Abstract. We investigate properties of \( FC \)-rings (i.e. rings \( R \) in which the centralizer \( C_R(a) \) of any element \( a \in R \) is of finite index in \( R \)) and, in particular, characterize left Artinian rings with a finite set of all derivations \( \text{Der}_R \) (respectively inner derivations \( \text{IDer}_R \)). We show that if \( R \) is a Jacobson radical ring in which its adjoint group \( R^\pi \) has a finite number of conjugacy classes, then

\[
R = R_{p_1} \oplus \cdots \oplus R_{p_t} \oplus D
\]
is a ring direct sum of Jacobson radical rings \( R_{p_i} \) and \( D \), where the additive group \( D^+ \) is a torsion-free divisible group, the adjoint group \( D^\pi \) is a group with a finite number of conjugacy classes, \( R^+_{p_i} \) is a finite \( p_i \)-group \( (i = 1, \ldots, t) \) and \( p_1, \ldots, p_t \) are pairwise distinct primes.

2010 Mathematics Subject Classification: 16P20; 16W25; 16N99; 17B60

Keywords: derivation, Artinian ring, Jacobson radical ring, \( FC \)-group, \( FC \)-ring, Lie ring

1. Introduction

Let \( R \) be an associative ring, not necessarily with identity. It is well-known that a group \( G \) is called an \( FC \)-group if the centralizer \( C_G(g) \) of any element \( g \in G \) is of finite index in \( G \) [19]. Naturally, if the centralizer

\[
C_R(a) = \{ r \in R \mid ar = ra \}
\]
of any element \( a \in R \) is of finite index in the additive group \( R^+ \) of \( R \), then \( R \) is called an \( FC \)-ring. Every finite ring and every commutative ring are \( FC \)-rings. Since

\[
\delta_a : R \ni r \mapsto ar - ra = [a, r] \in R
\]

(so-called an inner derivation of \( R \) induced by \( a \)) is an endomorphism of the group \( R^+ \), the kernel

\[
\ker \delta_a = \{ r \in R \mid ra = ar \} = C_R(a)
\]
is the centralizer of \( a \in R \) and the quotient group \( R^+ / \ker \delta_a \) is isomorphic to the image \( \text{Im} \delta_a \), we deduce that \( R \) is an \( FC \)-ring if and only if \( \text{Im} \delta_a \) is finite for any \( a \in R \). A map \( \delta : R \to R \) is said to be a derivation of \( R \) if

\[
\delta(a + b) = \delta(a) + \delta(b) \quad \text{and} \quad \delta(ab) = \delta(a)b + a \delta(b)
\]
for all \(a, b \in R\). Clearly, the zero map \(0_R : R \ni r \mapsto 0 \in R\) is a derivation of \(R\).

An algebraic operation “\(\circ\)" determined by the rule

\[ a \circ b = a + b + ab \]

for any \(a, b \in R\) is associative with the neutral element \(0 \in R\). The set of all invertible elements in \(R\) with respect to “\(\circ\)" is a group (which is called the adjoint group of \(R\) and denoted by \(R^\circ\)).

Notations. For a group \(G\), \(g^{-1}\) is the inverse of \(g \in G\), \(a^G := \{g^{-1}ag \mid g \in G\}\) is the conjugacy class of \(a \in G\), \(G'\) is the commutator subgroup of \(G\) (i.e. a subgroup generated by all multiplicative commutators \(g^{-1}h^{-1}gh\), where \(g, h \in G\)), \(Z_0(G) := 1\) is a trivial subgroup of \(G\), \(Z(G) = Z_1(G) := \{z \in G \mid zg = gz\} \text{ for all } g \in G\) is the center of \(G\), \(Z_{\alpha+1}(G)/Z_\alpha(G) = Z(G/Z_\alpha(G))\) and \(Z_\lambda(G) = \cup_{\beta<\lambda} Z_\beta(G)\), where \(\alpha\) is an ordinal and \(\lambda\) a limit ordinal. Recall that a group \(G\) is called:

- hypercentral (respectively nilpotent) if \(Z_\theta(G) = G\) for some ordinal (respectively non-negative integer) \(\theta\),
- locally nilpotent if every its finitely generated subgroup is nilpotent,
- simple in case \(G \neq 1\) and \(1, G\) are the only normal subgroups of \(G\),
- solvable if \(G^{(n)} = 1\) for some integer \(n \geq 0\), where \(G^{(0)} = G\) and \(G^{(m+1)} = (G^{(m)})'\) for any integer \(m \geq 1\),
- locally solvable if every its finitely generated subgroup is solvable.

Every nilpotent group is hypercentral. It is known that a group is hypercentral if and only if every its non-trivial homomorphic image has the non-trivial center. A group \(G\) with a finite number of conjugacy classes is called a \(v\)-group. A subgroup \(A\) of an (additive) abelian group \(G\) is called pure if \(A \cap nG = nA\) for any integer \(n\). An (additive) abelian group \(G\) is divisible if, for every positive integer \(n\) and every \(a \in G\), there exists \(x \in G\) such that \(nx = a\).

