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THE INCOMPLETE SRIVASTAVA'S TRIPLE HYPERGEOMETRIC FUNCTIONS γ_A^H AND Γ_A^H

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Abstract. Motivated mainly by certain interesting recent extensions of the generalized hypergeometric function [15], the second Appell function [6] and Srivastava's triple hypergeometric functions [9], we introduce here the family of incomplete Srivastava's triple hypergeometric functions γ_A^H and Γ_A^H. We then systematically investigate several properties of each of these incomplete Srivastava's triple hypergeometric functions including, for example, their various integral representations, transformation formula, reduction formula, derivative formula and recurrence relations. Various (known or new) special cases and consequences of the results presented here are also considered.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

The familiar *incomplete Gamma functions* $\gamma(s,x)$ and $\Gamma(s,x)$ defined by

$$\gamma(s,x) := \int_0^x t^{s-1}\, e^{-t}\, dt \qquad \big(\Re(s) > 0;\ x \geqq 0\big) \tag{1.1}$$

and

$$\Gamma(s,x) := \int_x^\infty t^{s-1}\, e^{-t}\, dt \qquad \big(x \geqq 0;\ \Re(s) > 0 \quad \text{when} \quad x = 0\big), \tag{1.2}$$

respectively, satisfy the following decomposition formula:

$$\gamma(s,x) + \Gamma(s,x) := \Gamma(s) \qquad \big(\Re(s) > 0\big). \tag{1.3}$$

Each of these functions plays an important role in the study of the analytic solutions of a variety of problems in diverse areas of science and engineering (see, *e.g.*, [1, 4, 7, 16, 17, 23]).

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Throughout this paper, $\mathbb{N}$, $\mathbb{Z}^-$ and $\mathbb{C}$ denote the sets of positive integers, negative integers and complex numbers, respectively,

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \qquad \text{and} \qquad \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}.$$

Moreover, the parameter $x \geqq 0$ used above in (1.1) and (1.2) *and* elsewhere in this paper is independent of $\Re(z)$ of the complex number $z \in \mathbb{C}$.

Recently, Srivastava *et al.* [15] introduced and studied in a rather systematic manner the following two families of generalized incomplete hypergeometric functions:

$$ {}_p\gamma_q\left[\begin{array}{r} (\alpha_1,x),\alpha_2,\cdots,\alpha_p; \\ \beta_1,\cdots,\beta_q; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1;x)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!} \tag{1.4}$$

and

$$ {}_p\Gamma_q\left[\begin{array}{r} (\alpha_1,x),\alpha_2,\cdots,\alpha_p; \\ \beta_1,\cdots,\beta_q; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{[\alpha_1;x]_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}, \tag{1.5}$$

where, in terms of the incomplete Gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ defined by (1.1) and (1.2), respectively, the *incomplete* Pochhammer symbols $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$ $(\lambda;\, \nu \in \mathbb{C};\, x \geqq 0)$ are defined as follows:

$$(\lambda;x)_\nu := \frac{\gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \qquad (\lambda,\nu \in \mathbb{C};\; x \geqq 0) \tag{1.6}$$

and

$$[\lambda;x]_\nu := \frac{\Gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \qquad (\lambda,\nu \in \mathbb{C};\; x \geqq 0), \tag{1.7}$$

so that, obviously, these incomplete Pochhammer symbols $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$ satisfy the following decomposition relation:

$$(\lambda;x)_\nu + [\lambda;x]_\nu := (\lambda)_\nu \qquad (\lambda;\, \nu \in \mathbb{C};\, x \geqq 0). \tag{1.8}$$

Here, and in what follows, $(\lambda)_\nu$ $(\lambda,\nu \in \mathbb{C})$ denotes the Pochhammer symbol (or the shifted factorial) which is defined (in general) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\; \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\; \lambda \in \mathbb{C}), \end{cases} \tag{1.9}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [17, p. 21 *et seq.*]).

