A GENERALIZATION OF J-QUASIPOLAR RINGS

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Abstract. In this paper we introduce a class of quasipolar rings which is a generalization of J-quasipolar rings. Let \( R \) be a ring with identity. An element \( a \in R \) is called \( \delta \)-quasipolar if there exists \( p^2 = p \in \text{comm}^2(a) \) such that \( a + p \) is contained in \( \delta(R) \), and the ring \( R \) is called \( \delta \)-quasipolar if every element of \( R \) is \( \delta \)-quasipolar. We use \( \delta \)-quasipolar rings to extend some results of \( J \)-quasipolar rings. Then some of the main results of \( J \)-quasipolar rings are special cases of our results for this general setting. We give many characterizations and investigate general properties of \( \delta \)-quasipolar rings.

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1. INTRODUCTION

Throughout this paper all rings are associative with identity unless otherwise stated. Let \( R \) be a ring. According to Koliha and Patricio [10], the commutant and double commutant of an element \( a \in R \) are defined by \( \text{comm}(a) = \{ x \in R \mid xa = ax \} \), \( \text{comm}^2(a) = \{ x \in R \mid xy = yx \text{ for all } y \in \text{comm}(a) \} \), respectively. If \( R^{qnil} = \{ a \in R \mid 1 + ax \in U(R) \text{ for every } x \in \text{comm}(a) \} \) and \( a \in R^{qnil} \), then \( a \) is said to be quasinilpotent (see [9]). The element \( a \) is called quasipolar if there exists \( p^2 = p \in R \) such that \( p \in \text{comm}^2(a) \), \( a + p \) is invertible in \( R \) and \( ap \in R^{qnil} \). Any idempotent \( p \) satisfying the above conditions is called a spectral idempotent of \( a \), and this term is borrowed from spectral theory in Banach algebra and it is unique for \( a \).

Quasipolar rings have been studied by many ring theorists (see [1, 2, 5–7, 9, 10] and [15]). In [7], the element \( a \in R \) is called nil-quasipolar if there exists \( p^2 = p \in \text{comm}^2(a) \) such that \( a + p \) is nilpotent, the idempotent \( p \) is called a nil-spectral idempotent of \( a \). The ring \( R \) is said to be nil-quasipolar if every element of \( R \) is nil-quasipolar. Recently, \( J \)-quasipolar rings are studied in [4]. The element \( a \) is called \( J \)-quasipolar if there exists \( p^2 = p \in R \) such that \( p \in \text{comm}^2(a) \) and \( a + p \in J(R) \), \( p \) is called a \( J \)-spectral idempotent of \( a \). The ring \( R \) is said to be \( J \)-quasipolar if

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every element of \( R \) is \( J \)-quasipolar. Motivated by these, we introduce a new class of quasipolar rings which is a generalization of \( J \)-quasipolar rings. By using \( \delta \)-quasipolar rings, we extend some results of \( J \)-quasipolar rings.

An outline of the paper is as follows: Section 2 deals with \( \delta \)-quasipolar rings. We prove various basic characterizations and properties of \( \delta \)-quasipolar rings. It is proven that every \( J \)-quasipolar ring is \( \delta \)-quasipolar. We supply an example to show that all \( \delta \)-quasipolar rings need not be \( J \)-quasipolar. Among others the \( \delta \)-quasipolarity of Dorroh extensions and some classes of matrix rings are investigated. In Section 3, we introduce an upper class of \( \delta \)-quasipolar rings, namely, weakly \( \delta \)-quasipolar rings. We show that every direct summand of a weakly \( \delta \)-quasipolar ring is weakly \( \delta \)-quasipolar and every direct product of weakly \( \delta \)-quasipolar rings is weakly \( \delta \)-quasipolar, and we give some properties of such rings.

In what follows, \( \mathbb{Z} \) and \( \mathbb{Q} \) denote the ring of integers and the ring of rational numbers and for a positive integer \( n \), \( \mathbb{Z}_n \) is the ring of integers modulo \( n \). For a positive integer \( n \), let \( \text{Mat}_n(R) \) denote the ring of all \( n \times n \) matrices and \( T_n(R) \) the ring of all \( n \times n \) upper triangular matrices with entries in \( R \). We write \( J(R) \) and \( \text{nil}(R) \) for the Jacobson radical of \( R \) and the set of nilpotent elements of \( R \), respectively.