For any ring \(R\), \([x, r] = xr - rx\) is an additive commutator of \(x, r \in R\), \(C(R)\) is the commutator ideal of \(R\) that is the ideal generated by the commutator set \([R, R] = \{[x, r] \mid x, r \in R\}\), \(J(R)\) is the Jacobson radical of \(R\), \(R^+\) is the additive group of \(R\), \(N(R)\) is the set of all nilpotent elements of \(R\), \(U(R)\) is the unit group of \(R\) with identity, \(Z(R)\) is the center of \(R\), \(F(R) = \{a \in R \mid a\text{ is of finite order in } R^+\}\) is the torsion part of \(R\), \(\exp F(R)\) is the exponent of the group \(F(R)^+\), \(g^{(-1)}\) is the inverse of \(g\) in the adjoint group \(R^\circ\), \(x^{(n)}\) is the \(n\)th power of \(x\) in the adjoint group \(R^\circ\), \(\text{ann } X = \{a \in R \mid aX = 0 = Xa\}\) is the annihilator of \(X \subseteq R\). By Der \(R\) we denote the set of all derivations of \(R\). A subring \(S\) is of finite index in \(R\) (i.e. \(|R : S| < \infty\)) if the additive subgroup \(S^+\) has a finite index in \(R^+\). Recall that a ring \(R\) is called:

- Jacobson radical if \(R^\circ = R\),
- nil if every its element \(x\) is nilpotent, i.e. there exists an integer \(n = n(x) > 0\) such that \(x^n = 0\); if there exists an integer \(n > 0\) such that \(x^n = 0\) for any \(x \in R\), then \(R\) is nil of bounded index,
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- nilpotent in case there is an integer $m > 0$ such that $x_1 x_2 \cdots x_m = 0$ for any $x_1, x_2, \ldots, x_m \in R$,
- locally nilpotent if every its finitely generated subring is nilpotent,
- reduced if it is without nonzero nilpotent elements,
- simple in case $R^2 \neq 0$ and $0, R$ are the only ideals of $R$,
- local if it has identity and the quotient ring $R/J(R)$ is simple,
- semiprime if it has no nonzero nilpotent ideals,
- left Artinian in case for every descending chain
  
  $$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

  of left ideals $I_j$ of $R$, there is an integer $n \geq 1$ with $I_{n+1} = I_n$ ($j = 1, 2, \ldots$).

Any unexplained terminology is standard as in [3, 10] and [19].

The purpose of this paper is to study associative $FC$-rings $R$ and some related topics. Obviously, commutative rings and, in particular, differentially trivial rings (i.e. $Der R = \{0_R\}$) are $FC$-rings. Associative rings $R$ with finite sets $Der R$, $IDer R$ are very related to $FC$-rings. Rings $R$ with the center $Z(R)$ of finite index are $FC$-rings. In [21, Problem 84] F. Szász asked: “In which rings $R$ the additive group $Z(R)^+$ of the center $Z(R)$ has a finite group-theoretic index with respect to $R^+$?” From Corollary 2 it follows that this problem is equivalent to study of rings with a finite set of all inner derivations $IDer R$. Y. Hirano [12, Proposition 1] has proved that the condition $[R : Z(R)] < \infty$ implies that the commutator ideal $C(R)$ is finite. F. Szász [21, Problem 83] asked: “In which rings the commutator ideal is finite or can be finitely generated?” H. Bell [5] has proved that if $I$ is a nonzero right ideal of finite index in a prime ring $R$ and $[I, I]$ is finite, then $R$ is either finite or commutative (see also [17, Corollary 1.2]). C. Lanski [16] has showed that if $T$ is a finite higher commutator of $R$ containing no nonzero nilpotent element, then $T$ generates a finite ideal of $R$.

We study left Artinian rings with the finite set of all derivations $Der R$ (respectively inner derivations $IDer R$) and prove the following with similar flavour.

**Theorem 1.** Let $R$ be a left Artinian ring. Then $IDer R$ (respectively $Der R$) is finite if and only if $R = A \oplus F$ is a direct sum of a finite ideal $F$ and a commutative (respectively differentially trivial) reduced ideal $A$.

An $FC$-ring $R$ has the adjoint $FC$-group. It is easy to see that a Jacobson radical ring $R$ is an $FC$-ring if and only if its adjoint group $R^\circ$ is an $FC$-group. In [21, Problem 88] F. Szász asked:“ Let

$$\hat{a} = \{(1-x)a(1-x)^{-1} \mid x \in R\}$$

in a Jacobson radical ring $R$. When is every class $\hat{a}$ finite, and when is a number of the classes $\hat{a}$ finite?” About Jacobson radical rings we prove the following

**Proposition 1.** If $R$ is a Jacobson radical ring with the adjoint $FC$-group $R^\circ$, then:
(1) every nonzero homomorphic image \( B \) of \( R \) is commutative or \( \operatorname{ann} B \) is nonzero,
(2) \( R^\circ \) is a hypercentral group,
(3) if \( R^+ \) is torsion-free, then \( R \) is commutative,
(4) if \( R \) is nonzero, then the commutator ideal \( C(R) \) is proper in \( R \); if, moreover, \( R \) is non-commutative, then \( R^2 \) is proper in \( R \).

