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Algebraic structures derived from BCK-algebras

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ALGEBRAIC STRUCTURES DERIVED FROM BCK-ALGEBRAS

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Abstract. Commutative BCK-algebras can be viewed as semilattices whose sections have antitone involutions and it is known that bounded commutative BCK-algebras are equivalent to MV-algebras. In the first part of this paper we assign to an arbitrary BCK-algebra a semilattice-like structure every section of which possesses a certain antitone mapping. The remaining part is devoted to algebras of the MV-language $\{\oplus, \neg, 0\}$ which are defined on bounded BCK-algebras in the same way as MV-algebras.

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1.

A *BCK-algebra* is an algebra $\mathcal{A} = (A; \rightarrow, 1)$ of type $(2,0)$ satisfying the following quasi-identities:

(I) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,
(II) $x \rightarrow ((x \rightarrow y) \rightarrow y) = 1$,
(III) $x \rightarrow x = 1$,
(IV) $x \rightarrow 1 = 1$,
(V) $(x \rightarrow y = 1 \;\&\; y \rightarrow x = 1) \;\Rightarrow\; x = y$.

BCK-algebras were introduced by Y. Imai and K. Iséki [5–7] and form an algebraic semantics for C. A. Meredith's logic.

The relation $\leq$ on A given by

$$x \leq y \quad \Leftrightarrow \quad x \rightarrow y = 1 \tag{1}$$

is a partial order on A with 1 as the top element, but the poset $(A; \leq)$ has no particular properties because any poset $(P; \leq)$ with 1 can be made a BCK-algebra $(P; \rightarrow, 1)$ by setting $x \rightarrow y := 1$ for $x \leq y$, and $x \rightarrow y := y$ otherwise.

By a *bounded BCK-algebra* we mean an algebra $\mathcal{A} = (A; \rightarrow, 0, 1)$, where $(A; \rightarrow, 1)$ is a BCK-algebra with the bottom element 0.

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In every BCK-algebra $(A;\to,1)$, the following hold for all $x,y,z \in A$:

$$x \leq y \quad\Rightarrow\quad y \to z \leq x \to z, \tag{2}$$

$$x \leq y \quad\Rightarrow\quad z \to x \leq z \to y, \tag{3}$$

$$x \to (y \to z) = y \to (x \to z), \qquad \text{(exchange)} \tag{4}$$

$$y \leq x \to y, \tag{5}$$

$$1 \to x = x, \tag{6}$$

$$x \to y \leq (z \to x) \to (z \to y), \tag{7}$$

$$((x \to y) \to y) \to y = x \to y. \tag{8}$$

A *commutative BCK-algebra* is a BCK-algebra that satisfies the identity

$$(x \to y) \to y = (y \to x) \to x. \tag{9}$$

In this case, the underlying poset is a join-semilattice in which $x \vee y = (x \to y) \to y$. Commutative BCK-algebras form a variety that is axiomatized by the identities (9), (4), (III) and (6).

We have proved in [1,2] that commutative BCK-algebras (named here *weak implication algebras* and defined in a slightly different way) can be characterized as join-semilattices whose sections (= principal order filters) posses antitone involutions.

Recall that by a *semilattice with sectionally antitone involutions* we mean a structure $\mathcal{S} = (S;\vee,(^a)_{a\in S},1)$, where $(S;\vee)$ is a join-semilattice with the greatest element 1, and for every $a \in S$, the mapping $x \mapsto x^a$ is an antitone involution on the section $[a,1] = \{x \in S : a \leq x\}$. If $\mathcal{A} = (A;\to,1)$ is a commutative BCK-algebra, then $\mathcal{S}(\mathcal{A}) = (A;\vee,(^a)_{a\in A},1)$, where $x \vee y = (x \to y) \to y$ and $x^a = x \to a$ ($x \in [a,1]$), is a semilattice with sectionally antitone involutions in which we have $x \to y = (x \vee y)^y$. On the other hand, given a structure $\mathcal{S} = (S;\vee,(^a)_{a\in S},1)$, we define a new algebra $\mathcal{A}(\mathcal{S}) = (S;\to,1)$ via $x \to y = (x \vee y)^y$. Then $\mathcal{A}(\mathcal{S})$ is a BCK-algebra if and only if it satisfies identity (4), and if this is the case, then $\mathcal{A}(\mathcal{S})$ is a commutative BCK-algebra.

