THE INCOMPLETE SRIVASTAVA’S TRIPLE HYPERGEOMETRIC FUNCTIONS $\gamma^H_A$ AND $\Gamma^H_A$

JUNESANG CHOI AND RAKESH K. PARMAR

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Abstract. Motivated mainly by certain interesting recent extensions of the generalized hypergeometric function [15], the second Appell function [6] and Srivastava’s triple hypergeometric functions [9], we introduce here the family of incomplete Srivastava’s triple hypergeometric functions $\gamma^H_A$ and $\Gamma^H_A$. We then systematically investigate several properties of each of these incomplete Srivastava’s triple hypergeometric functions including, for example, their various integral representations, transformation formula, reduction formula, derivative formula and recurrence relations. Various (known or new) special cases and consequences of the results presented here are also considered.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

The familiar incomplete Gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$ defined by

$$
\gamma(s, x) := \int_0^x t^{s-1} e^{-t} \, dt \quad (\Re(s) > 0; \ x \geq 0)
$$

and

$$
\Gamma(s, x) := \int_x^\infty t^{s-1} e^{-t} \, dt \quad (x \geq 0; \ \Re(s) > 0 \ \text{when} \ x = 0),
$$

respectively, satisfy the following decomposition formula:

$$
\gamma(s, x) + \Gamma(s, x) := \Gamma(s) \quad (\Re(s) > 0).
$$

Each of these functions plays an important role in the study of the analytic solutions of a variety of problems in diverse areas of science and engineering (see, e.g., [1, 4, 7, 16, 17, 23]).
Throughout this paper, $\mathbb{N}$, $\mathbb{Z}^-$ and $\mathbb{C}$ denote the sets of positive integers, negative integers and complex numbers, respectively,

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad \text{and} \quad \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}. $$

Moreover, the parameter $x \geq 0$ used above in (1.1) and (1.2) and elsewhere in this paper is independent of $\Re(z)$ of the complex number $z \in \mathbb{C}$.

Recently, Srivastava et al. [15] introduced and studied in a rather systematic manner the following two families of generalized incomplete hypergeometric functions:

$$p \gamma_q \left[ \frac{(\alpha_1, x), \alpha_2, \ldots, \alpha_p;}{\beta_1, \ldots, \beta_q; \ z} \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_{\nu}n!(\alpha_2)_{\nu}n! \cdots (\alpha_p)_{\nu}n!}{(\beta_1)_{\nu}n!(\beta_2)_{\nu}n! \cdots (\beta_q)_{\nu}n!} z^n \quad (1.4)$$

and

$$p \Gamma_q \left[ \frac{(\alpha_1, x), \alpha_2, \ldots, \alpha_p;}{\beta_1, \ldots, \beta_q; \ z} \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_{\nu}n!(\alpha_2)_{\nu}n! \cdots (\alpha_p)_{\nu}n!}{(\beta_1)_{\nu}n!(\beta_2)_{\nu}n! \cdots (\beta_q)_{\nu}n!} \frac{z^n}{n!} \quad (1.5)$$

where, in terms of the incomplete Gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$ defined by (1.1) and (1.2), respectively, the incomplete Pochhammer symbols $(\lambda; x)_{\nu}$ and $[\lambda; x]_{\nu}$ ($\lambda; v \in \mathbb{C}; x \geq 0$) are defined as follows:

$$(\lambda; x)_{\nu} := \frac{\gamma(\lambda + \nu, x)}{\Gamma(\lambda)} \quad (\lambda, v \in \mathbb{C}; x \geq 0) \quad (1.6)$$

and

$$[\lambda; x]_{\nu} := \frac{\Gamma(\lambda + \nu, x)}{\Gamma(\lambda)} \quad (\lambda, v \in \mathbb{C}; x \geq 0), \quad (1.7)$$

so that, obviously, these incomplete Pochhammer symbols $(\lambda; x)_{\nu}$ and $[\lambda; x]_{\nu}$ satisfy the following decomposition relation:

$$(\lambda; x)_{\nu} + [\lambda; x]_{\nu} := (\lambda)_{\nu} \quad (\lambda; v \in \mathbb{C}; x \geq 0). \quad (1.8)$$