We give a partial answer on the second part of [21, Problem 88] in the following

**Proposition 2.** Let \( R \) be a Jacobson radical ring. Then the adjoint group \( R^\circ \) is a \( \nu \)-group if and only if
\[
R = R_{p_1} \bigoplus \cdots \bigoplus R_{p_t} \bigoplus D
\]
is a ring direct sum of Jacobson radical rings \( R_{p_i} \) and \( D \), where \( D^+ \) is a torsion-free divisible group, \( D^0 \) is a \( \nu \)-group, \( R_{p_i} \) is a finite \( p_i \)-group \( (i = 1, \ldots, t) \) and \( p_1, \ldots, p_t \) are pairwise distinct primes.

F. Szász [20] has investigated properties of infinite Jacobson radical rings \( R \) whose adjoint groups \( R^\circ \) have only two conjugacy classes. If, moreover, \( R^2 = R \), then it is a simple domain by Propositions 1 and 2 from [20]. We make this result more precise in the following

**Corollary 1.** If \( R \) is a simple Jacobson radical ring, then the following hold:
(1) if \( R^\circ \) is a \( \nu \)-group, then either \( R \) is a domain or contains a nonzero nilpotent element,
(2) if the adjoint group \( R^\circ \) has only two conjugacy classes, then \( R \) is a domain with the torsion-free divisible additive group \( R^+ \) and the simple adjoint group \( R^\circ \).

2. Preliminaries

For a convenience of the reader and in order to have the paper more self-contained in this section we collect some results needed in the next.

**Lemma 1** ([1], Lemma 2.4(1)). If \( R \) is a nil-ring and \( p \) is prime, then the additive group \( R^+ \) is a \( p \)-group if and only if the adjoint group \( R^\circ \) is a \( p \)-group.

**Lemma 2** (see [2], Corollary 1). Let \( G \) be a subgroup of the adjoint group of a radical ring. If \( G \) has finite exponent, then it is locally nilpotent.

If \( (R, +, \cdot) \) is an associative ring, then \( R \) is a Lie ring with respect to the addition “+” and the Lie multiplication “\([\cdot, \cdot]\)" (denoted by \( R^L \)) defined by the rule \( [a, b] = a \cdot b - b \cdot a \) for any \( a, b \in R \). Then the center \( Z(R) \) of \( R \) is an ideal of the Lie ring \( R^L \).

**Lemma 3.** If \( R \) is an associative ring, then there is a Lie ring isomorphism
\[
\operatorname{IDer} R \ni \partial_a \mapsto a + Z(R) \in R^L / Z(R).
\]
Proof. Immediate.

From this we have the following

**Corollary 2.** Let $R$ be a ring. Then the set $\text{IDer} R$ is finite if and only if $|R : Z(R)| < \infty$.

**Lemma 4.** Let $R$ be a ring, $I$ an ideal and $S$ a subring of $R$. If the set of all inner derivations $\text{IDer} R$ is finite, then sets $\text{IDer} S$ and $\text{IDer}(R/I)$ are finite.

Proof. Straightforward.

**Lemma 5** ([13], Theorem 1). Let $S$ be a subring of a ring $R$. If $S$ has a finite index in $R$, then there exists an ideal $I$ of $R$ contained in $S$ such that $R/I$ is a finite ring.

If $R$ has identity $1$, then
\[ R^0 \ni a \mapsto 1 + a \in U(R) \]

is a group isomorphism of the adjoint group $R^0$ and the unit group $U(R)$ of $R$. As proved in [8], a division ring $D$ with multiplicative $FC$-group $U(D)$ is commutative.

**Lemma 6.** A local ring $R$ is an $FC$-ring if and only if its adjoint group $R^0$ is an $FC$-group.

Proof. The adjoint group of any $FC$-ring is an $FC$-group. A commutative ring is an $FC$-ring. Therefore we assume that a local ring $R$ is not commutative and $R^0$ is an $FC$-group. Since $R$ has a proper ideal of finite index in view of Lemma 5, we conclude that the quotient ring $R/J(R)$ is finite. Moreover, $J(R)^0 \cong 1 + J(R)$ is a subgroup of the unit group $U(R)$,

\[ |J(R) : C_{J(R)}(a)| = |1 + J(R) : C_{1 + J(R)}(a)| < \infty \]

for any $a \in U(R)$ and therefore $|R : C_R(a)| < \infty$. Inasmuch $R = J(R) \cup U(R)$ and $J(R)$ is a Jacobson radical $FC$-ring, we deduce that $R$ is an $FC$-ring.