Our first objective is to give a similar description for general BCK-algebras, i. e. to an arbitrary BCK-algebra we assign a semilattice-like structure the sections of which have certain antitone mappings, and also conversely, we describe the reverse passage from such structures to BCK-algebras.

Theorem 1. *Let $\mathcal{A} = (A;\to,1)$ be a BCK-algebra. Define a binary term operation $\sqcup$ on A by*

$$x \sqcup y = (x \to y) \to y,$$

and for every $a \in A$, a unary operation ^{a} on the section $[a,1] = \{x \in A : a \leq x\}$ by

$$x^a = x \to a.$$

Then the structure $\mathcal{S}(\mathcal{A}) = (A;\sqcup,(^a)_{a\in A},1)$ satisfies the following quasi-identities:

(i) $x \sqcup x = x$,

(ii) $(x \sqcup y = y \ \& \ y \sqcup x = x) \ \Rightarrow \ x = y$,
(iii) $x \sqcup y = (x \sqcup y) \sqcup y = x \sqcup (x \sqcup y) = y \sqcup (x \sqcup y)$,
(iv) $(x \sqcup z) \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$,
(v) $x \sqcup 1 = 1$,
(vi) $x^x = 1$, $1^x = x$,
(vii) $x \sqcup y = (x \sqcup y)^{yy} = ((x \sqcup y)^y \sqcup y)^y$,
(viii) $(x \sqcup y)^y \sqcup ((x \sqcup z) \sqcup (y \sqcup z))^{y \sqcup z} = ((x \sqcup z) \sqcup (y \sqcup z))^{y \sqcup z}$,
(ix) $((x \sqcup z)^z \sqcup (y \sqcup z))^{y \sqcup z} = ((y \sqcup z)^z \sqcup (x \sqcup z))^{x \sqcup z}$,
(x) $((x \sqcup y) \sqcup x)^x = (x \sqcup y)^x$.

Proof. First note that $x \sqcup y \in [y, 1]$ by (5), hence using (8) we have

$$(x \sqcup y)^y = ((x \to y) \to y) \to y = x \to y. \tag{10}$$

Further

$$x \leq y \quad \Leftrightarrow \quad x \to y = 1 \quad \Leftrightarrow \quad x \sqcup y = y. \tag{11}$$

Indeed, $x \to y = 1$ implies $x \sqcup y = (x \to y) \to y = 1 \to y = y$, and conversely, if $y = x \sqcup y = (x \to y) \to y$, then $1 = y \to y = ((x \to y) \to y) \to y = x \to y$.

Now, we can verify the properties (i)–(x) by direct computations:

(i) $x \sqcup x = (x \to x) \to x = 1 \to x = x$.

(ii) If $x \sqcup y = y$ and $y \sqcup x = x$, then $x \to y = 1$ and $y \to x = 1$ by (11), so that $x = y$ by axiom (V).

(iii) We have $(x \sqcup y) \sqcup y = (((x \to y) \to y) \to y) \to y = (x \to y) \to y = x \sqcup y$, and the equalities $x \sqcup (x \sqcup y) = y \sqcup (x \sqcup y) = x \sqcup y$ follow from (11) as $x, y \leq x \sqcup y$.

(iv) Since $x \leq x \sqcup y$ implies $x \sqcup z = (x \to z) \to z \leq ((x \sqcup y) \to z) \to z = (x \sqcup y) \sqcup z$, we have $(x \sqcup z) \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$ by (11).

(v) $x \sqcup 1 = (x \to 1) \to 1 = 1$.

(vi) $x^x = x \to x = 1$ and $1^x = 1 \to x = x$.