2. \( \delta \)-QUASIPOLAR RINGS

In this section we introduce the concept of \( \delta \)-quasipolar rings and investigate some properties of such rings. We show that every quasipolar ring need not be \( \delta \)-quasipolar (Example 2). It is proven that every \( J \)-quasipolar ring is \( \delta \)-quasipolar and the converse does not hold in general (see Example 3). Among others we extend some results of \( J \)-quasipolar rings for this general setting.

A right ideal \( I \) of the ring \( R \) is said to be \( \delta \)-small in \( R \) if whenever \( R = I + K \) with \( R/K \) singular right \( R \)-module for any right ideal \( K \) then \( R = K \). In [16], the ideal \( \delta(R) \) is introduced as a sum of \( \delta \)-small right ideals of \( R \). We begin with the equivalent conditions for \( \delta(R) \) which is proved in [16, Theorem 1.6] for an easy reference for the reader.

**Lemma 1.** Given a ring \( R \), each of the following sets is equal to \( \delta(R) \).

1. \( R_1 = \text{the intersection of all essential maximal right ideals of } R \).
2. \( R_2 = \text{the unique largest } \delta \)-small right ideal of \( R \).
3. \( R_3 = \{ x \in R \mid xR + K_R = R \text{ implies } K_R \text{ is a direct summand of } R_R \} \).
4. \( R_4 = \bigcap \{ \text{ideals } P \text{ of } R \mid R/P \text{ has a faithful singular simple module} \} \).
5. \( R_5 = \{ x \in R \mid \text{for all } y \in R \text{ there exists a semisimple right ideal } Y \text{ of } R \text{ such that } (1 + xy)R \oplus Y = R_R \} \).

Now we give our main definition.
**Definition 1.** Let $R$ be a ring. An element $a \in R$ is called $\delta$-quasipolar if there exists $p^2 = p \in \text{comm}^2(a)$ such that $a + p \in \delta(R)$ and $p$ is called a $\delta$-spectral idempotent. The ring $R$ is called $\delta$-quasipolar if every element of $R$ is $\delta$-quasipolar.

The following are examples for $\delta$-quasipolar rings.

**Example 1.** (1) Every semisimple ring and every Boolean ring is $\delta$-quasipolar.
(2) Since $\delta(\mathbb{Q}) = \mathbb{Q}$, $\mathbb{Q}$ is $\delta$-quasipolar. On the other hand, $\mathbb{Z}$ is not $\delta$-quasipolar since $\delta(\mathbb{Z}) = 0$.

One may suspects that every quasipolar ring is $\delta$-quasipolar. But the following example erases the possibility.

**Example 2.** Let $p$ be a prime integer with $p \geq 3$ and $R = \mathbb{Z}(p)$ the localization of $\mathbb{Z}$ at the ideal $(p)$. By [4, Example 2.8], $R$ is a quasipolar ring. Since $J(R) = \delta(R)$, it is not $\delta$-quasipolar.

Let $S_r$ denote the right socle of the ring $R$, that is, $S_r$ is the sum of minimal right ideals of $R$. We now prove that the class of $J$-quasipolar rings is a subclass of $\delta$-quasipolar rings.

**Lemma 2.** If $R$ is a $J$-quasipolar ring, then $R$ is $\delta$-quasipolar. The converse holds if $S_r \subseteq J(R)$.

**Proof.** The first assertion is clear since $J(R) \subseteq \delta(R)$. Assume that $R$ is $\delta$-quasipolar. If $S_r \subseteq J(R)$, then $J(R)/S_r = J(R)/S_r = \delta(R)/S_r$ by [16, Corollary 1.7] and we have $J(R) = \delta(R)$. Hence, $R$ is $J$-quasipolar.

The converse of Lemma 2 is not true in general as the following example shows.

**Example 3.** Let $F$ be a field and consider the ring $R = \begin{bmatrix} F & F \\ F & F \end{bmatrix}$. Then $R$ is a semisimple ring and $R = \delta(R)$ and $J(R) = 0$. Hence $R$ is $\delta$-quasipolar and it is not $J$-quasipolar.

**Lemma 3.** Let $R$ be a ring. Then we have the following.

1. If $a, u \in R$ and $u$ is invertible, then $a$ is $\delta$-quasipolar if and only if $u^{-1} au$ is $\delta$-quasipolar.
2. The element $a \in R$ is $\delta$-quasipolar if and only $-1 - a$ is $\delta$-quasipolar.
3. If $R$ is a $\delta$-quasipolar ring with $\delta(R) = J(R)$, then the spectral idempotent for any invertible element in $R$ is the identity of $R$.