As already observed by Srivastava *et al.* [15], the definitions (1.4) and (1.5) readily yield the following decomposition formula:

$$\begin{aligned} {}_p\gamma_q\left[\begin{array}{r} (\alpha_1,x),\alpha_2,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q; \end{array} z\right] + {}_p\Gamma_q\left[\begin{array}{r} (\alpha_1,x),\alpha_2,\ldots,\alpha_p; \\ \beta_1,\cdots,\beta_q; \end{array} z\right] \\ = {}_pF_q\left[\begin{array}{r} \alpha_1,\alpha_2,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q; \end{array} z\right] \end{aligned} \tag{1.10}$$

for the familiar generalized hypergeometric function ${}_pF_q$.

More recently, Çetinkaya [6] introduced and studied various properties of the following two families of the incomplete second Appell hypergeometric functions γ_2 and Γ_2:

$$\gamma_2[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2] = \sum_{m,p=0}^{\infty} \frac{(\alpha;x)_{m+p}(\beta_1)_m(\beta_2)_p}{(\gamma_1)_m(\gamma_2)_p} \frac{x_1^m}{m!} \frac{x_2^p}{p!} \tag{1.11}$$

and

$$\Gamma_2[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2] = \sum_{m,p=0}^{\infty} \frac{[\alpha;x]_{m+p}(\beta_1)_m(\beta_2)_p}{(\gamma_1)_m(\gamma_2)_p} \frac{x_1^m}{m!} \frac{x_2^p}{p!}. \tag{1.12}$$

Very recently, Choi *et al.* [9] introduced and studied various properties of the following two families of the incomplete Srivastava's triple hypergeometric functions γ_B^H and Γ_B^H as follows:

$$\gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3] = \sum_{m,n,p=0}^{\infty} \frac{(\alpha;x)_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p}}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p} \frac{x_1^m}{m!} \frac{x_2^n}{n!} \frac{x_3^p}{p!} \tag{1.13}$$

and

$$\Gamma_B^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x_1,x_2,x_3] = \sum_{m,n,p=0}^{\infty} \frac{[\alpha;x]_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p}}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p} \frac{x_1^m}{m!} \frac{x_2^n}{n!} \frac{x_3^p}{p!} \tag{1.14}$$

$$\left(x \geqq 0;\ |x_1| < r,\ |x_2| < s,\ |x_3| < t,\ r+s+t+2\sqrt{rst} = 1 \text{ when } x = 0\right).$$

In a sequel to the aforementioned work by Srivastava *et al.* [15], Çetinkaya [6] and Choi *et al.* [9] and motivated essentially by these families of incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$, incomplete second Appell functions γ_2 and Γ_2 and incomplete Srivastava's triple hypergeometric functions γ_B^H and Γ_B^H (see, for details, [6, 9, 15] and the references cited therein), we aim here at systematically investigating the family of the incomplete Srivastava's triple hypergeometric functions γ_A^H and Γ_A^H to present various representations and formulas, for example, various definite and semi-definite integral representations involving the Laguerre polynomials, Bessel and modified Bessel functions, transformation formula, reduction formula, derivative formula and recurrence relations. For various other investigations involving generalizations of the hypergeometric function ${}_pF_q$, which were motivated essentially by the pioneering work of Srivastava *et al.* [15], the interested reader may refer to recent papers on the subject (see, for example, [8, 19–22] *and* the references cited in each of these papers).

2. The incomplete Srivastava's triple hypergeometric functions

In terms of the incomplete Pochhammer symbol $(\lambda;x)_\upsilon$ and $[\lambda;x]_\upsilon$ defined by (1.6) and (1.7), we introduce the following incomplete Srivastava's triple hypergeometric functions γ_A^H and Γ_A^H: For $\alpha,\beta_1,\beta_2\in\mathbb{C}$ and $\gamma_1,\gamma_2\in\mathbb{C}\setminus\mathbb{Z}_0^-$,

$$\begin{aligned}&\gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3]\\&\qquad=\sum_{m,n,p=0}^{\infty}\frac{(\alpha;x)_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p}}{(\gamma_1)_m(\gamma_2)_{n+p}}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}\end{aligned}\tag{2.1}$$

$$(x\geqq 0;\ |x_1|<r,\ |x_2|<s,\ |x_3|<t,\ r+s+t=1+st \text{ when } x=0)$$

and

$$\begin{aligned}&\Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3]\\&\qquad=\sum_{m,n,p=0}^{\infty}\frac{[\alpha;x]_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p}}{(\gamma_1)_m(\gamma_2)_{n+p}}\frac{x_1^m}{m!}\frac{x_2^n}{n!}\frac{x_3^p}{p!}\end{aligned}\tag{2.2}$$