Here, and in what follows, $(\lambda)_{\nu}$ ($\lambda, v \in \mathbb{C}$) denotes the Pochhammer symbol (or the shifted factorial) which is defined (in general) by

$$(\lambda)_{\nu} := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (v = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (v = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \quad (1.9)$$

it being understood conventionally that $(0)_0 := 1$ and assumed tacitly that the $\Gamma$-quotient exists (see, for details, [17, p. 21 et seq.]).

As already observed by Srivastava et al. [15], the definitions (1.4) and (1.5) readily yield the following decomposition formula:

$$p \gamma_q \left[ \frac{(\alpha_1, x), \alpha_2, \ldots, \alpha_p;}{\beta_1, \ldots, \beta_q; \ z} \right] + p \Gamma_q \left[ \frac{(\alpha_1, x), \alpha_2, \ldots, \alpha_p;}{\beta_1, \ldots, \beta_q; \ z} \right] = p \Gamma_q \left[ \frac{\alpha_1, \alpha_2, \ldots, \alpha_p;}{\beta_1, \ldots, \beta_q; \ z} \right] \quad (1.10)$$
for the familiar generalized hypergeometric function $pF_q$.

More recently, Çetinkaya [6] introduced and studied various properties of the
two families of the incomplete second Appell hypergeometric functions $y_2$
and $\Gamma_2$:

$$y_2((\alpha, x), \beta_1, \beta_2; y_1, y_2; x_1, x_2) = \sum_{m, p=0}^{\infty} \frac{(\alpha; x)_m p(\beta_1)_m (\beta_2)_p}{(y_1)_m (y_2)_p} \frac{x_1^m x_2^p}{m! \ p!}$$

(1.11)

and

$$\Gamma_2((\alpha, x), \beta_1, \beta_2; y_1, y_2; x_1, x_2) = \sum_{m, p=0}^{\infty} \frac{(\alpha; x)_m p(\beta_1)_m (\beta_2)_p}{(y_1)_m (y_2)_p} \frac{x_1^m x_2^p}{m! \ p!}$$

(1.12)

Very recently, Choi et al. [9] introduced and studied various properties of the
following two families of the incomplete Srivastava’s triple hypergeometric functions
$\gamma_B^H$ and $\Gamma_B^H$ as follows:

$$\gamma_B^H((\alpha, x), \beta_1, \beta_2; y_1, y_2, y_3; x_1, x_2, x_3) = \sum_{m, n, p=0}^{\infty} \frac{(\alpha; x)_m p(\beta_1)_m (\beta_2)_n p(\beta_3)_n}{(y_1)_m (y_2)_n (y_3)_p} \frac{x_1^m x_2^n (\beta_3)_n}{m! \ n! \ p!}$$

(1.13)

and

$$\Gamma_B^H((\alpha, x), \beta_1, \beta_2; y_1, y_2, y_3; x_1, x_2, x_3) = \sum_{m, n, p=0}^{\infty} \frac{(\alpha; x)_m p(\beta_1)_m (\beta_2)_n p(\beta_3)_n}{(y_1)_m (y_2)_n (y_3)_p} \frac{x_1^m x_2^n (\beta_3)_n}{m! \ n! \ p!}$$

(1.14)

$$\left( x \geq 0; \ |x_1| < r, \ |x_2| < s, \ |x_3| < t, \ r + s + t + 2 \sqrt{rst} = 1 \ \text{when} \ x = 0 \right).$$

