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THE BERNSTEIN CUBATURE FORMULA REVISED

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Abstract. Let $a,b,c,d \in \mathbb{R}$ be given such that $a < b, c < d$. Starting with the Bernstein bivariate approximation formula on $[a,b] \times [c,d]$ a corresponding composite Bernstein type cubature formula is constructed. Its coefficients and an upper bound estimation for the remainder term are established. Numerical examples and comparisons with other known cubature formulas are also provided.

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1. Preliminaries

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. It is well known (see for example [1,2,5,6,8,9]) that the Bernstein bivariate operators $B_{m,n} : C[0,1] \times [0,1] \to C[0,1] \times [0,1]$ are defined for any $f \in C[0,1] \times [0,1]$, any $(x,y) \in [0,1] \times [0,1]$ and any $m,n \in \mathbb{N}$ by:

$$B_{m,n} f(x,y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) f\left(\frac{k}{m}, \frac{j}{n}\right) \tag{1.1}$$

where

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \qquad p_{n,j}(x) = \binom{n}{j} y^j (1-y)^{n-j} \tag{1.2}$$

are the fundamental univariate Bernstein's polynomials.

Note that the bivariate polynomials (1.1) are known as the Bernstein bivariate polynomials of degree (m,n).

For any $f \in C[0,1] \times [0,1]$, $(x,y) \in [0,1] \times [0,1]$, $m,n \in \mathbb{N}$, the equality

$$f(x,y) = B_{m,n} f(x,y) + R_{m,n} f(x,y) \tag{1.3}$$

is known as the Bernstein bivariate approximation formula, $R_{m,n} f$ being its remainder term.

Regarding the remainder term of (1.3) were established the following results.

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Theorem 1 ([5]). *For any $f \in C[0,1]\times[0,1]$ the remainder term of (1.3) can be represented under the form:*

$$
\begin{aligned}
R_{m,n}f(x,y) = & -\frac{x(1-x)}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{n} p_{m-1,k}(x)p_{n,j}(y)\left[\begin{matrix} x, \frac{k}{m}, \frac{k+1}{m} \\ \frac{j}{n} \end{matrix}\; ; f\right] \\
& -\frac{y(1-y)}{n}\sum_{k=0}^{m}\sum_{j=0}^{n-1} p_{m,k}(x)p_{n-1,j}(y)\left[\begin{matrix} \frac{k}{m} \\ y, \frac{j}{k}, \frac{j+1}{n} \end{matrix}\; ; f\right] \\
& +\frac{xy(1-x)(1-y)}{mn}\sum_{k=0}^{m-1}\sum_{j=0}^{n-1} p_{m-1,k}(x)p_{n-1,j}(y)\left[\begin{matrix} x, \frac{k}{m}, \frac{k+1}{m} \\ y, \frac{j}{n}, \frac{j+1}{n} \end{matrix}\; ; f\right],
\end{aligned}
\tag{1.4}
$$

for any $(x,y)\in[0,1]\times[0,1]$, where the brackets denote bivariate divided differences.

Theorem 2 ([5]). *Let $p,q \in \mathbb{N}_0$, $a \le x_0 < x_1 < \cdots < x_p \le b, c \le y_0 < y_1 < \cdots < y_q \le d$ and $f : [a,b]\times[c,d]$ be given. Suppose $f \in C^{(p-1,q-1)}[a,b]\times[c,d]$ and there exists $\dfrac{\partial^{p+q} f}{\partial x^p \partial y^q}$ on $]a,b[\times]c,d[$. Then, there exists $(\xi,\eta)\in]a,b[\times]c,d[$ such that*

$$
\left[\begin{matrix} x_0, x_1, \ldots, x_p \\ y_0, y_1, \ldots, y_q \end{matrix}\; ; f\right] = \frac{1}{p!q!}\frac{\partial^{p+q} f}{\partial x^p \partial y^q}(\xi,\eta). \tag{1.5}
$$