**Proof.** (1) Assume that $a$ is $\delta$-quasipolar. Let $p^2 = p \in \text{comm}^2(a)$ such that $a + p \in \delta(R)$. Let $x \in \text{comm}(u^{-1} au)$. Then $(uxu^{-1})a = a(uxu^{-1})$. Since $p \in \text{comm}^2(a)$, $(uxu^{-1})p = p(uxu^{-1})$. Hence $(u^{-1} pu)^2 = u^{-1} pu \in \text{comm}^2(u^{-1} au)$. Since $\delta(R)$ is an ideal of $R$, $u^{-1}(a + p)u = u^{-1} au + u^{-1} pu \in \delta(R)$. Thus $u^{-1} au$ is $\delta$-quasipolar. Conversely, if $u^{-1} au$ is $\delta$-quasipolar, then by the preceding proof
\( u(u^{-1}au)u^{-1} = a \) is \( \delta \)-quasipolar.

(2) Assume that \( a \) is \( \delta \)-quasipolar. Let \( p^2 = p \in \text{comm}^2(a) \) such that \( a + p = r \in \delta (R) \). Then \( -1 - a + (1 - p) = -r \in \delta (R) \). Then \( 1 - p \in \text{comm}^2(-1 - a) \) and \( 1 - p \) is the spectral idempotent of \(-1 - a\). Conversely, if \(-1 - a\) is \( \delta \)-quasipolar, then from what we have proved that \(-1 - (-1 - a) = a\) is quasipolar.

(3) Assume that \( \delta (R) = J(R) \). Then \( \delta \)-quasipolarity of \( R \) implies \( J \)-quasipolarity of \( R \). So its proof can be directly obtained from [4, Example 2.2].

In [4, Corollary 2.3], it is proved that if \( R \) is a \( J \)-quasipolar ring, then \( 2 \in J(R) \). In this direction we prove the following.

**Lemma 4.** If \( R \) is a \( \delta \)-quasipolar ring, then \( 2 \in \delta (R) \).

**Proof.** For the identity \( 1 \), there exists \( p^2 = p \in R \) such that \( 1 + p \in \delta (R) \). Multiplying the latter by \( p \), we have \( 2p \in \delta (R) \). So \( 2 = 2(1 + p) - 2p \in \delta (R) \).

Lemma 4 can be used to determine whether given rings are \( \delta \)-quasipolar.

**Example 4.** (1) The ring \( \mathbb{Z}_3 \) is a semisimple ring and \( \delta \)-quasipolar but the ring \( R = \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix} \) is not \( \delta \)-quasipolar since \( \delta (R) = \begin{bmatrix} 0 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix} \) and 2 does not contained in \( \delta (R) \).

(2) Let \( R = \{(a_{ij}) \in \mathbb{T}_n(\mathbb{Z}_3) \mid a_{11} = a_{22} = \cdots = a_{nn}\} \). \( \mathbb{Z}_3 \) is \( \delta \)-quasipolar but \( R \) is not since \( \delta (R) = \{(a_{ij}) \in \mathbb{T}_n(\mathbb{Z}_3) \mid a_{11} = a_{22} = \cdots = a_{nn} = 0\} \) and 2 does not contained in \( \delta (R) \).

Recall that a ring \( R \) is called *local* if it has only one maximal left ideal, equivalently, maximal right ideal.

**Proposition 1.** Let \( R \) be a local ring. If \( R/J(R) \cong \mathbb{Z}_2 \), then \( R \) is \( \delta \)-quasipolar.

**Proof.** Let \( a \in R \). If \( a \in J(R) \), it is clear. Assume that \( a \notin J(R) \). Since \( R \) is local, \( a \) is invertible. Hence \( a + 1 \in \delta (R) \) by \( \delta (R) = J(R) \).

A ring \( R \) is said to be *clean* [12] if for each \( a \in R \) there exists \( e^2 = e \in R \) such that \( a - e \) is invertible, and \( R \) is called *strongly clean* [13] provided that every element of \( R \) can be written as the sum of an idempotent and an invertible element that commute.