(vii) According to (10), $(x \sqcup y)^{yy} = (x \to y) \to y = x \sqcup y$ and $((x \sqcup y)^y \sqcup y)^y = ((x \to y) \sqcup y)^y = (x \to y) \to y = x \sqcup y$.

(viii) Due to (10), we have

$$\begin{aligned} ((x \sqcup z) \sqcup (y \sqcup z))^{y \sqcup z} &= (x \sqcup z) \to (y \sqcup z) \\ &= ((x \to z) \to z) \to ((y \to z) \to z) \\ &= (y \to z) \to (((x \to z) \to z) \to z) \\ &= (y \to z) \to (x \to z), \end{aligned}$$

whence

$$\begin{aligned} (x \sqcup y)^y \sqcup ((x \sqcup z) \sqcup (y \sqcup z))^{y \sqcup z} &= (x \to y) \sqcup ((y \to z) \to (x \to z)) \\ &= (y \to z) \to (x \to z) \\ &= ((x \sqcup z) \sqcup (y \sqcup z))^{y \sqcup z} \end{aligned}$$

by (I) and (11).

(ix) Again, in view of (10),

$$\begin{aligned}((x \sqcup z)^z \sqcup (y \sqcup z))^{y \sqcup z} &= (x \to z) \to (y \sqcup z) \\ &= (x \to z) \to ((y \to z) \to z) \\ &= (y \to z) \to ((x \to z) \to z) \\ &= (y \to z) \to (x \sqcup z) \\ &= ((y \sqcup z)^z \sqcup (x \sqcup z))^{x \sqcup z}.\end{aligned}$$

(x) We have $((x \sqcup y) \sqcup x)^x = (((x \sqcup y) \to x) \to x) \to x = (x \sqcup y) \to x = (x \sqcup y)^x$. □

Lemma 2. *Let* $(A; \sqcup)$ *be a groupoid satisfying the quasi-identities* (i)–(iv) *of Theorem 1. Then the binary relation defined by*

$$x \leq y \quad \Leftrightarrow \quad x \sqcup y = y \tag{12}$$

is a partial order on A *such that, for every* $x, y \in A$, $x \sqcup y$ *is a common upper bound of* x, y.

Proof. By (i) and (ii), $\leq$ is reflexive and antisymmetric. For transitivity, assume that $x \sqcup y = y$ and $y \sqcup z = z$. Then $x \sqcup z = (x \sqcup z) \sqcup z = (x \sqcup z) \sqcup (y \sqcup z) = (x \sqcup z) \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z = y \sqcup z = z$ by (iii) and (iv). Thus $\leq$ is a partial order on A. Moreover, from (iii) we conclude that $x, y \leq x \sqcup y$. □

Therefore, any BCK-algebra induces a semilattice-like structure with a join-like operation $\sqcup$. Another kind of generalizations of join-semilattices was introduced by J. Ježek and R. Quackenbush [8]:

A *directoid* is a groupoid $(A; \sqcup)$ satisfying the identities

(a) $x \sqcup x = x$,
(b) $(x \sqcup y) \sqcup x = x \sqcup y$,
(c) $y \sqcup (x \sqcup y) = x \sqcup y$,
(d) $x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$.

The relation $\leq$ given by (12) is a partial order. The binary operation $\sqcup$ assigns to a pair (x, y) a common upper bound of $\{x, y\}$ in such a way that $x \sqcup y = y \sqcup x = y$ provided $x \leq y$.

Observe that this is the point where directoids differ from our semilattice-like structures since in our case $x \leq y$ does not imply $y \sqcup x = y$.