Theorem 3 ([5]). *Suppose $f \in C^{2,2}[0,1]\times[0,1]$ and there exists $\dfrac{\partial^4 f}{\partial x^2 \partial y^2}$, bounded on $]0,1[\times]0,1[$. The following*

$$
|(R_{m,n}f)(x,y)| \le \frac{x(1-x)}{2m}M_1[f] + \frac{y(1-y)}{2n}M_2[f] + \frac{xy(1-x)(1-y)}{4mn}M_3[f] \tag{1.6}
$$

holds, for any $(x,y)\in[0,1]\times[0,1]$ and any $m,n \in \mathbb{N}$, where

$$
\begin{aligned}
M_1(f) &= \sup_{(x,y)\in]0,1[\times]0,1[}\left|\frac{\partial^2 f(x,y)}{\partial x^2}\right|, \\
M_2(f) &= \sup_{(x,y)\in]0,1[\times]0,1[}\left|\frac{\partial^2 f(x,y)}{\partial y^2}\right|, \\
M_3(f) &= \sup_{(x,y)\in]0,1[\times]0,1[}\left|\frac{\partial^4 f(x,y)}{\partial x^2 \partial y^2}\right|.
\end{aligned}
\tag{1.7}
$$

Applying the above results, in [4] was obtained the following Bernstein type cubature formula.

Theorem 4 ([3]). *Let* $f \in C^{2,2}[0,1]\times[0,1]$ *be given such that* $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$ *and* $\dfrac{\partial^4 f}{\partial x^2 \partial y^2}$ *are bounded on* $]0,1[\times]0,1[$*. Then, the following Bernstein type cubature formula*

$$\int_0^1 \int_0^1 f(x,y)dx\,dy = \sum_{i=0}^{m}\sum_{j=0}^{n} A_{i,j} f\left(\frac{i}{m},\frac{j}{n}\right) + R_{m,n}[f] \tag{1.8}$$

holds, where the coefficients are expressed by

$$A_{ij} = \frac{1}{(m+1)(n+1)}, \quad (\forall)\, i = \overline{0,m},\ (\forall)\, j = \overline{0,n} \tag{1.9}$$

and the remainder term satisfies

$$|R_{m,n}[f]| \le \frac{1}{12m} M_1[f] + \frac{1}{12n} M_2[f] + \frac{1}{144mn} M_3[f]. \tag{1.10}$$

Note that the cubature formula (1.8) has the degree of exactness $(1,1)$ i.e. it is exact for the bivariate polynomials $\epsilon_{i,j}(x,y) = x^i y^j, 0 \le i \le j \le 1, i,j \in \mathbb{N}_0, i+j \le 1$.
Theorem 1.4 was generalized as follows.

Theorem 5 ([3]). *Let* $f \in C^{2,2}[0,1]\times[0,1]$ *be given such that* $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$ *and* $\dfrac{\partial^4 f}{\partial x^2 \partial y^2}$ *are bounded on* $]0,1[\times]0,1[$*. The following composite Bernstein cubature formula*

$$\int_0^1 \int_0^1 f(x,y)dx\,dy = \frac{1}{mn(p+1)(q+1)} \sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{h=0}^{p}\sum_{l=0}^{q} f\left(\frac{kp-p+h}{mp}, \frac{jq-q+l}{nq}\right) + R_{m,n}[f]. \tag{1.11}$$

holds, were $R_{m,n}[f]$ *satisfies* (1.10).

Using (1.11) it is possible to approximate $\int_0^1\int_0^1 f(x,y)dx\,dy$ which the desired precision ε, imposing the condition $|R_{m,n}[f]| < \varepsilon$.

The focus of the present paper is to extend the results from [4] in order to obtain a composite Bernstein type cubature formula on any bidimensional interval $[a,b]\times[c,d]$. Finally, numerical examples and comparisons with other known cubature formulas will be provided.

2. Main results

Let $a,b,c,d \in \mathbb{R}$ be given such that $a < b, c < d$ and $\mathbb{N}$ be the set of positive integers.

Lemma 1. *The Bernstein bivariate polynomial associated to $f \in C[a,b]\times[c,d]$ is expressed for any $(x,y) \in [a,b]\times[c,d]$ and any $m,n \in \mathbb{N}$ by*

$$\overline{B}_{m,n} f(x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} \overline{p}_{m,k}(x)\overline{p}_{n,j}(y) f\left(a + k\cdot\frac{b-a}{m}, c + j\cdot\frac{d-c}{n}\right) \tag{2.1}$$

where

$$\overline{p}_{m,k}(x) = \frac{1}{(b-a)^m}\binom{m}{k}(x-a)^k(b-x)^{m-k}, \tag{2.2}$$

$$\overline{p}_{n,j}(y) = \frac{1}{(d-c)^n}\binom{n}{j}(y-c)^j(d-y)^{n-j}$$

are the Bernstein fundamental polynomials.