**Example 5.** Let \( R = \{(q_1, q_2, q_3, \ldots, q_n, a, a, a, \ldots) \mid n \geq 1; q_i \in \mathbb{Q}; a \in \mathbb{Z}(2)\} \). Then \( R \) is strongly clean but not quasipolar (see [15, Example 3.4(3)]). Therefore \( R \) is not \( J \)-quasipolar since every \( J \)-quasipolar ring is quasipolar. On the other hand, since \( S_r = 0 \) and \( \delta (R)/S_r = J(R)/S_r \), \( \delta (R) = J(R) \). Thus \( R \) is not \( \delta \)-quasipolar.

In [4, Theorem 2.9], it is shown that if the ring \( R \) is \( J \)-quasipolar, then \( R/J(R) \) is Boolean and idempotents in \( R/J(R) \) lift \( R \). We have the following result for \( \delta \)-quasipolar rings.
Theorem 1. If $R$ is a $\delta$-quasipolar ring, then $R/\delta(R)$ is a Boolean ring and idempotents in $R/\delta(R)$ lift to $R$.

Proof. Let $\overline{a} \in R/\delta(R)$. There exists $p^2 = p \in \text{comm}^2(-1 + a)$ such that $-1 + a + p \in \delta(R)$. Hence $\overline{a} = \overline{1 - p}$ is an idempotent in $R/\delta(R)$ and $R/\delta(R)$ is a Boolean ring. Let $\overline{p}^2 = \overline{a} \in R/\delta(R)$. Then there exists $p^2 = p \in \text{comm}^2(-a)$ such that $-a + p \in \delta(R)$. This yields $\overline{a} = \overline{p}$, as asserted. □

The concept of $\delta_r$-clean rings are defined in [8]. A ring $R$ is called $\delta_r$-clean if for every element $a \in R$ there exists an idempotent $e \in R$ such that $a - e \in \delta(R)$. A ring is abelian if all idempotents are central.

Lemma 5. If $R$ is a $\delta$-quasipolar ring, then it is $\delta_r$-clean. The converse holds if $R$ is abelian.

Proof. Let $R$ be a $\delta$-quasipolar ring and $a \in R$. There exists $p^2 = p \in \text{comm}^2(-1 + a)$ such that $-1 + a + p \in \delta(R)$. Then $a - (1 - p) \in \delta(R)$. For the converse, assume that $R$ is abelian. Let $a \in R$. There exists an idempotent $e$ such that $1 + a - e \in \delta(R)$. By assumption, $1 - e$ is a central idempotent and so $1 - e \in \text{comm}^2(a)$. □

Recall that a ring $R$ is exchange if for every $a \in R$, there exists an idempotent $e \in R$ such that $1 - e \in (1 - a)R$. Namely, von Neumann regular rings and clean rings are exchange.

Corollary 1. Let $R$ be a $\delta$-quasipolar ring. Then

1. $R$ is an exchange ring.
2. $R/\delta(R)$ is a clean ring.

Proof. (1) Let $R$ be a $\delta$-quasipolar ring. By Lemma 5, $R$ is a $\delta_r$-clean ring. By [8, Theorem 2.2(2)], every $\delta_r$-clean ring is an exchange ring.
(2) By Theorem 1, $R/\delta(R)$ is Boolean, therefore, it is clean. □

Corollary 2. Consider following conditions for a ring $R$.

1. $R$ is $\delta$-quasipolar and $\delta(R) = 0$.
2. $R$ is Boolean.
3. $R$ is von Neumann regular and $\delta$-quasipolar.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) Assume that $R$ is $\delta$-quasipolar and $\delta(R) = 0$. By Theorem 1, $R$ is Boolean.
(2) $\Rightarrow$ (3) Assume that $R$ is Boolean. Then it is commutative with characteristic 2 and $a^2 + a = 0 \in \delta(R)$ and $a^2 = a = a^3$ for all $a \in R$. Hence $R$ is von Neumann regular and $\delta$-quasipolar. □

Strongly $J$-clean rings were introduced by Chen in [3]. For a ring $R$ the element $a \in R$ is called $J$-clean if $a$ is the sum of an idempotent and a radical element in...
its Jacobson radical. The ring $R$ is called $J$-clean if every element is a sum of an idempotent and a radical element.

**Theorem 2.** If $R$ is an abelian $J$-clean ring, then it is $\delta$-quasipolar.

**Proof.** Let $a \in R$. Then we have $-a \in R$. Since $R$ is $J$-clean, there exist $e^2 = e \in R$ and $j \in J(R)$ such that $-a = e + j$. Hence $a + e \in J(R)$. Since $R$ is abelian, $e^2 = e \in comm^2(a)$ and $J(R) \subseteq \delta(R)$, $R$ is $\delta$-quasipolar as asserted. \[\square\]

All $\delta$-quasipolar rings need not be Boolean and the converse statement of Theorem 2 is not true in general.