Lemma 3. *Let* $\mathcal{A} = (A; \to, 1)$ *be a BCK-algebra and* $\sqcup$ *be the binary operation defined in Theorem 1. Then the following conditions are equivalent:*

(a) $(A; \sqcup)$ *is a directoid;*
(b) $\mathcal{A}$ *is a commutative BCK-algebra;*
(c) $(A; \sqcup)$ *is a join-semilattice.*

Proof. (a) $\Rightarrow$ (b). Assume that $(A;\sqcup)$ is a directoid. Remember that $x \leq y$ iff $x \to y = 1$ iff $x \sqcup y = y$. Since $(A;\sqcup)$ is a directoid, $x \leq y$ entails $x \sqcup y = y \sqcup x = y$, so $\mathcal{A}$ satisfies the quasi-identity

$$x \leq y \quad \Rightarrow \quad y = (y \to x) \to x.$$

Hence $x \leq (x \to y) \to y$ yields $(x \to y) \to y = (((x \to y) \to y) \to x) \to x$. But from $y \leq (x \to y) \to y$ it follows $(y \to x) \to x \leq (((x \to y) \to y) \to x) \to x$, and hence $(y \to x) \to x \leq (x \to y) \to y$. The converse inequality is obtained by interchanging x and y, thus $\mathcal{A}$ is a commutative BCK-algebra.

(b) $\Rightarrow$ (c). As we already know, if $\mathcal{A}$ is a commutative BCK-algebra, then $x \sqcup y = (x \to y) \to y$ is the supremum of $\{x, y\}$, hence $(A;\sqcup)$ is a join-semilattice.

(c) $\Rightarrow$ (a). Clearly, every join-semilattice is a directoid. □

Theorem 4. *Let $\mathcal{S} = (S;\sqcup,(^a)_{a\in S},1)$ be a structure—where $\sqcup$ is a binary operation on S and for each $a \in S$, $^a : x \mapsto x^a$ is unary operation on $\{x \in S : a \sqcup x = x\}$, and 1 is a distinguished element of S—satisfying the quasi-identities* (i)–(ix) *from Theorem 1. Define a new binary operation $\to$ on S by*

$$x \to y = (x \sqcup y)^y.$$

Then $\mathcal{A}(\mathcal{S}) = (S;\to,1)$ is a BCK-algebra.

Proof. The definition of $\to$ is correct since $y \sqcup (x \sqcup y) = x \sqcup y$ by (iii). Furthermore, we note that

$$x \sqcup y = y \quad \Leftrightarrow \quad x \to y = 1. \tag{13}$$

Indeed, if $x \sqcup y = y$, then $x \to y = (x \sqcup y)^y = y^y = 1$, and conversely, $1 = x \to y = (x \sqcup y)^y$ implies $y = 1^y = (x \sqcup y)^{yy} = x \sqcup y$.

Now, we verify the axioms of BCK-algebras:

(I) By (viii) we have

$$\begin{aligned}(x \to y) \sqcup ((x \sqcup z) \to (y \sqcup z)) &= (x \sqcup y)^y \sqcup ((x \sqcup z) \sqcup (y \sqcup z))^{y\sqcup z}\\ &= ((x \sqcup z) \sqcup (y \sqcup z))^{y\sqcup z}\\ &= (x \sqcup z) \to (y \sqcup z),\end{aligned}$$

so $(x \to y) \to ((x \sqcup z) \to (y \sqcup z)) = 1$. Further, using (vii) and (ix), we obtain

$$\begin{aligned}(x \to z) \to ((y \to z) \to z) &= (x \to z) \to ((y \sqcup z)^z \sqcup z)^z\\ &= (x \to z) \to (y \sqcup z)\\ &= ((x \sqcup z)^z \sqcup (y \sqcup z))^{y\sqcup z}\\ &= ((y \sqcup z)^z \sqcup (x \sqcup z))^{x\sqcup z}\\ &= (y \to z) \to (x \sqcup z)\\ &= (y \to z) \to ((x \to z) \to z).\end{aligned}$$

When we replace x by $x \to z$, we get

$$\begin{aligned}(x \sqcup z) \to (y \sqcup z) &= ((x \to z) \to z) \to ((y \to z) \to z)\\ &= (y \to z) \to (((x \to z) \to z) \to z)\\ &= (y \to z) \to (x \to z)\end{aligned}$$

since $((x \to z) \to z) \to z = (((x \sqcup z)^z \sqcup z)^z \sqcup z)^z = ((x \sqcup z) \sqcup z)^z = (x \sqcup z)^z = x \to z$. Altogether, we have proved

$$(x \to y) \to ((y \to z) \to (x \to z)) = (x \to y) \to ((x \sqcup z) \to (y \sqcup z)) = 1.$$