Proof. The vectorial function $(t,s) \to \left(\frac{x-a}{b-a}, \frac{y-c}{d-c}\right)$ transform the bidimensional interval $[a,b]\times[c,d]$ into $[0,1]\times[0,1]$. Taking the above and definitions (1.1),(1.2) into account one arrives to (2.1), (2.2). □

For any $f \in C[a,b]\times[c,d], m,n \in \mathbb{N}$ the equality

$$f(x,y) = \overline{B}_{m,n} f(x,y) + \overline{R}_{m,n} f(x,y) \tag{2.3}$$

is the Bernstein approximation formula on $[a,b]\times[c,d]$. Applying Theorem 1.3, for the remainder term of (2.3) follows

Lemma 2. *If $f \in C^{2,2}[a,b]\times[c,d]$ such that $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^4 f}{\partial x^2 \partial y^2}$ are bounded on $]a,b[\times]c,d[$, the following*

$$\begin{aligned} |R_{m,n} f(x,y)| \le{} & \frac{(x-a)(b-x)}{2m(b-a)^2} M_1[f] \\ & + \frac{(y-c)(d-y)}{2n(d-c)^2} M_2[f] \\ & + \frac{(x-a)(b-x)(y-c)(d-y)}{4mn(b-a)^2(d-c)^2} M_3[f] \end{aligned} \tag{2.4}$$

holds, for any $(x,y) \in [a,b]\times[c,d]$, where

$$M_1[f] = \sup_{(x,y)\in I\times J} \left| \frac{\partial^2 f}{\partial x^2}(x,y) \right|, \tag{2.5}$$
$$M_2[f] = \sup_{(x,y)\in I\times J} \left| \frac{\partial^2 f}{\partial y^2}(x,y) \right|,$$
$$M_3[f] = \sup_{(x,y)\in I\times J} \left| \frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \right|$$

and $I \times J = [a,b] \times [c,d]$.

Consider now the bidimensional interval $[a,b]\times[c,d]$ divided in $m \cdot n$ equally spaced subintervals

$$I_k \times J_j = \left[a+(k-1)\cdot\frac{b-a}{m}, a+k\cdot\frac{b-a}{m} \right] \times \left[c+(j-1)\cdot\frac{d-c}{n}, c+j\cdot\frac{d-c}{n} \right],$$
$$k=\overline{1,m},\ j=\overline{1,n}.$$

In each such type of interval one considers the distinct knots (x_h, y_l), $h=\overline{0,p}, l=\overline{0,q}$, where

$$x_h = a+(kp-p+h)\cdot\frac{b-a}{mp},\ y_l = c+(jq-q+l)\cdot\frac{d-c}{nq},\ h=\overline{0,p},\ l=\overline{0,q}. \tag{2.6}$$

Applying Lemma 2.1 on $I_k \times J_j$, follows the following bivariate Bernstein type polynomial

$$\overline{B}_{p,k,q,j}; f(x,y) = \frac{m^p \cdot n^q}{(b-a)^p (d-c)^q} \sum_{k=0}^{p} \sum_{l=0}^{q} \binom{p}{k}\binom{q}{l} \tag{2.7}$$
$$\left(x-a-(k-1)\cdot\frac{b-a}{m} \right)^k \left(a+k\cdot\frac{b-a}{m}-x \right)^{p-k} \left(y-c-(j-1)\cdot\frac{d-c}{n} \right)^l \left(a+j\cdot\frac{d-c}{n}-y \right)^{q-l}$$
$$f\left(a+(kp-p+h)\cdot\frac{b-a}{mp}, c+(jq-q+l)\cdot\frac{d-c}{nq} \right).$$