**Lemma 7** ([19], Theorem 14.5.9). If $G$ is an $FC$-group, then the commutator subgroup $G'$ is torsion.

**Lemma 8** ([15], Theorem 2). Let $R$ be a semiprime ring and $S = \{x \in R \mid x^2 = 0\}$. If the cardinality $\text{card} S$ is finite, then $R = A \oplus F$ is a direct sum of ideals $A$ and $F$. $A$ is reduced and $F$ is finite. In particular, $R$ has only finitely many nilpotent elements.

**Lemma 9** (see [11, 18]). Let $A$ be an algebra over a field of characteristic zero. Suppose that there is a positive integer $n$ such that $a^n = 0$ for all $a \in A$. Then there is an integer $N$ such that $a_1a_2\cdots a_N = 0$ for all $a_1, a_2, \ldots, a_N \in A$.

Theorem 1 of [23] implies the next

**Lemma 10.** A finite Jacobson radical ring is nilpotent.
Lemma 11 ([15], Lemma 3). If $R$ is a finite ring which is not nilpotent, then $R$ contains a nonzero idempotent.

Lemma 12 ([19], 4.3.8). A pure subgroup $A$ of finite index of an abelian group $G$ is a direct summand.

A ring $R$ is a subdirect product of some rings $S_i$ ($i \in I$) if, for any $i \in I$, $S_i \cong R/K_i$, where $K_i$ is an ideal of $R$ and

$$\bigcap_{i \in I} K_i = 0.$$

Lemma 13 (see [4]). A ring $R$ is reduced if and only if it is a subdirect product of domains.

Lemma 14 ([10], §1.4, Corollary 2). If $R$ is a left Artinian ring and its Jacobson radical $J(R) = 0$ is zero, then $R$ contains identity.

If $I$ is an ideal of a ring $R$, then we say that an idempotent $g + I$ of the quotient ring $R/I$ can be lifted (to $e$) modulo $I$ in case there is an idempotent $e \in R$ such that $g + I = e + I$.

Lemma 15 ([3], Proposition 27.1). If $I$ is a nil-ideal of a ring $R$, then idempotents lift modulo $I$.

Lemma 16 ([9], Theorem 18.13). A left Artinian ring is left Noetherian (i.e. every its ideal is finitely generated).

A commutative ring $V$ is called a $v$-ring if it is complete (in the $J(V)$-adic topology), discrete, unramified valuation ring of characteristic $0$ with the quotient ring $V/J(V)$ of prime characteristic $p$ [6, p.79]. Then $J(V) = pV$, $V/pV$ is a field, $p^kV/p^{k+1}$ and $V/pV$ are isomorphic as $(V/pV)$-linear spaces.

Lemma 17. Let $R$ be a complete (in the $J(R)$-adic topology) local Noetherian commutative ring of prime power characteristic $p^n$. Then the following hold:

1. (see [6, Theorem 9]) if $n = 1$, then there exists a subfield $C$ of $R$ such that $R = J(R) + C$ is a group direct sum,

2. (see [6, Theorem 11]) if $n \geq 2$, then there exists a subring $C$ of $R$ such that $R = J(R) + C$ is a group sum, where $C \cong V/p^nV$ for some $v$-ring $V$ and $J(R) \cap V = pV$.

Any local Artinian ring $R$ is a complete local Noetherian ring.

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Lemma 18. If $R$ is an infinite ring with a finite set of all inner derivations $\text{IDer} R$, then the following hold:
(1) the centralizer $C_R(a)$ is a subring of finite index in $R$ for any $a \in R$ (i.e. $R$ is an FC-ring),
(2) the adjoint group $R^o$ is an FC-group,
(3) if $R$ is a simple ring, then $R$ is a field,
(4) $R$ contains a central ideal $I$ of finite index such that $I \cdot C(R) = 0$,
(5) the commutator ideal $C(R)$ is finite.

Proof. (1) Since the set $\{ \partial_r(a) \mid r \in R \}$ is finite, the index $|R : C_R(a)|$ is finite.
(2) It follows from the part (1).
(3) It holds from Corollary 2 in view of Lemma 5.
(4) Corollary 2 and Lemma 5 imply that $R$ contains an ideal $I$ of finite index such that $I \leq Z(R)$. Moreover, for any $r, t \in R$ and $i \in I$ we obtain that
\[(rt)i = r(ti) = (ti)r = t(ri) = t(ri) = (tr)i,\]
and so $(rt- tr)i = 0$. As a consequence, $I \cdot C(R) = 0$.
(5) By Corollary 2, the center $Z(R)$ is of finite index in $R$ and so, by the part (4), the annihilator $\text{ann}_R \partial_x(y)$ has a finite index in $R$. Then the commutator ideal
\[C(R) = \sum_{x,y \in R \setminus Z(R)} R\partial_x(y)R\]
is finite.

