of groups (Russian)*. Moscow: Nauka, 1967.

[15] C. Lanski, "Rings with few nilpotents," *Houston J. Math.*, vol. 18, no. 4, pp. 577–590, 1992, doi: https://www.math.uh.edu/ hjm/restricted/archive/v018n4/0577LANSKI.pdf.

[16] C. Lanski, "Finite higher commutators in associative rings," *Bull. Aust. Math. Soc.*, vol. 89, no. 3, pp. 503–509, 2014, doi: https://doi.org/10.1017/S0004972713000890.

[17] T.-K. Lee, "Prime rings with finiteness properties on one-sided ideals," *Proc. Edinburgh Math. Soc. (2)*, vol. 45, pp. 507–511, 2002, doi: https://doi.org/10.1017/S0013091501000050.

[18] M. Nagata, "On the nilpotency of nil-algebras," *J. Math. Soc. Japan*, vol. 4, pp. 296–301, 1952, doi: 10.2969/jmsj/00430296.
[19] D. J. S. Robinson, *A Course in the Theory of Groups*, 2nd ed., ser. Graduate Text in Mathematics. New York, Berlin: Springer, 1980, vol. 80, doi: 1980.10.1007/978-1-4419-8594-1.
[20] F. A. Szász, "On some simple jacobson radical rings," *Math. Japan.*, vol. 18, pp. 225–228, 1973.
[21] F. A. Szász, *Radikals of Rings*. Chichester: John Wieley and Sons Ltd., 1981.
[22] S. N. Černikov, "On the theory of locally soluble groups (russian)," *Rec. Mat. [Mat. Sbornik] N.S.*, vol. 13(55), pp. 317–333, 1943, doi: http://mi.mathnet.ru/msb6182.
[23] J. F. Watters, "On the adjoint group of a radical ring," *J. London Math. Soc. (2)*, vol. 43, pp. 725–729, 1968, doi: 10.1112/jlms/s1-43.1.725.

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