On any bidimensional interval

$$I_k \times J_j = \left[a+(k-1)\cdot\frac{b-a}{m}, a+k\cdot\frac{b-a}{m} \right] \times \left[c+(j-1)\cdot\frac{d-c}{n}, c+j\cdot\frac{d-c}{n} \right]$$

holds the bivariate Bernstein approximation formula

$$f(x,y) = \overline{B}_{p,k,q,j} f(x,y) + \overline{R}_{p,k,q,j} f(x,y). \tag{2.8}$$

Applying Lemma 2.2 we get the following upper bound estimation for the remainder term of (2.8)

$$|\overline{R}_{p,k,q,j} f(x,y)| \tag{2.9}$$

$$\leq \frac{m\left(x-a-(k-1)\cdot\frac{b-a}{m}\right)\left(a+k\cdot\frac{b-a}{m}-x\right)}{2(b-a)^2}M_1''[f]+$$
$$+\frac{n\left(y-c-(j-1)\cdot\frac{d-c}{n}\right)\left(c+j\cdot\frac{d-c}{n}-y\right)}{2(d-c)^2}M_2''[f]+$$
$$+\frac{mn\left(x-a-(k-1)\cdot\frac{b-a}{m}\right)\left(a+k\cdot\frac{b-a}{m}-x\right)\left(y-c-(j-1)\cdot\frac{d-c}{n}\right)\left(c+j\cdot\frac{d-c}{n}-y\right)}{4(b-a)^2(d-c)^2}$$
$$M_3''[f]$$

where

$$M_1''[f]=\sup_{(x,y)\in I_k\times J_j}\left|\frac{\partial^2 f}{\partial x^2}(x,y)\right|, \tag{2.10}$$
$$M_2''[f]=\sup_{(x,y)\in I_k\times J_j}\left|\frac{\partial^2 f}{\partial y^2}(x,y)\right|,$$
$$M_3''[f]=\sup_{(x,y)\in I_k\times J_j}\left|\frac{\partial^4 f}{\partial x^2\partial y^2}(x,y)\right|.$$

Theorem 6. *Let* $I_k\times J_j=\left[a+(k-1)\cdot\frac{b-a}{m},a+k\cdot\frac{b-a}{m}\right]\times\left[c+(j-1)\cdot\frac{d-c}{n},c+j\cdot\frac{d-c}{n}\right]$ *and* x_k,y_l *given at* (2.6). *The coefficients of the Bernstein type cubature formula*

$$\iint\limits_{I_k\times J_j} f(x,y)dxdy=\sum_{h=0}^{p}\sum_{l=0}^{q}\overline{A}_{h,k,l,j}f(x_k,y_l)+\overline{R}_{kj}[f] \tag{2.11}$$

are expressed by

$$\overline{A}_{h,k,l,j}=\frac{(b-a)(d-c)}{mn(p+1)(q+1)}. \tag{2.12}$$

Proof. Integrating (2.8) on $I_k\times J_j$ and taking (2.7) into account, yields

$$\overline{A}_{h,k,l,j}=\frac{m^p\cdot n^q}{(b-a)^p(d-c)^q}\binom{p}{k}\binom{q}{l}\iint\limits_{I_k\times J_j}\left(x-a-(k-1)\cdot\frac{b-a}{m}\right)^k\left(a+k\cdot\frac{b-a}{m}-x\right)^{p-k}$$
$$\cdot\left(y-c-(j-1)\cdot\frac{d-c}{n}\right)^l\left(c+j+\frac{d-c}{n}-y\right)^{q-l}dxdy.$$

Denoting

$$A_p=\frac{m^p}{(b-a)^p}\binom{p}{h}\int\limits_{I_k}\left(x-a-(k-1)\cdot\frac{b-a}{m}\right)^k\left(a+k\cdot\frac{b-a}{m}-x\right)^{p-k}dx$$

respectively

$$A_q = \frac{n^q}{(d-c)^q}\binom{q}{l}\int\limits_{J_j}\left(y-c-(j-1)\cdot\frac{d-c}{n}\right)^l\left(c+j+\frac{d-c}{n}-y\right)^{q-l}dy$$

we have

$$\overline{A}_{h,k,l,j} = A_p\cdot A_q.$$

To computing A_p one makes the change of variable $x = a+(k-1)\frac{b-a}{m}+t\frac{b-a}{m}$, which leads to