$$(x\geqq 0;\ |x_1|<r,\ |x_2|<s,\ |x_3|<t,\ r+s+t=1+st \text{ when } x=0).$$

In view of (1.8), these incomplete Srivastava's triple hypergeometric functions satisfy the following decomposition formula:

$$\begin{aligned}&\gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3]+\Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3]\\&\qquad=H_A[\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3],\end{aligned}\tag{2.3}$$

where H_A is the familiar Srivastava's triple hypergeometric functions (see, for details, [10–14, 17]).

Remark 1. It is interesting to note that the special cases of (2.1) and (2.2) when $x_2=0$ reduce to the known incomplete second Appell hypergeometric functions (1.11) and (1.12). Also, the special cases of (2.1) and (2.2) when $x_2=0$ and $x_3=0$ or $x_1=0$ are seen to yield the known incomplete families of Gauss hypergeometric functions [15].

In view of the formula (2.3), it is sufficient to discuss properties and characteristics of one of the incomplete Srivastava's triple hypergeometric functions γ_A^H and Γ_A^H.

3. Integral representations of Γ_A^H

In this section, we apply (1.2) and (1.7) to present certain integral representations of the incomplete Srivastava's triple hypergeometric functions Γ_A^H. We also obtain various integral representations involving Laguerre polynomial, Bessel and modified Bessel functions.

Theorem 1. *The following integral representation for Γ_A^H in (2.2) holds true:*

$$\Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3] = \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)} \tag{3.1}$$
$$\times \int_x^\infty \int_0^\infty e^{-s-t}t^{\alpha-1}s^{\beta_1-1}\, {}_0F_1(-;\gamma_1;x_1st)\, {}_1F_1(\beta_2;\gamma_2;x_2s+x_3t)\,dt\,ds$$
$$\big(x \geqq 0;\ \max\{\Re(x_2),\Re(x_3)\} < 1,\ \min\{\Re(\alpha),\Re(\beta_1)\} > 0 \text{ when } x=0\big).$$

Proof of Theorem 1. Using the integral representations of the incomplete Pochhammer symbol $[\alpha;x]_{m+p}$ by considering (1.2) and (1.7), the classical Pochhammer symbol $(\beta_1)_{m+n}$ and using the elementary series identity [18, p. 52, Eq. 1.6(2)]:

$$\sum_{m_1,m_2=0}^{\infty} \Omega(m_1+m_2)\,\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!} = \sum_{m=0}^{\infty} \Omega(m)\,\frac{(x_1+x_2)^m}{m!}, \tag{3.2}$$

in (2.2), we are led to the desired result. □

Theorem 2. *The following triple integral representation for Γ_A^H in (2.2) holds true:*

$$\Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3] = \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)}$$
$$\times \int_x^\infty \int_0^\infty \int_0^\infty e^{-s-t}t^{\alpha-1}s^{\beta_1-1}u^{\beta_2-1} \tag{3.3}$$
$$\times\, {}_0F_1(-;\gamma_1;x_1st)\, {}_0F_1(-;\gamma_2;x_2us+x_3ut)\,dt\,ds\,du$$
$$\big(x \geqq 0;\ \min\{\Re(\alpha),\Re(\beta_1),\Re(\beta_2)\} > 0 \text{ when } x=0\big).$$

Proof of Theorem 2. Using the elementary integral representation [2, p. 678, Eq.(4)]:

$$ {}_1F_1(\lambda;\mu;z) = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-t}\, {}_0F_1(-;\mu;zt)\,dt \qquad (\Re(\lambda)>0) \tag{3.4}$$

in (3.1), we are led to the desired integral representation. □

The Laguerre polynomial $L_n^{(\alpha)}(x)$ of order (index) α and degree n in x, Bessel function $J_\nu(z)$ and the modified Bessel function $I_\nu(z)$ are expressible in terms of hypergeometric functions as follows (see, *e.g.*, [2]; see also [1, p. 265, Eq. (3.2)] and [5,23]):