**Example 6.** The ring $\mathbb{Z}_3$ is semisimple and so $\mathbb{Z}_3 = \delta(\mathbb{Z}_3)$. Therefore $\mathbb{Z}_3$ is $\delta$-quasipolar, but it is neither Boolean nor $J$-clean.

In [4, Proposition 2.11], it is shown that a ring $R$ is local and $J$-quasipolar if and only if $R$ is $J$-quasipolar with only trivial idempotents if and only if $R/J(R) \cong \mathbb{Z}_2$. We have the following for $\delta$-quasipolar rings.

**Proposition 2.** Let $R$ be a ring with only trivial idempotents. Then $R$ is $\delta$-quasipolar if and only if $R/\delta(R) \cong \mathbb{Z}_2$.

**Proof.** Assume that $R$ is $\delta$-quasipolar. Let $a \in R$. There exists an idempotent $p \in comm^2(a)$ such that $-a + p \in \delta(R)$. By hypothesis $p = 1$ or $p = 0$. If $\delta(R) = 0$, then $R/\delta(R) \cong \mathbb{Z}_2$. Suppose that $\delta(R) \neq 0$. For any $a \in R \setminus \delta(R)$, $\bar{a} = \bar{1} \in R/\delta(R)$. Hence $R/\delta(R) \cong \mathbb{Z}_2$. Conversely, suppose that $R/\delta(R)$ is isomorphic to $\mathbb{Z}_2$ by isomorphism $f$. Let $a \in R \setminus \delta(R)$. Then $f(\bar{-a}) = \bar{1} \in \mathbb{Z}_2$. Then $f(\bar{-a}) = \bar{f}(\bar{1})$ implies $-\bar{a} = \bar{1} \in \text{Ker}f = 0$. Hence $-\bar{a} = \bar{1}$. That is, $a + 1 \in \delta(R)$. Thus $R$ is $\delta$-quasipolar. \[\square\]

Recall that a ring $R$ is called strongly $\pi$-regular if for every element $a$ of $R$ there exist a positive integer $n$ (depending on $a$) and an element $x$ of $R$ such that $a^n = a^{n+1}x$, equivalently, an element $y$ of $R$ such that $a^n = ya^{n+1}$. In spite of the fact that $J(R)$ is contained in both $\delta(R)$ and $R^{qnil}$, no comparisons between $\delta(R)$ and $R^{qnil}$ exist. Strongly $\pi$-regular rings play crucial role in this direction.

**Proposition 3.** Let $R$ be a $\delta$-quasipolar ring and $\delta(R) = J(R)$. Then $R$ is strongly $\pi$-regular if and only if $J(R) = R^{qnil} = \text{nil}(R) = \delta(R)$.

**Proof.** Necessity. Let $a \in R^{qnil}$. Then for any $x \in comm(a)$, $1 - ax$ is invertible. By hypothesis, there exist a positive integer $m$ and $b \in R$ such that $a^m = a^{m+1}b$. Since $b \in comm(a)$ by [11, Page 347, Exercise 23.6(1)], $a^m = 0$. Hence $a \in \text{nil}(R)$ and so $R^{qnil} \subseteq \text{nil}(R)$. To prove $\text{nil}(R) \subseteq \delta(R)$, let $a \in \text{nil}(R)$. By hypothesis there exists $p^2 = p \in comm^2(1 - a)$ such that $1 - a + p \in \delta(R)$. Since $1 - a$ is invertible, $p = 1$ by Lemma 3 (3). Hence $2 - a \in \delta(R)$. Also $2 \in \delta(R)$ by Lemma 4, we then have $a \in \delta(R)$.\[\square\]
Sufficiency. Let \( a \in R \). There exists \( p^2 = p \in \text{comm}^2(-1 + a) \) such that \(-1 + a + p \in \delta(R)\). Set \( u = -1 + a + p \in \text{nil}(R) \). Then \( a + p \) is nilpotent so that \( a^n p = 0 \) for some positive integer \( n \). So \( a^n = a^n(1-p) = (u + (1-p))^n(1-p) = (u + 1)^n(1-p) = (a + p)^n(1-p) = (1-p)(a + p)^n \). By [13, Proposition 1], \( a \) is strongly \( \pi \)-regular. This completes the proof. \( \square \)

Let \( R \) and \( V \) be rings and \( V \) be an \((R, R)\)-bimodule that is also a ring with \((vw)r = v(wr), (vr)w = v(rw), \) and \((rw)v = r(vw)\) for all \( v, w \in V \) and \( r \in R \). The Dorroh extension \( D(R, V) \) of \( R \) by \( V \) defined as the ring consisting of the additive abelian group \( R \oplus V \) with multiplication \((r, v)(s, w) = (rs, rw + vs + vw)\) where \( r, s \in R \) and \( v, w \in V \).