Proof of Proposition 1. (1) Assume that $B$ is a non-commutative homomorphic image of $R$ and $a \in B \setminus Z(B)$. Then $C_B(a)$ is of finite index in $B$ and, by Lemma 5, the centralizer $C_B(a)$ contains a proper ideal $I$ of $B$ such that $|B : I| < \infty$. If $i \in I$ and $r \in B$, then
\[ari = ria = rai, iar = air = ira\]
and so
\[[a, r]i = 0 = i[a, r].\]
This gives that $[B, B] \subseteq \text{ann} I$. Since $B/I$ is a finite radical ring, it is nilpotent by Lemma 10. We have that $B^n \leq I$ for some positive integer $n$ and $\text{ann} I \leq \text{ann}(B^n)$. From this it holds that $\text{ann} B \neq 0$. Then every non-trivial homomorphic image of $R$ is commutative or has a non-trivial annihilator.
(2) Since $\text{ann} R \subseteq Z(R)$, we deduce that every proper quotient group of $R^o$ has the non-trivial center. This means that the adjoint group $R^o$ is hypercentral [19, Exercises 12.2.2].
(3) By Lemma 7, every torsion-free FC-group is abelian.
(4) The commutator ideal of a commutative ring $R$ is zero (and so it is proper in $R \neq 0$). Assume that $R$ is non-commutative. As in the part (1), we can prove that $R$ contains a proper ideal of finite index and therefore $R^2$ is proper in $R$ in view of Lemma 10. Obviously that $C(R) \subseteq R^2$. Hence $C(R)$ is proper in a non-commutative ring $R$.\[\square\]
4. Rings with a finite set of derivations

Lemma 19. If $e$ is an idempotent of a commutative ring $R$, then $d(e) = 0$ for any $d \in \text{Der} R$.

Proof. In fact, $d(e) = d(e^2) = d(e)e + ed(e)$ implies that $ed(e) = ed(e)e + ed(e)$ and so $d(e)e = ed(e) = e^2d(e) = ed(e)e = 0$. Hence $d(e) = 0$. □

Lemma 20. Let $R$ be a ring. If the set $\text{Der} R$ (respectively $\text{IDer} R$) is finite, then $\text{Der} R = \{0_R\}$ (respectively $R$ is commutative) or $\delta(R) \subseteq F(R)$ for any $\delta \in \text{Der} R$ (respectively $\delta \in \text{IDer} R$).

Proof. Assume that $\delta(a) \neq 0$ for some $\delta \in \text{Der} R$ and $a \in R$. Since the set
$$\{n\delta(a) \mid n \text{ is an integer}\}$$
is finite, the torsion part $F(R) \neq 0$ is nonzero and $\delta(a) \in F(R)$. □

Lemma 21. Any finite semiprime commutative ring $R$ is differentially trivial.

Proof. By Lemma 14, $R$ contains identity 1 and, by the Artin-Wedderburn structure theorem, $R = R_1 \oplus \cdots \oplus R_m$ is a ring direct sum of finite fields $R_i$ ($i = 1, \ldots, m$). Since $1 = f_1 + \cdots + f_m$, where $f_i$ is identity of $R_i$ and $R_i = f_iR$, we obtain that
$$d(R_i) = d(f_i)R + f_id(R) = f_id(R) \subseteq f_iR = R_i$$
for any $d \in \text{Der} R$ by Lemma 19. Every finite field is differentially trivial and we conclude that $d(R) = 0$. Hence $R$ is differentially trivial. □

Proposition 3. Let $R$ be a reduced ring (respectively a ring with the torsion-free additive group $R^+$). Then the following hold:

1. the set of all inner derivations $\text{IDer} R$ is finite if and only if $R$ is commutative,
2. the set of all derivations $\text{Der} R$ is finite if and only if $R$ is differentially trivial.

Proof. If the additive group $R^+$ is torsion-free and $\text{IDer} R$ (respectively $\text{Der} R$) is finite, the assertion follows in view of Lemma 20. Therefore we suppose that $R$ is reduced.

1. Assume that the set $\text{IDer} R$ is finite. By Lemmas 13 and 20, $R$ is a subdirect product of domains $D$ with finite sets $\text{IDer} D$ of inner derivations. If $D$ is finite, then it is a field. If $D$ is infinite, then it does not contain a proper ideal of finite index and therefore it is commutative in view of Lemma 18(4). This implies that $R$ is commutative.

The converse is clear.

2. If $\text{Der} R$ is finite, then $R$ is commutative in view of the part (1). Assume that $d(a) \neq 0$ for some $d \in \text{Der} R$ and $a \in R$. The rule $rd : R \ni x \mapsto rd(x) \in R$ determines a derivation $rd$ of $R$ and so $Rd \subseteq \text{Der} R$. Then the set $Rd(a)$ is a finite
ring and Lemma 11 implies that there exists a nonzero idempotent \( e \in Rd(a) \) such that

\[
Re \leq Rd(a)
\]

and \( e = td(a) \) for some \( t \in R \). By Lemma 21, \( Re \) is differentially trivial. This gives that

\[
0 = d(ae) = d(a)e = d(a)td(a).
\]

Then \( d(a)t \) is a nilpotent element and therefore \( e = 0 \), a contradiction. Hence \( \text{Der} R = \{0_R\} \).