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_1F_1(-n;\alpha+1;x), \tag{3.5}$$

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left(\text{—}\,;\nu+1;-\frac{1}{4}z^2\right) \qquad (\nu \in \mathbb{C}\setminus\mathbb{Z}^-) \tag{3.6}$$

and

$$I_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)} \, {}_0F_1\left(\text{---}; \nu+1; \frac{1}{4}z^2 \right) \qquad (\nu \in \mathbb{C} \setminus \mathbb{Z}^-). \tag{3.7}$$

Now, applying the relationships (3.5) to (3.1), (3.6) and (3.7) to (3.1), and (3.5), (3.6) and (3.7) to (3.1), respectively, we can deduce certain interesting integral representations for the incomplete Srivastava's triple hypergeometric function in (2.2) asserted by Corollaries 1, 2 and 3 below. Their proofs are omitted.

Corollary 1. *The following integral representation for Γ_A^H in (2.2) holds true:*

$$\begin{aligned} \Gamma_A^H[(\alpha,x),\beta_1,-m;\gamma_1,\gamma_2+1;x_1,x_2,x_3] &= \frac{m!}{(\gamma_2+1)_m \Gamma(\alpha)\Gamma(\beta_1)} \\ \times \int_x^\infty \int_0^\infty e^{-s-t} t^{\alpha-1} s^{\beta_1-1} \, {}_0F_1(-;\gamma_1;x_1 st) \, L_m^{(\gamma_2)}(x_2 s + x_3 t) dt\, ds. \end{aligned} \tag{3.8}$$

Corollary 2. *Each of the following double integral representations holds true:*

$$\begin{aligned} \Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1+1,\gamma_2;-x_1,x_2,x_3] &= \frac{\Gamma(\gamma_1+1)\, x_1^{-\frac{\gamma_1}{2}}}{\Gamma(\alpha)\Gamma(\beta_1)} \\ \times \int_x^\infty \int_0^\infty e^{-s-t} t^{\alpha-\frac{\gamma_1}{2}-1} s^{\beta_1-\frac{\gamma_1}{2}-1} J_{\gamma_1}(2\sqrt{x_1 st}) \, {}_1F_1(\beta_2;\gamma_2;x_2 s + x_3 t)\, dt\, ds \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1+1,\gamma_2;x_1,x_2,x_3] &= \frac{\Gamma(\gamma_1+1)\, x_1^{-\frac{\gamma_1}{2}}}{\Gamma(\alpha)\Gamma(\beta_1)} \\ \times \int_x^\infty \int_0^\infty e^{-s-t} t^{\alpha-\frac{\gamma_1}{2}-1} s^{\beta_1-\frac{\gamma_1}{2}-1} I_{\gamma_1}(2\sqrt{x_1 st}) \, {}_1F_1(\beta_2;\gamma_2;x_2 s + x_3 t)\, dt\, ds, \end{aligned} \tag{3.10}$$

provided that the involved integrals are convergent.

Corollary 3. *Each of the following double integral representations holds true:*

$$\begin{aligned} \Gamma_A^H[(\alpha,x),\beta_1,-m;\gamma_1+1,\gamma_2+1;-x_1,x_2,x_3] &= \frac{m!\,\Gamma(\gamma_1+1)\, x_1^{-\frac{\gamma_1}{2}}}{(\gamma_2+1)_m \Gamma(\alpha)\Gamma(\beta_1)} \\ \times \int_x^\infty \int_0^\infty e^{-s-t} t^{\alpha-\frac{\gamma_1}{2}-1} s^{\beta_1-\frac{\gamma_1}{2}-1} J_{\gamma_1}(2\sqrt{x_1 st}) \, L_m^{(\gamma_2)}(x_2 s + x_3 t)\, dt\, ds \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \Gamma_A^H[(\alpha,x),\beta_1,-m;\gamma_1+1,\gamma_2+1;x_1,x_2,x_3] &= \frac{m!\,\Gamma(\gamma_1+1)\, x_1^{-\frac{\gamma_1}{2}}}{(\gamma_2+1)_m \Gamma(\alpha)\Gamma(\beta_1)} \\ \times \int_x^\infty \int_0^\infty e^{-s-t} t^{\alpha-\frac{\gamma_1}{2}-1} s^{\beta_1-\frac{\gamma_1}{2}-1} I_{\gamma_1}(2\sqrt{x_1 st}) \, L_m^{(\gamma_2)}(x_2 s + x_3 t)\, dt\, ds, \end{aligned} \tag{3.12}$$

provided that the involved integrals are convergent.