In a sequel to the aforementioned work by Srivastava et al. [15], Çetinkaya [6] and
Choi et al. [9] and motivated essentially by these families of incomplete hyper-
geometric functions $p\gamma_q$ and $p\Gamma_q$, incomplete second Appell functions $y_2$ and $\Gamma_2$ and
incomplete Srivastava’s triple hypergeometric functions $\gamma_B^H$ and $\Gamma_B^H$ (see, for details,
[6, 9, 15] and the references cited therein), we aim here at systematically investigat-
ing the family of the incomplete Srivastava’s triple hypergeometric functions $\gamma_A^H$ and
$\Gamma_A^H$ to present various representations and formulas, for example, various definite
and semi-definite integral representations involving the Laguerre polynomials, Bessel
and modified Bessel functions, transformation formula, reduction formula, derivative
formula and recurrence relations. For various other investigations involving general-
izations of the hypergeometric function $pF_q$, which were motivated essentially by
the pioneering work of Srivastava et al. [15], the interested reader may refer to recent
papers on the subject (see, for example, [8, 19–22] and the references cited in each of
these papers).
2. The Incomplete Srivastava’s Triple Hypergeometric Functions

In terms of the incomplete Pochhammer symbol \((\lambda;x)_n\) and \([\lambda;x]_n\) defined by (1.6) and (1.7), we introduce the following incomplete Srivastava’s triple hypergeometric functions \(\gamma_A^H\) and \(\Gamma_A^H\): For \(\alpha, \beta_1, \beta_2 \in \mathbb{C}\) and \(\gamma_1, \gamma_2 \in \mathbb{C} \setminus \mathbb{Z}^+_0\),

\[
\gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha;x)_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p} x_1^m x_2^n x_3^p}{(\gamma_1)m(\gamma_2)n+p m! n! p!} \tag{2.1}
\]

\((x \geq 0; \ |x_1| < r, \ |x_2| < s, \ |x_3| < t, \ r+s+t = 1+st \text{ when } x = 0)\)

and

\[
\Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3] = \sum_{m,n,p=0}^{\infty} \frac{[\alpha;x]_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p} x_1^m x_2^n x_3^p}{(\gamma_1)m(\gamma_2)n+p m! n! p!} \tag{2.2}
\]

\((x \geq 0; \ |x_1| < r, \ |x_2| < s, \ |x_3| < t, \ r+s+t = 1+st \text{ when } x = 0)\).

In view of (1.8), these incomplete Srivastava’s triple hypergeometric functions satisfy the following decomposition formula:

\[
\gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3] + \Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3] = H_A[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3], \tag{2.3}
\]

where \(H_A\) is the familiar Srivastava’s triple hypergeometric functions (see, for details, [10–14, 17]).

Remark 1. It is interesting to note that the special cases of (2.1) and (2.2) when \(x_2 = 0\) reduce to the known incomplete second Appell hypergeometric functions (1.11) and (1.12). Also, the special cases of (2.1) and (2.2) when \(x_2 = 0\) and \(x_3 = 0\) or \(x_1 = 0\) are seen to yield the known incomplete families of Gauss hypergeometric functions [15].

In view of the formula (2.3), it is sufficient to discuss properties and characteristics of one of the incomplete Srivastava’s triple hypergeometric functions \(\gamma_A^H\) and \(\Gamma_A^H\).

3. Integral Representations of \(\Gamma_A^H\)

In this section, we apply (1.2) and (1.7) to present certain integral representations of the incomplete Srivastava’s triple hypergeometric functions \(\Gamma_A^H\). We also obtain various integral representations involving Laguerre polynomial, Bessel and modified Bessel functions.
Theorem 1. The following integral representation for $\Gamma_A^H$ in (2.2) holds true:

\[
\Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] = \frac{1}{\Gamma(\alpha) \Gamma(\beta_1) \Gamma(\beta_2)} \times \int_x^\infty \int_0^\infty \int_0^\infty e^{-s-t}t^{\alpha-1} \beta_1-1 \beta_2-1 \left(\begin{array}{c} 1 \\ 0 \end{array}\right) F_1(-; \gamma_1; x_1 st) \times F_1(-; \gamma_2; x_2 s + x_3 t) dt \, ds
\]

\[
(x \geq 0; \max\{|\Re(x_2), \Re(x_3)| < 1, \min\{|\Re(\alpha), \Re(\beta_1)| > 0\} \text{ when } x = 0).
\]