Uniquely clean rings were introduced by Nicholson and Zhou in [14]. A ring \( R \) is uniquely clean in case for any \( a \in R \) there exists a unique idempotent \( e \in R \) such that \( a - e \in R \) is invertible. In [8], among others, uniquely \( \delta_r \)-clean rings are studied. A ring \( R \) is called uniquely \( \delta_r \)-clean if for every element \( a \in R \) there exists a unique idempotent \( e \in R \) such that \( a - e \in \delta(R) \). Uniquely clean Dorroh extensions in [14, Proposition 7] and uniquely \( \delta_r \)-clean Dorroh extensions in [8, Proposition 3.11] are considered. Now we consider \( \delta \)-quasipolar Dorroh extensions.

**Proposition 4.** Let \( R \) be a ring. Then we have the following.

1. If \( D(R, V) \) is \( \delta \)-quasipolar, then \( R \) is \( \delta \)-quasipolar.
2. If the following conditions are satisfied, then \( D(R, V) \) is \( \delta \)-quasipolar.
   (i) \( R \) is \( \delta \)-quasipolar;
   (ii) \( e^2 = e \in R \), then \( ev = ve \) for all \( v \in V \);
   (iii) \( V = \delta(V) \).

**Proof.** (1) Let \( r \in R \). There exists \( e^2 = e \in D(R, V) \) such that \( e \in \text{comm}^2(r, 0) \) and \((r, 0) + e \in \delta(D(R, V)) \). Since \( e \in D(R, V) \), \( e \) has the form such that \((p, v)^2 = (p, v) \) and \( p^2 = p \). Then \( e = (p, v) \in \text{comm}^2(r, 0) \) implies that \( p \in \text{comm}^2(r) \) and \( r + p \in \delta(R) \) since \((r + p, v) \in \delta(D(R, V)) \) and by [8, Proposition 3.11]. Hence \( R \) is \( \delta \)-quasipolar.

(2) Assume that (i), (ii) and (iii) hold. Let \((r, v) \in D(R, V) \). There exists \( p^2 = p \in \text{comm}^2(r) \) such that \( r + p \in \delta(R) \). By (iii), \((0, V) \subseteq \delta(D(R, V)) \). Then \((r, v) + (p, 0) = (r + p, v) \in \delta(D(R, V)) \). To see that \((p, 0) \in \text{comm}^2((r, v)) \), let \((a, b) \in D(R, V) \) and \((a, b)(r, v) = (r, v)(a, b) \). Then \( ar = ra \) and so \( ap = pa \) since \( p \in \text{comm}^2(r) \). Also \( pb = bp \) by (ii). Therefore we have \((p, 0)(a, b) = (a, b)(p, 0) \) that is \((p, 0) \in \text{comm}^2((r, v)) \). \( \square \)

As an application of Dorroh extensions we consider the following example. This example also shows that in Proposition 4 (2), the conditions (i), (ii) and (iii) are not superfluous.

**Example 7.** Consider the ring \( D(\mathbb{Z}, \mathbb{Q}) \). Then \( D(\mathbb{Z}, \mathbb{Q}) \cong \mathbb{Z} \times \mathbb{Q} \). Then \( \delta(\mathbb{Z} \times \mathbb{Q}) = (0) \times \mathbb{Q} \). Since \( \mathbb{Z} \) is not \( \delta \)-quasipolar, \( D(\mathbb{Z}, \mathbb{Q}) \) is not \( \delta \)-quasipolar.
Let $R$ and $S$ be any ring and $M$ an $(R, S)$-bimodule. Consider the ring of the formal upper triangular matrix ring $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$. It is well known that $\delta(T) \subseteq \begin{bmatrix} \delta(R) & M \\ 0 & \delta(S) \end{bmatrix}$. However, if $M = R = S = F$ is a field, then $\delta(T) = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$.