4. Transformation and reduction formula of Γ_A^H

In this section, we present a transformation formula and a reduction formula for the incomplete Srivastava's triple hypergeometric functions Γ_A^H.

Theorem 3. *The following transformation formula for Γ_A^H holds true:*

$$\begin{aligned}\Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3] &= (1-x_2)^{-\beta_2}(1-x_3)^{-\alpha} \\ \times\, \Gamma_A^H\Big[\big(\alpha,x(1-x_3)\big),\beta_1,\gamma_2-\beta_2;\gamma_1,\gamma_2; &\frac{x_1}{(1-x_2)(1-x_3)},\frac{x_2}{x_2-1},\frac{x_3}{x_3-1}\Big].\end{aligned} \tag{4.1}$$

Proof of Theorem 3. If we first apply Kummer's transformation formula (see, *e.g.*, [2, p. 125, Eq. (2)]):

$$ {}_1F_1(\alpha;\beta;z) = e^z\ {}_1F_1(\beta-\alpha;\beta;-z) \tag{4.2}$$

to (3.1), we find that

$$\begin{aligned}&\Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3] \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)}\int_x^\infty\int_0^\infty e^{-s(1-x_2)-t(1-x_3)}t^{\alpha-1}s^{\beta_1-1} \\ &\quad\times\ {}_0F_1(-;\gamma_1;x_1st)\ {}_1F_1(\gamma_2-\beta_2;\gamma_1;-x_2s-x_3t)\,dt\,ds.\end{aligned} \tag{4.3}$$

The substitution $t(1-x_3)=u,\ \ s(1-x_2)=v$ in (4.3), leads to

$$\begin{aligned}\Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3] &= \frac{(1-x_2)^{-\beta_1}(1-x_3)^{-\alpha}}{\Gamma(\alpha)\Gamma(\beta_1)} \\ \times\int_{x(1-x_3)}^\infty\int_0^\infty e^{-u-v}u^{\alpha-1}v^{\beta_1-1}\ {}_0F_1\left(-;\gamma_1;\frac{x_1uv}{(1-x_2)(1-x_3)}\right) \\ \times\ {}_1F_1\left(\gamma_2-\beta_2;\gamma_1;\frac{x_2v}{(x_2-1)}+\frac{x_3u}{(x_3-1)}\right)du\,dv.\end{aligned} \tag{4.4}$$

which, in view of (3.1), is easily seen to be the same as the right-hand side of (4.1). □

Theorem 4. *The following reduction formula for Γ_A^H holds true:*

$$\begin{aligned}\Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\beta_2;x_1,x_2,x_3] &= (1-x_2)^{-\beta_1}(1-x_3)^{-\alpha} \\ \times\ {}_2\Gamma_1\left[\begin{array}{r}(\alpha,x(1-x_3)),\beta_1;\\ \gamma_1;\end{array}\ \frac{x_1}{(1-x_2)(1-x_3)}\right].\end{aligned} \tag{4.5}$$

Proof of Theorem 4. Setting $\gamma_2=\beta_2$ in the integral representation (3.1), we have

$$\begin{aligned}&\Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\beta_2;x_1,x_2,x_3] \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)}\int_x^\infty\int_0^\infty e^{-s(1-x_2)-t(1-x_3)}t^{\alpha-1}s^{\beta_1-1}\ {}_0F_1(-;\gamma_1;x_1st)\,dt\,ds.\end{aligned} \tag{4.6}$$

Setting $t(1-x_3)=u,\ s(1-x_2)=v$ and using (3.4) in (4.6), we obtain

$$\begin{aligned}\Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\beta_2;x_1,x_2,x_3] &= \frac{(1-x_2)^{-\beta_1}(1-x_3)^{-\alpha}}{\Gamma(\alpha)}\\ \times\int_{x(1-x_3)}^{\infty} e^{-u}u^{\alpha-1}\,{}_1F_1\left(\beta_1;\gamma_1;\frac{x_1u}{(1-x_2)(1-x_3)}\right)du.\end{aligned} \tag{4.7}$$