Proof of Theorem 1. Using the integral representations of the incomplete Pochhammer symbol $[\alpha; x]_{m+p}$ by considering (1.2) and (1.7), the classical Pochhammer symbol $(\beta_1)_{m+n}$ and using the elementary series identity [18, p. 52, Eq. 1.6(2)]:

\[
\sum_{m_1, m_2=0}^\infty \Omega(m_1 + m_2) \frac{x_1^{m_1} x_2^{m_2}}{m_1! m_2!} = \sum_{m=0}^\infty \Omega(m) \frac{(x_1 + x_2)^m}{m!},
\]

in (2.2), we are led to the desired result.

Theorem 2. The following triple integral representation for $\Gamma_A^H$ in (2.2) holds true:

\[
\Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] = \frac{1}{\Gamma(\alpha) \Gamma(\beta_1) \Gamma(\beta_2)} \times \int_x^\infty \int_0^\infty \int_0^\infty e^{-s-t}t^{\alpha-1} \beta_1-1 \beta_2-1 \left(\begin{array}{c} 1 \\ 0 \end{array}\right) F_1(-; \gamma_1; x_1 st) \times F_1(-; \gamma_2; x_2 us + x_3 ut) dt \, ds \, du
\]

\[
(x \geq 0; \min\{|\Re(\alpha), \Re(\beta_1), \Re(\beta_2)| > 0\} \text{ when } x = 0).
\]

Proof of Theorem 2. Using the elementary integral representation [2, p. 678, Eq.(4)]:

\[
1 F_1(\lambda; \mu; z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-t} F_1(-; \mu; zt) dt \quad (\Re(\lambda) > 0)
\]

in (3.1), we are led to the desired integral representation.

The Laguerre polynomial $L_n^{(\alpha)}(x)$ of order (index) $\alpha$ and degree $n$ in $x$, Bessel function $J_\nu(z)$ and the modified Bessel function $I_\nu(z)$ are expressible in terms of hypergeometric functions as follows (see, e.g., [2]; see also [1, p. 265, Eq. (3.2)] and [5, 23]):

\[
L_n^{(\alpha)}(x) = \frac{(\alpha + 1)_n}{n!} 1 F_1(-n; \alpha + 1; x),
\]

\[
J_\nu(z) = \frac{(\frac{1}{2})^\nu}{\Gamma(\nu + 1)} 0 F_1\left(-; \nu + 1; -\frac{z^2}{4}\right) \quad (\nu \in \mathbb{C} \setminus \mathbb{Z}^-)
\]
and
\[ I_v(z) = (\frac{\Gamma}{\Gamma(v+1)}) \text{e}^{-\frac{1}{4}z^2} {}_0F_1\left( 1; v + 1; \frac{1}{4}z^2 \right) \quad (v \in \mathbb{C} \setminus \mathbb{Z}^-). \tag{3.7} \]

Now, applying the relationships (3.5) to (3.1), (3.6) and (3.7) to (3.1), and (3.5), (3.6) and (3.7) to (3.1), respectively, we can deduce certain interesting integral representations for the incomplete Srivastava’s triple hypergeometric function in (2.2) asserted by Corollaries 1, 2 and 3 below. Their proofs are omitted.