The following example illustrates the $\delta$-quasipolarity of full matrix rings and upper triangular matrix rings depend on the coefficient ring.

**Example 8.**

1. Consider the ring $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. Then $J(R) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ and $\delta(R) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. $R$ is $\delta$-quasipolar.

2. As noted in Example 4, the ring $\mathbb{Z}_3$ is semisimple and therefore $\delta$-quasipolar. However, the ring $\begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix}$ is not $\delta$-quasipolar.

3. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \text{Mat}_2(\mathbb{Z})$. For any $P^2 = P \in \text{comm}^2(A)$, the matrix $P$ has the form $P = \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$ with $x^2 = x$ and $2xy = y$ where $x, y \in \mathbb{Z}$. This would imply that $P$ is the zero matrix or the identity matrix. Since $\delta(\mathbb{Z}) = 0$, $\delta(\text{Mat}_2(\mathbb{Z})) = 0$. In consequence, $A + P$ cannot be in $\delta(\text{Mat}_2(\mathbb{Z}))$. Therefore $\text{Mat}_2(\mathbb{Z})$ is not $\delta$-quasipolar.

4. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in T_2(\mathbb{Z})$. The idempotents of $T_2(\mathbb{Z})$ are zero, identity, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & y \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix}$ where $y$ is an arbitrary integer. Since $A$ commutes with only zero, identity, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, among these idempotents there is no idempotent $P$ such that $A + P \in \delta(T_2(\mathbb{Z}))$ since $\delta(T_2(\mathbb{Z})) = \begin{bmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$. Hence $T_2(\mathbb{Z})$ is not $\delta$-quasipolar.

3. Weakly $\delta$-Quasipolar Rings

In this section, we introduce an upper class of $\delta$-quasipolar rings, namely, weakly $\delta$-quasipolar rings, and we give some properties of such rings.

**Definition 2.** Let $R$ be a ring and $a \in R$. The element $a$ is called weakly $\delta$-quasipolar if there exists $p^2 = p \in \text{comm}(a)$ such that $a + p \in \delta(R)$, and $p$ is called a weakly $\delta$-spectral idempotent. A ring $R$ is called weakly $\delta$-quasipolar if every element of $R$ is weakly $\delta$-quasipolar.
An element of a ring is called strongly $J$-clean [3] provided that it can be written as the sum of an idempotent and an element in its Jacobson radical that commute. A ring is strongly $J$-clean in case each of its elements is strongly $J$-clean.

Example 9. (1) Every semisimple ring and every Boolean ring is weakly $\delta$-quasipolar, since $\delta$-quasipolar rings are weakly $\delta$-quasipolar.
(2) Every strongly $J$-clean ring is weakly $\delta$-quasipolar.

Proposition 5. Let $f : R \to S$ be a surjective ring homomorphism. If $R$ is weakly $\delta$-quasipolar, then $S$ is weakly $\delta$-quasipolar.

Proof. Let $s \in S$ with $s = f(r)$ where $r \in R$. There exists an idempotent $p \in \text{comm}(r)$ such that $r + p \in \delta(R)$. Let $q = f(p)$. Then $q^2 = q \in \text{comm}(f(r)) = \text{comm}(s)$. By [16], $f(\delta(R)) \subseteq \delta(S)$. Then $s + q = f(r) + f(p) = f(r + p) \in f(\delta(R)) \subseteq \delta(S)$. Hence $S$ is weakly $\delta$-quasipolar. □

Corollary 3. Every direct summand of a weakly $\delta$-quasipolar ring is weakly $\delta$-quasipolar.

Proposition 6. Let $R = \prod_{i=1}^{n} R_i$ be a finite direct product of rings. $R$ is weakly $\delta$-quasipolar if and only if each $R_i$ is weakly $\delta$-quasipolar for $i = 1, 2, \ldots, n$.

Proof. One way is clear from Corollary 3. We may assume that $n = 2$ and $R_1$ and $R_2$ are weakly $\delta$-quasipolar. Let $a = (x_1, x_2) \in R$. There exist idempotents $p_i \in \text{comm}(x_i)$ such that $x_i + p_i \in \delta(R_i)$ for $i = 1, 2$. Then $p = (p_1, p_2)$ is an idempotent in $R$ and $p \in \text{comm}(a)$ and $a + p \in \delta(R)$. Hence $R$ is weakly $\delta$-quasipolar. □

In [8], Gurgun and Ozcan introduce and investigate properties of $\delta_r$-clean rings. Motivated by this work strongly $\delta_r$-clean rings can be defined as follows.