The converse is clear.

**Corollary 3.** Let \( R \) be a semiprime ring. Then \( \text{IDer} R \) (respectively \( \text{Der} R \)) is finite if and only if \( R = A \oplus F \) is a direct sum of a finite ideal \( F \) and a commutative (respectively differentially trivial) reduced ideal \( A \) (in particular, \( A = 0 \)).

**Proof.** Assume that \( R \) is infinite and \( \text{IDer} R \) (respectively \( \text{Der} R \)) is finite. If \( a \in Z(R) \cap N(R) \), then \( aR \) is a nilpotent ideal of \( R \) and therefore \( |N(R)| \leq |R : Z(R)| \). Since the index \( |R : Z(R)| \) is finite by Corollary 2, we deduce that

\[
R = A \oplus F
\]

is a direct sum of a finite ideal \( F \) and a reduced ideal \( A \) by Lemma 8 (in particular, \( A = 0 \)). If \( A \neq 0 \), then \( A \) is a commutative (respectively differentially trivial) ring by Proposition 3.

The converse is clear.

**Corollary 4.** A semiprime Jacobson radical FC-ring is commutative.

**Proof.** In view of Lemma 10, \( R \) does not contain a nonzero finite ideal and so \( C(R) = 0 \) by Lemma 18(5). Hence \( R \) is commutative.

**Lemma 22.** If \( R \) is a commutative Artinian ring such that \( |R : J(R)| < \infty \), then it is finite.

**Proof.** Since \( R \) is a ring direct sum of local Artinian rings of prime power characteristics by the Artin-Wedderburn structure theorem, we may assume that \( R \) is local Artinian of characteristic \( p^n \) for some prime \( p \) and an integer \( n \geq 1 \). By Lemma 17, \( R = J(R) + C \) is a group sum, where either \( C \) is a field (and consequently it is finite) or \( C \cong V/p^nV \) for some \( v \)-ring \( V \) and \( n \geq 2 \). Since \( R/J(R) \cong V/pV \) is a field, \( p^kV/p^{k+1}V \) and \( V/pV \) are isomorphic as \( (V/pV) \)-linear spaces \((k = 1, \ldots, n-1)\), we deduce that \( C \) is finite. In view of Lemma 16,

\[
J(R) = \sum_{s=1}^{t} j_s R
\]
for some integer $t \geq 1$ and elements $j_1, \ldots, j_s \in R$. Then

$$J(R) = \sum_{s=1}^{t} (j_s J(R) + j_s C)$$

$$= \sum_{s=1}^{t} \left( j_s \sum_{l=1}^{t} (j_l J(R) + j_l C) + j_s C \right)$$

$$\ldots = \sum_{u=1}^{w} g_w C$$

for some integer $w \geq 1$ and $g_1, \ldots, g_w \in J(R)$. Thus $R$ is finite. □

As usual, if $e$ is an idempotent of a ring $R$ (not necessarily with 1), then we write

$$eR(1-e) := \{er-ere \mid r \in R\}, \quad (1-e)Re := \{re-ere \mid r \in R\}$$

and

$$(1-e)R(1-e) := \{r-er-re+ere \mid r \in R\}.$$ 

**Proof of Theorem 1.** a) Suppose that the set $\text{IDer } R$ is finite and $R$ is infinite. By Lemma 18, the commutator ideal $C(R)$ is finite. The quotient ring

$$R/J(R) = F_1 \bigoplus A_1$$

is semisimple and so, by the Artin-Wedderburn structure theorem, it is a direct sum of ideals $F_1$ and $A_1$, where $F_1$ is finite. If $A_1 = 0$, then $R/C(R)$ (and consequently $R$) is finite by Lemma 22, a contradiction with the assumption. Thus $A_1 \neq 0$ and we may assume that $A_1$ does not contain a nonzero finite ideal and so, by the Artin-Wedderburn structure theorem, $A_1$ is a ring direct sum of finitely many infinite semisimple Artinian rings (which are fields). An idempotent that is identity of $A_1$ can be lifted to an idempotent $e \in R$ by Lemma 15. We denote $eRe$ by $A$ and $A \cap \text{ann } C(R)$ by $C_1$. Then $|A : C_1| < \infty$ in view of Lemma 18(4) and $J(A) = A \cap J(R)$. Since $A_1$ does not contain a proper ideal of finite index and the quotient ring

$$A/J(A) \cong (A + J(R))/J(R) \cong A_1$$

is a ring direct sum of finitely many infinite fields, we deduce that

$$A/J(A) = (C_1 + J(A))/J(A) \text{ and } A = J(A) + C_1.$$ 

The Jacobson radical $J(A)$ does not contain $e$ and, as a consequence, $A = C_1 \leq \text{ann } C(R)$. Since $e \in A$, we have that $eC(R) = C(R)e = 0$ and hence