**Corollary 1.** The following integral representation for \( H_A \) in (2.2) holds true:
\[
H_A(\alpha; x; \gamma) = \frac{m!}{(\gamma_2 + 1)_m \Gamma(\alpha) \Gamma(\beta_1)} \int_x^\infty \int_0^\infty e^{-s-t} a^{-y_1} \mathcal{J}_{\gamma_1}(2\sqrt{st}) \text{I} F_1(\gamma_1 \gamma_2 s + x_3) dt ds.
\tag{3.8}
\]

**Corollary 2.** Each of the following double integral representations holds true:
\[
H_A(\alpha; x; \gamma) = \frac{\Gamma(\gamma_1 + 1) x_3^{-\gamma_2}}{\Gamma(\alpha) \Gamma(\beta_1)} \int_x^\infty \int_0^\infty e^{-s-t} a^{-y_1} \mathcal{J}_{\gamma_1}(2\sqrt{st}) \text{I} F_1(\beta_2 \gamma_2 x_2 s + x_3 t) dt ds.
\tag{3.9}
\]

and
\[
H_A(\alpha; x; \gamma) = \frac{\Gamma(\gamma_1 + 1) x_3^{-\gamma_2}}{\Gamma(\alpha) \Gamma(\beta_1)} \int_x^\infty \int_0^\infty e^{-s-t} a^{-y_1} \mathcal{J}_{\gamma_1}(2\sqrt{st}) \text{I} F_1(\beta_2 \gamma_2 x_2 s + x_3 t) dt ds.
\tag{3.10}
\]
provided that the involved integrals are convergent.

**Corollary 3.** Each of the following double integral representations holds true:
\[
H_A(\alpha; x; \gamma) = \frac{m! \Gamma(\gamma_1 + 1) x_3^{-\gamma_2}}{(\gamma_2 + 1)_m \Gamma(\alpha) \Gamma(\beta_1)} \int_x^\infty \int_0^\infty e^{-s-t} a^{-y_1} \mathcal{J}_{\gamma_1}(2\sqrt{st}) \text{I} F_1(\gamma_2 \gamma_2 x_2 s + x_3 t) dt ds.
\tag{3.11}
\]

and
\[
H_A(\alpha; x; \gamma) = \frac{m! \Gamma(\gamma_1 + 1) x_3^{-\gamma_2}}{(\gamma_2 + 1)_m \Gamma(\alpha) \Gamma(\beta_1)} \int_x^\infty \int_0^\infty e^{-s-t} a^{-y_1} \mathcal{J}_{\gamma_1}(2\sqrt{st}) \text{I} F_1(\gamma_2 \gamma_2 x_2 s + x_3 t) dt ds.
\tag{3.12}
\]
provided that the involved integrals are convergent.
4. Transformation and Reduction Formula of $\Gamma_A^H$

In this section, we present a transformation formula and a reduction formula for the incomplete Srivastava’s triple hypergeometric functions $\Gamma_A^H$.

**Theorem 3.** The following transformation formula for $\Gamma_A^H$ holds true:

$$
\Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] = (1-x_2)^{-\beta_2}(1-x_3)^{-\alpha}
$$

$$
\times \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \int_x^\infty \int_0^\infty e^{-s(1-x_2)-r(1-x_3)} \frac{x_1}{s - 1} \frac{x_2}{r - 1} \frac{x_3}{s - r - 1} \cdot
\begin{align*}
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \int_x^\infty \int_0^\infty e^{-s(1-x_2)-r(1-x_3)} \frac{x_1}{s - 1} \frac{x_2}{r - 1} \frac{x_3}{s - r - 1} \cdot
\end{align*}

**Proof of Theorem 3.** If we first apply Kummer’s transformation formula (see, e.g., [2, p. 125, Eq. (2)):

$$
1 F_1(\alpha; \beta; z) = e^z 1 F_1(\beta - \alpha; \beta; -z)
$$

to (3.1), we find that

$$
\Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] = (1-x_2)^{-\beta_2}(1-x_3)^{-\alpha}
$$

$$
\times \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \int_x^\infty \int_0^\infty e^{-s(1-x_2)-r(1-x_3)} \frac{x_1}{s - 1} \frac{x_2}{r - 1} \frac{x_3}{s - r - 1} \cdot
\begin{align*}
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \int_x^\infty \int_0^\infty e^{-s(1-x_2)-r(1-x_3)} \frac{x_1}{s - 1} \frac{x_2}{r - 1} \frac{x_3}{s - r - 1} \cdot
\end{align*}