Definition 3. An element $x \in R$ is called strongly $\delta_r$-clean provided that there exist an idempotent $e \in R$ and an element $w \in \delta_r$ such that $x = e + w$ and $ew = we$. A ring $R$ is called strongly $\delta_r$-clean in case every element in $R$ is strongly $\delta_r$-clean.

Any strongly $J$-clean ring is strongly $\delta_r$-clean. But the converse need not be true, for example any commutative semisimple ring which is not a Boolean ring is such a ring.

Note that in the following theorem it is proved that the notions of strongly $\delta_r$-clean rings and weakly $\delta$-quasipolar rings coincide.

Theorem 3. Let $R$ be a ring. Then $R$ is a weakly $\delta$-quasipolar ring if and only if it is strongly $\delta_r$-clean.

Proof. Let $R$ be a weakly $\delta$-quasipolar ring and $a \in R$. There exists $p^2 = p \in \text{comm}(-1 + a)$ such that $-1 + a + p \in \delta(R)$. Then $a - (1 - p) \in \delta(R)$ and $a(1 - p) = (1 - p)a$. Hence $R$ is a strongly $\delta_r$-clean ring. Conversely, assume that $R$ is a strongly
\( \delta \)-clean ring. Let \( a \in R \). Since \( -a \in R \), by assumption there exists an idempotent \( p \in R \) such that \( -a - p \in \delta(R) \) and \( (-a)p = p(-a) \). So \( R \) is a weakly \( \delta \)-quasipolar ring.

Theorem 3 states that the weakly \( \delta \)-quasipolarity of a ring is equivalent to the strongly \( \delta \)-cleanness of this ring. The following example reveals that a weakly \( \delta \)-quasipolar element is different from a strongly \( \delta \)-clean element.

Example 10. Let \( R = \mathbb{Z} \) and \( a = 1 \in R \). There exists no idempotent \( p \) such that \( a + p \in \delta(R) \). Then \( a \) is not weakly \( \delta \)-quasipolar. Let \( p = \frac{1}{2} \in R \). Since \( a \in \delta(R) \), \( a \) is strongly \( \delta \)-clean. On the other hand, if \( a = -1 \in R \), then there exists no idempotent \( p \) such that \( a - p \in \delta(R) \). Then \( a \) is not strongly \( \delta \)-clean. Let \( p = 1 \in R \). Since \( a \in \delta(R) \), \( a \) is weakly \( \delta \)-quasipolar.

Theorem 4. Let \( R \) be a local ring with non-zero maximal ideal. Then the following are equivalent.

1. \( R \) is weakly \( \delta \)-quasipolar;
2. \( R \) is strongly \( J \)-clean;
3. \( R \) is uniquely clean;
4. \( R/J(R) \cong \mathbb{Z}_2 \);
5. \( R/\delta(R) \cong \mathbb{Z}_2 \).

Proof. Let \( R \) be a local ring with non-zero maximal ideal.

1. \( \Leftrightarrow \) (2) Assume that \( R \) is weakly \( \delta \)-quasipolar. Let \( a \in R \). There exists \( p^2 = p \in comm(-1 + a) \) such that \( -1 + a + p \in \delta(R) \). Then \( a - (1 - p) \in \delta(R) \). Since \( p \in comm(-1 + a) \), \( p = ap \). Hence \( R \) is strongly \( J \)-clean by \( J(R) = \delta(R) \). Similarly, the rest is clear.

2. \( \Leftrightarrow \) (3) follows from [3, Lemma 4.2].

3. \( \Leftrightarrow \) (4) follows from [14, Theorem 15].

1. \( \Rightarrow \) (5) Let \( R \) be weakly \( \delta \)-quasipolar and \( \overline{0} \neq \overline{a} = a + \delta(R) \in R/\delta(R) \), we show that \( \overline{a} = \overline{0} \). Then there exists an idempotent \( p \in R \) such that \( -a + p \in \delta(R) \) and \( p^2 = p \in comm(-a) \). Since \( R \) is a local, \( p = 0 \) or \( p = 1 \). If \( p = 0 \), this contradicts \( \overline{0} \neq \overline{a} \). Therefore \( p = 1 \). It follows that \( \overline{a} = \overline{0} \).

(5) \( \Rightarrow \) (1) It follows from Proposition 2.

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References


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