$$\begin{aligned}A_p &= \frac{m^p}{(b-a)^p}\binom{p}{h}\frac{(b-a)^p}{m^p}\cdot\frac{b-a}{m}\int\limits_0^1 t^h(1-t)^{p-h}dt\\ &= \binom{p}{h}\frac{b-a}{m}B(h+1,p-h+1)\end{aligned}$$

were $B(h+1,p-h+1)$ is the Euler function of first kind. Taking its well known properties into account, one arrives to

$$A_p = \binom{p}{h}\frac{b-a}{m}\cdot\frac{h!(p-h)!}{(p+1)!} = \frac{b-a}{m(p+1)}.$$

In a similar way one obtains $A_q = \frac{d-c}{n(q+1)}$ and then, taking the equality $\overline{A}_{h,k,l,j} = A_p\cdot A_q$ into account follows (2.12). □

Theorem 7. *Let $f\in C^{2,2}[a,b]\times[c,d]$ such that exist $\dfrac{\partial^4 f}{\partial x^2\partial y^2}$ on $I_k\times J_j$ and $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}, \dfrac{\partial^4 f}{\partial x^2\partial y^2}$ are bounded on $I_k\times J_j$, $k=\overline{1,m}$, $j=\overline{1,n}$. Then, the following upper bound estimation for the remainder term of* (2.11)

$$|\overline{R}_{k.j}[f]| \le \frac{(b-a)(d-c)}{12m^2n}M_1''[f]+\frac{(b-a)(d-c)}{12mn^2}M_2''[f]+ \\ +\frac{(b-a)(d-c)}{144m^2n^2}M_3''[f] \tag{2.13}$$

holds, where $M_1''[f], M_2''[f], M_3''[f]$ are defined at (2.10).

Proof. The inequality (2.13) follows integrating the approximation formula (2.8) and taking the inequalities (2.9) into account. □

The main result of the paper is the following

Theorem 8. *Let* $f \in C^{2,2}[a,b]\times[c,d[$ *be given such that exist* $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ *on* $]a,b[\times]c,d[$ *and* $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^4 f}{\partial x^2 \partial y^2}$ *are bounded on* $]a,b[\times]c,d[$. *Then, the following composite Bernstein type cubature formula*

$$\int_a^b \int_c^d f(x,y)dxdy = \frac{(b-a)(d-c)}{mn(p+1)(q+1)} \sum_{k=i}^{m} \sum_{j=i}^{n} \sum_{h=0}^{p} \sum_{l=0}^{q} f(x_h, y_l) + R_{m,n}[f] \tag{2.14}$$

holds, where x_h, y_l *are defined at* (2.6) *while the remainder term verifies the inequality* (1.10).

Proof. Adding the Bernstein type cubature formulas (2.11) for $k = \overline{1,m}$, $j = \overline{1,n}$ one arrives to (2.14). □

Remark 1.

(i) For $a = 0, b = 1$ one refinds the results from [7].
(ii) It is immediate that $\lim_{m,n\to\infty} |R_{m,n}[f]| = 0$, which proves that

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \frac{(b-a)(d-c)}{mn(p+1)(q+1)}$$

$$\sum_{k=i}^{m} \sum_{j=i}^{n} \sum_{h=0}^{p} \sum_{l=0}^{q} f\left(a + (kp-p+h)\cdot\frac{b-a}{mp}, c + (jq-q+l)\cdot\frac{d-c}{nq}\right) =$$

$$= \int_a^b \int_c^d f(x,y)dxdy.$$

3. Numerical examples

In [9] the authors introduce cubature formulas for two-variables function with boundary-layer components to evaluate the integral

$$I_f = \int_a^b \int_c^d f(x,y)dxdy \tag{3.1}$$

For a uniform grid on the domain $[a,b]\times[c,d]$ with nodes $(x_i, y_j), i = 0,1,\ldots,m$, $j = 0,1,\ldots,n$, with steps h_1 and h_2 in x and y, respectively, let $u_{i,j} = f(x_i, y_j)$.