The substitution $t(1-x_3) = u, s(1-x_2) = v$ in (4.3), leads to

$$
\Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] = (1-x_2)^{-\beta_2}(1-x_3)^{-\alpha}
$$

$$
\times \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \int_x^\infty \int_0^\infty e^{-s(1-x_2)-r(1-x_3)} \frac{x_1}{s - 1} \frac{x_2}{r - 1} \frac{x_3}{s - r - 1} \cdot
\begin{align*}
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \int_x^\infty \int_0^\infty e^{-s(1-x_2)-r(1-x_3)} \frac{x_1}{s - 1} \frac{x_2}{r - 1} \frac{x_3}{s - r - 1} \cdot
\end{align*}

which, in view of (3.1), is easily seen to be the same as the right-hand side of (4.1).

**Theorem 4.** The following reduction formula for $\Gamma_A^H$ holds true:

$$
\Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \beta_2; x_1, x_2, x_3] = (1-x_2)^{-\beta_1}(1-x_3)^{-\alpha}
$$

$$
\times 2 \Gamma_1[(\alpha, x(1-x_3)), \beta_1; \gamma_1; (1-x_2)(1-x_3)].
$$

**Proof of Theorem 4.** Setting $\gamma_2 = \beta_2$ in the integral representation (3.1), we have

$$
\Gamma_A^H[(\alpha, x), \beta_1, \beta_2; \gamma_1, \beta_2; x_1, x_2, x_3]
$$

$$
= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \int_x^\infty \int_0^\infty e^{-s(1-x_2)-r(1-x_3)} \frac{x_1}{s - 1} \frac{x_2}{r - 1} \frac{x_3}{s - r - 1} \cdot
\begin{align*}
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \int_x^\infty \int_0^\infty e^{-s(1-x_2)-r(1-x_3)} \frac{x_1}{s - 1} \frac{x_2}{r - 1} \frac{x_3}{s - r - 1} \cdot
\end{align*}
Setting \( t(1-x_3) = u, \ s(1-x_2) = v \) and using (3.4) in (4.6), we obtain
\[
\Gamma_A^H [(\alpha, x), \beta_1, \beta_2; \gamma_1, \beta_2; x_1, x_2, x_3] = \frac{(1-x_2)^{-\beta_1}(1-x_3)^{-\alpha}}{\Gamma(a)} \times \int_{x(1-x_3)}^\infty e^{-u^{a-1}} \ {}_1F_1 \left( \beta_1; \gamma_1; \frac{x_1 u}{(1-x_2)(1-x_3)} \right) \, du.
\] (4.7)

Finally, using the known result in Srivastava et al. [15, p. 665, Eq. (3.6)]:
\[
2 \Gamma_1 \left[ \binom{a}{b; c; z} \right] = \frac{1}{\Gamma(a)} \int_x^\infty e^{-t^a-1} \ {}_1F_1 (b; c; zt) \, dt
\]
in (4.7), we are led to the desired result (4.5).

5. DERIVATIVE FORMULA AND RECURRENCE RELATIONS OF \( \Gamma_A^H \)

Differentiating, partially, both sides of (2.2) with respect to \( x_1, x_2, x_3 \), \( m, n \) and \( p \) times, respectively, we obtain a derivative formula for the incomplete Srivastava’s triple hypergeometric function \( \Gamma_A^H \) given in the following theorem.