$$rea - erea = (re-er)ea = 0 \text{ and } aer - aere = ae(er-re) = 0$$

for any $a \in A$ and $r \in R$. If

$$I = (1-e)R(1-e) + eR(1-e) + (1-e)Re + C(R),$$
then \( I \) is an ideal of \( R \) and \( R = I + A \). Inasmuch \( IA = 0 = AI \), we conclude that \( A \) is an ideal of \( R \). Moreover, if \( z \in I \cap A \), then

\[
e xe = z = (u - eu - we + eue) + (ev - eve) + (we - ewe) + c
\]

for some \( x, u, v, w \in R \), \( c \in C(R) \) and \( exe = e^2xe^2 = eze = ece = 0 \) what forces that \( I \cap A = 0 \). Hence

\[
R = I \bigoplus A
\]

is a ring direct sum, where \( A \) is commutative.

\( b) \) Now assume that the set \( \text{Der} R \) is finite. Then the set \( \text{IDer} R \) is also finite and (4.1) it follows. Consequently, \( \text{Der} A \) is finite. If \( \delta \in \text{Der} A \), then the rule

\[
b\delta : A \ni r \mapsto b\delta(r) \in A
\]

determines a derivation \( b\delta \) of \( A \) for any \( b \in A \). Since sets \( A\delta \) and \( A\delta(a) \) are finite, the index \( |A : \text{ann} \delta(a)| < \infty \) for any \( a \in A \). But \( A/J(A) \) is a ring direct sum of finitely many infinite fields and so \( A \) does not contain a proper ideal of finite index. This yields that \( \delta = 0_A \) is zero and consequently \( A \) is a differentially trivial ring.

The converse is clear.

5. ON JACOBSON RADICAL RINGS WITH TWO CONJUGACY CLASSES

It is well known that any linear group (over a field) and any SI-group (i.e. group which possess a normal series with abelian factors) with a finite number of conjugacy classes are finite (see e.g. [19] or [14]). A result of Cohn [7] says that there exists a radical ring whose adjoint group has only two conjugacy classes and it is a torsion-free simple group.

**Lemma 23.** If \( R \) is a Jacobson radical ring with the adjoint \( v \)-group \( R^o \), then:

1. \( R \) has only the finite number of two-sided ideals,
2. \( R \) has a simple homomorphic image,
3. The center \( Z(R) \) is finite.

**Proof.** (1) If \( I \) is an ideal of \( R \), then \( I^o \) is a normal subgroup of \( R^o \). Since every normal subgroup is a set-theoretic sum of some conjugacy classes, the assertion holds.

(2) It follows from the part (1).

(3) If \( a \in Z(R) \), then the conjugacy class \( a^G \) contains only one \( a \).

**Lemma 24.** Let \( R \) be a Jacobson radical ring and \( R^2 \neq R \). If the adjoint group \( R^o \) has only two conjugacy classes, then \( R \) is a zero-ring that contains only two elements.

**Proof.** Assume \( G = R^o \) has only exactly two conjugacy classes. Now, \( \{0\} \) is a conjugacy class by itself, and the other class is \( a^G \) for some \( 0 \neq a \in G \). Then \((R^2)^o\)
is a normal subgroup of $G$, because $R^2$ is an ideal of $R$, thus $(R^2)^c$ is the union of conjugacy classes. As $0 \in R^2$, we have that $(R^2)^c$ is either $\{0\}$ or $G$. Since the ring $R$ is Jacobson radical, we obtain that $R^2$ is either $\{0\}$ or $R$. From $R^2 \neq R$ it follows that $R^2 = \{0\}$, i.e. $R$ is a zero-ring. Hence “$\circ$” is simply the addition, and then a number of conjugacy classes is card $R$ from the commutativity of the addition “$+$”.

Lemma 25 ([22], Corollary, p. 332). Let $G$ be a locally solvable group. If $G$ has a finite number of conjugacy classes, then it is finite.

Corollary 5. Any Jacobson radical ring $R$ with the torsion adjoint $v$-group $R^\circ$ is finite.

Proof. Since elements of the same conjugacy class have the same order, $R^\circ$ is of finite exponent. Then, by Lemma 2, the adjoint group $R^\circ$ is locally nilpotent. By Lemma 25, $R$ is finite. □

Proof of Proposition 2. a) By Corollary 5, the torsion part $F(R)$ is finite and so $D^\circ$ is a $v$-group.

b) Since $F(R)$ is an ideal of $R$ and $R^\circ$ is a $v$-group, we deduce that the exponent $\exp F(R)$ is finite. Then $F(R)$ is a ring direct sum of finitely many $p$-components for pairwise distinct primes $p$.