Let us denote:

Trapezoidal Rule (TR)

$$TR_f = \frac{h_1 h_2}{4} \sum_{i,j} (u_{i+1,j+1} + u_{i,j+1} + u_{i,j} + u_{i+1,j}),\ \ 0 \le i < m,\ \ 0 \le j < n$$

Simpson's Rule (SR)

$$SR_f = \frac{h_1h_2}{9}\sum_{i,j}(16u_{i,j} + 4u_{i+1,j} + 4u_{i-1,j} + 4u_{i,j+1} + 4u_{i,j-1} + 4u_{i+1,j+1}$$
$$+u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1}),\ \ i = 1,3,\cdots,m-1,\ \ j = 1,3,\cdots,n-1$$

Bernstein Rule (BR)

$$BR_f = \frac{(b-a)(d-c)}{mn(p+1)(q+1)}$$
$$\sum_{k=i}^{m}\sum_{j=i}^{n}\sum_{h=0}^{p}\sum_{l=0}^{q} f\left(a+(kp-p+h)\cdot\frac{b-a}{mp}, c+(jq-q+l)\cdot\frac{d-c}{nq}\right).$$

To demonstrate the accuracy of our new numerical cubature formula, we compare the *Bernstein Rule (BR)* with *Trapezoidal Rule (TR)* and *Simpson's Rule (SR)* by using the following test functions:

$$f_1 : [0,2]\times[0,2] \to \mathbb{R},\ f_1(x,y) = e^{-(x^2+y^2)}$$

$$f_2 : [0,5]\times[0,3] \to \mathbb{R},\ f_2(x,y) = e^{-2(x+y)}\sin(4x+4y)$$

$$f_\varepsilon[0,1]\times[0,1] \to \mathbb{R},\ f_\varepsilon(x,y) = (1-e^{\frac{-x}{\varepsilon}})(1-e^{\frac{-2y}{\varepsilon}})(1-x)(1-y)+$$
$$+cos(\frac{\pi x}{2})e^{-y},\ \ \varepsilon \in (,1].$$

The last function f_ε is studied in [9] for the values of $\varepsilon = 1,\ \varepsilon = 10^{-1},\ \varepsilon = 10^{-2},\ \varepsilon = 10^{-3},\ \varepsilon = 10^{-4}$ and $\varepsilon = 10^{-5}$, by using Trapezoidal Rule (TR), Simpson's Rule (SR) and the new methods introduced by the author.

Let us denote the values of the integrals by

$$I_{f_1} = \int_0^2\int_0^2 f_1(x,y)dxdy,\quad I_{f_2} = \int_0^5\int_0^3 f_2(x,y)dxdy,$$
$$I_{f_\varepsilon} = \int_0^1\int_0^1 f_\varepsilon(x,y)dxdy$$

and the errors by

$$eTR_f = |I_f - TR_f|,\ \ eSR_f = |I_f - SR_f|,\ \ eBR_f = |I_f - BR_f|.$$

The numerical results obtained for $m = 64$, $n = 64$, $p = 5$ and $q = 5$ are presented in Table 1 and Table 2. In these tables, $e-m$ means 10^{-m}.

The Mathcad 14.0 package was used to generate these numerical results.

Table 1. Errors for f_1 and f_2

	eTR_f	eSR_f	eBR_f
f_1	1.05e-5	6.84e-9	2.10e-6
f_2	1.40e-5	1.89e-6	1.47e-6

Table 2. Errors for f_ε

ε	eTR_{f_ε}	eSR_{f_ε}	eBR_{f_ε}
1	2.68e-5	6.65e-11	5.37e-6
10^{-1}	2.87e-4	1.42e-6	5.76e-5
10^{-2}	2.71e-3	6.95e-4	5.54e-4
10^{-3}	7.01e-3	4.43e-3	1.91e-3
10^{-4}	7.76e-3	5.11e-3	2.59e-3
10^{-5}	7.76e-3	5.18e-3	2.59e-3

4. CONCLUSIONS

In Table 1 and Table 2 one can see that Bernstein cubature formula revised is better than trapezoidal cubature formula for all functions studied. Even if the Simpson's cubature formula is of order (2,2) and Bernstein cubature formula is of order (1,1), the results obtained by using our revised formula are better for the cases of functions f_2 and f_ε, with $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$.

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