**Theorem 5.** The following derivative formula for \( \Gamma_A^H \) holds true:
\[
\frac{\partial^{m+n+p}}{\partial x_1^m \partial x_2^n \partial x_3^p} \Gamma_A^H [(\alpha, x), \beta_1, \beta_2; \gamma_1, \beta_2; x_1, x_2, x_3] = \frac{\binom{(\alpha+m+p)(\beta_1+n+p)}{(\gamma_1)m(\gamma_2)n+p}}{(\gamma_1)_{m+n+p}} \times \Gamma_A^H [(\alpha+m+p, x), \beta_1 + m + n, \beta_2 + n + p; \gamma_1 + m, \gamma_2 + n + p; x_1, x_2, x_3].
\] (5.1)

Next we give recurrence relations for the incomplete Srivastava triple hypergeometric function \( \Gamma_A^H \).

**Theorem 6.** The following recurrence relation for \( \Gamma_A^H \) holds true:
\[
\Gamma_A^H [(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2, x_3] = \Gamma_A^H [(\alpha, x), \beta_1, \beta_2; \gamma_1 - 1, \gamma_2; x_1, x_2, x_3] \\
+ \frac{a\beta_1 x_1}{\gamma_1(1-\gamma_1)} \Gamma_A^H [(\alpha+1, x), \beta_1 + 1, \beta_2; \gamma_1 + 1, \gamma_2; x_1, x_2, x_3]
\] (5.2)

**Proof of Theorem 6.** Using the well-known contiguous relation for the function \( {}_0F_1 \) (see [3, p. 12]):
\[
0F_1 (-; \gamma - 1; x) - _0F_1 (-; \gamma; x) - \frac{x}{\gamma (\gamma - 1)} _0F_1 (-; \gamma + 1; x) = 0
\]
in the integral representation (3.1), we are led to the desired result.

**Theorem 7.** The following recurrence relation for \( \Gamma_A^H \) holds true:
\[
(\gamma_2 - \beta_2 - 1) \Gamma_A^H [(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2] \\
= (\gamma_2 - 1) \Gamma_A^H [(\alpha, x), \beta_1, \beta_2; \gamma_1, \gamma_2 - 1] - \beta_2 \Gamma_A^H [(\alpha, x), \beta_1, \beta_2 + 1; \gamma_1, \gamma_2].
\] (5.3)
where the variables which are not explicitly mentioned are assumed to be unchanged in value.

Proof of Theorem 7. Using the well-known contiguous relation for the function $1F_1$ (see [2, p. 124, Eq.(6)):

$$(c-b-1)\ 1F_1(b;c;x) = (c-1)\ 1F_1(b;c-1;x) - b\ 1F_1(b+1;c;x)$$

in the integral representation (3.1), we are led to the desired result. □

6. Concluding Remarks and Observations

In our present investigation, with the help of the incomplete Pochhammer symbols $(\lambda;x)_v$ and $[\lambda;x]_v$, we have introduced the incomplete Srivastava triple hypergeometric function $P^H_A$, whose special cases when $x_2 = 0$ reduces to the incomplete Appell functions of two variables (see [6]) and when $x_2 = 0$, $x_3 = 0$ or $x_4 = 0$ reduces to the incomplete Gauss hypergeometric function (see [15]), respectively, and investigated their diverse properties such mainly as integral representations, derivative formula, reduction formula and recurrence relation. The special cases of the results presented here when $x = 0$ would reduce to the corresponding well-known results for the Srivastava’s triple hypergeometric function $H_A$ (see, for details, [10–14, 17]).

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1964.


Authors’ addresses

**Junesang Choi**
Department of Mathematics, Dongguk University, Gyeongju 780-714, Republic of Korea
*E-mail address*: junesang@mail.dongguk.ac.kr

**Rakesh K. Parmar**
Department of Mathematics, Government College of Engineering and Technology, Bikaner-334004, Rajasthan, India
*E-mail address*: rakeshparmar27@gmail.com