c) Suppose that the additive group $R^+$ is torsion-free. Assume that $qR$ is proper in $R$ for some prime $q$. Since the set

$$\{nR \mid n \text{ is a positive integer}\}$$

is finite by Lemma 23(1), we conclude that

$$q^l R = q^k R$$

for some positive integers $l, k$, where $l > k$. If $a \in R \setminus qR$, then $q^k a = q^l b$ for some $b \in R$ and

$$q^k (a - q^{l-k} b) = 0.$$ 

This yields $a = q^{l-k} b \in qR$, a contradiction. Hence $R^+$ is divisible.

d) Let $R$ be any Jacobson radical ring with the adjoint $v$-group $R^\circ$. Then $R^+ / F(R)$ is a divisible group and, by Lemma 12,

$$R^+ = F(R) \bigoplus D$$

is a group direct sum, where $D$ is a divisible group. If $c, d \in D$, then $cd = f + h$ for some $f \in F(R)$ and $h \in D$. Since $c = nc_1$ and $h = nh_1$ for some $c_1, h_1 \in D$, where $n$ is a positive integer, we conclude that

$$f \in \bigcap_{n=1}^{\infty} nR.$$
Therefore \( f = 0 \) and \( D \) is a ring. Moreover \( F(R)D = 0 = DF(R) \). Then \( R = F(R) \oplus D \) is a ring direct sum and so \( F(R) \) is a ring with a finite number of conjugacy classes.

6. SOME COROLLARIES

If \( B \) is the two-element zero ring, then \( B \) is Jacobson radical and its adjoint group \( B^o \) has only two conjugacy classes. But \( B \) is not simple because \( B^2 = 0 \).

Proof of Corollary 1. (1) Assume that \( R \) is not a domain and does not contain a nonzero nilpotent element. If there exists a nonzero element \( a \in R \) with a nonzero right annihilator \( A = \text{ann}_r a \), then \( A \neq R, (Aa)^2 = 0 \) and, as a consequence, \( Aa = 0 \). This means that \( A \) is contained in the two-sided annihilator \( \text{ann}_a \). Since \( R \) does not contain a nonzero nilpotent element, \( \text{ann}_a \) is a nonzero proper ideal of \( R \), a contradiction.

(2) In view of the part (1), assume that \( R \) contains a nonzero nilpotent element \( x \in R \) such that \( x^2 = 0 \). Then any \( 0 \neq y \in R \) is contained in the class \( xR^o \) and so there exists \( a \in R \) such that
\[
y = a^{(-1)} \circ x \circ a.
\]
Since
\[
y^2 = (x + xa + a^{(-1)}x + a^{(-1)}xa)(x + xa + a^{(-1)}x + a^{(-1)}xa)
\]
\[
= (xa^{(-1)}x + xa^{(-1)}xa + xaa^{(-1)}x + xaa^{(-1)}x) + (a^{(-1)}xa^{(-1)}x + a^{(-1)}xaa^{(-1)}x) + (a^{(-1)}xaxa + a^{(-1)}xaaa^{(-1)}x)
\]
\[
= x(a^{(-1)} \circ a)x + x(a^{(-1)} \circ a)x + a^{(-1)}x(a^{(-1)} \circ a)x + a^{(-1)}x(a^{(-1)} \circ a)x
\]
\[
= 0,
\]
we obtain that \( R \) is a nil-ring of bounded index. If \( F(R) \neq 0 \), then the \( p \)-part
\[
F_p(R) = \{a \in R \mid a \text{ is of finite order } p^n \text{ for some non-negative integer } n\}
\]
is a nonzero ideal of \( R \) for some prime \( p \) and therefore \( R = F_p(R) \). Hence \( R^+ \) is a \( p \)-group. If \( F(R) = 0 \), then \( R^+ \) is torsion-free.

a) If \( R^+ \) is a \( p \)-group, then, by Lemma 1, \( R^o \) is a \( p \)-group and, by Corollary 5, \( R \) is finite. Then \( R \) is a nilpotent ring, a contradiction with the simplicity of \( R \).

b) Let \( R^+ \) be a torsion-free group. By Proposition 2, \( R \) is a Q-algebra and, by Lemma 9, it is nilpotent, a contradiction.

Hence \( R \) is reduced and, by the part (1), \( R \) is a domain. As a consequence, \( R^o \) is a simple group. By Proposition 2, \( R^+ \) is torsion-free divisible.

Corollary 6. If \( R \) is an infinite nil-ring, then \( R^2 \neq R \) or \( R^o \) is not a \( v \)-group.
Proof. By contrary. If $R^\circ$ is a $v$-group and $R^2 = R$, then, in view of Lemma 23(2), there exists an ideal $I$ of $R$ with the simple homomorphic image $R/I$. If $(R/I)^+$ is a torsion group, then $(R/I)^+$ is torsion by Lemma 1 and so $R/I$ is a finite nilpotent ring, which leads to a contradiction. If $(R/I)^+$ is a torsion-free group, then, as in the proof of Corollary 1, $R/I$ is a nil $Q$-algebra of bounded index and it is nilpotent by Lemma 9, a contradiction with $R^2 = R$. □

ACKNOWLEDGEMENT

The author wishes to thank the referee for useful remarks and suggestions.

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