Finally, using the known result in Srivastava *et al.* [15, p. 665, Eq. (3.6)]:

$${}_2\Gamma_1\left[\begin{array}{c}(a,x);b\\ c;\end{array} z\right]=\frac{1}{\Gamma(a)}\int_x^{\infty}e^{-t}t^{a-1}\,{}_1F_1(b;c;zt)\,dt$$

in (4.7), we are led to the desired result (4.5). □

5. Derivative formula and recurrence relations of Γ_A^H

Differentiating, partially, both sides of (2.2) with respect to x_1, x_2 and x_3, m, n and p times, respectively, we obtain a derivative formula for the incomplete Srivastava's triple hypergeometric function Γ_A^H given in the following theorem.

Theorem 5. *The following derivative formula for Γ_A^H holds true:*

$$\begin{aligned}&\frac{\partial^{m+n+p}}{\partial x_1^m\partial x_2^n\partial x_3^p}\Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3]=\frac{(\alpha)_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p}}{(\gamma_1)_m(\gamma_2)_{n+p}}\\ &\times\Gamma_A^H[(\alpha+m+p,x),\beta_1+m+n,\beta_2+n+p;\gamma_1+m,\gamma_2+n+p;x_1,x_2,x_3].\end{aligned} \tag{5.1}$$

Next we give recurrence relations for the incomplete Srivastava triple hypergeometric function Γ_A^H.

Theorem 6. *The following recurrence relation for Γ_A^H holds true:*

$$\begin{aligned}\Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2,x_3]&=\Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1-1,\gamma_2;x_1,x_2,x_3]\\ &+\frac{\alpha\beta_1x_1}{\gamma_1(1-\gamma_1)}\Gamma_A^H[(\alpha+1,x),\beta_1+1,\beta_2;\gamma_1+1,\gamma_2;x_1,x_2,x_3]\end{aligned} \tag{5.2}$$

Proof of Theorem 6. Using the well-known contiguous relation for the function ${}_0F_1$ (see [3, p. 12]):

$${}_0F_1(-;\gamma-1;x)-{}_0F_1(-;\gamma;x)-\frac{x}{\gamma(\gamma-1)}\,{}_0F_1(-;\gamma+1;x)=0$$

in the integral representation (3.1), we are led to the desired result. □

Theorem 7. *The following recurrence relation for Γ_A^H holds true:*

$$\begin{aligned}&(\gamma_2-\beta_2-1)\Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2]\\ &=(\gamma_2-1)\Gamma_A^H[(\alpha,x),\beta_1,\beta_2;\gamma_1,\gamma_2-1]-\beta_2\Gamma_A^H[(\alpha,x),\beta_1,\beta_2+1;\gamma_1,\gamma_2],\end{aligned} \tag{5.3}$$

where the variables which are not explicitly mentioned are assumed to be unchanged in value.

Proof of Theorem 7. Using the well-known contiguous relation for the function ${}_1F_1$ (see [2, p. 124, Eq.(6)]):

$$(c-b-1)\,{}_1F_1(b;c;x) = (c-1)\,{}_1F_1(b;c-1;x) - b\,{}_1F_1(b+1;c;x)$$

in the integral representation (3.1), we are led to the desired result. □

6. CONCLUDING REMARKS AND OBSERVATIONS

In our present investigation, with the help of the incomplete Pochhammer symbols $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$, we have introduced the incomplete Srivastava triple hypergeometric function Γ_A^H, whose special cases when $x_2 = 0$ reduces to the incomplete Appell functions of two variables (see [6]) and when $x_2 = 0$, $x_3 = 0$ or $x_1 = 0$ reduces to the incomplete Gauss hypergeometric function (see [15]), respectively, and investigated their diverse properties such mainly as integral representations, derivative formula, reduction formula and recurrence relation. The special cases of the results presented here when $x = 0$ would reduce to the corresponding well-known results for the Srivastava's triple hypergeometric function H_A (see, for details, [10–14, 17]).

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