(II) We have $(x \to y) \to y = ((x \sqcup y)^y \sqcup y)^y = x \sqcup y$, hence $x \sqcup ((x \to y) \to y) = x \sqcup (x \sqcup y) = x \sqcup y = (x \to y) \to y$ and by (13) we obtain $x \to ((x \to y) \to y) = 1$.

(III) Due to (13), from $x \sqcup x = x$ it follows $x \to x = 1$.

(IV) Analogously, $x \sqcup 1 = 1$ gives $x \to 1 = 1$.

(V) If $x \to y = 1$ and $y \to x = 1$, then $x \sqcup y = y$ and $y \sqcup x = x$ which imply $x = y$. □

Remark 5. Observe that in Theorem 4 we did not employ the identity (x) of Theorem 1. Actually, this identity is useful in order to establish a one-to-one correspondence between BCK-algebras and semilattice-like structures with sectionally antitone mappings:

Theorem 6. *Let $\mathcal{A} = (A; \to, 1)$ be a BCK-algebra and let $\mathcal{S} = (S; \sqcup, (^a)_{a \in S}, 1)$ be an algebra as in Theorem 4 satisfying* (i)–(x) *of Theorem 1. Then $\mathcal{A}(\mathcal{S}(\mathcal{A})) = \mathcal{A}$ and $\mathcal{S}(\mathcal{A}(\mathcal{S})) = \mathcal{S}$.*

Proof. Let $\mathcal{S}(\mathcal{A}) = (A; \sqcup, (^a)_{a \in A}, 1)$ be the structure satisfying (i)–(x) which is assigned to a given BCK-algebra $\mathcal{A}$ by Theorem 1. Then in $\mathcal{A}(\mathcal{S}(\mathcal{A})) = (A; \rightsquigarrow, 1)$ we have $x \rightsquigarrow y = (x \sqcup y)^y = ((x \to y) \to y) \to y = x \to y$, so that $\mathcal{A}(\mathcal{S}(\mathcal{A})) = \mathcal{A}$.

Conversely, let $\mathcal{S} = (S; \sqcup, (^a)_{a \in S}, 1)$ be a structure that satisfies (i)–(x) of Theorem 1, $\mathcal{A}(\mathcal{S}) = (S; \to, 1)$ its corresponding BCK-algebra (cf. Theorem 4) and $\mathcal{S}(\mathcal{A}(\mathcal{S})) = (S; \Cup, (r_a)_{a \in S}, 1)$. Then $x \Cup y = (x \to y) \to y = ((x \sqcup y)^y \sqcup y)^y = x \sqcup y$, and for $x \in [a, 1]$, $r_a(x) = x \to a = (x \sqcup a)^a = ((a \sqcup x) \sqcup a)^a = (a \sqcup x)^a = x^a$ in view of (x), and hence $\mathcal{S}(\mathcal{A}(\mathcal{S})) = \mathcal{S}$. □

Corollary 7. *Let $\mathcal{S} = (S; \sqcup, (^a)_{a \in S}, 1)$ be an algebra satisfying* (i)–(x) *of Theorem 1. Then the relation defined by* (12) *is a partial order on S, 1 is the greatest element of S and for every $x, y \in S$, $x, y \leq x \sqcup y$. Moreover, for each $a \in S$, $x \mapsto x^a$ is an antitone mapping on $[a, 1] = \{x \in S : a \leq x\}$.*

2.

The MV-algebras were introduced by C. C. Chang [3] as an algebraic counterpart of the Łukasiewicz many-valued propositional logic. Here we use the present simplest definition from [4]:

An MV-algebra is an algebra $\mathcal{M} = (M; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(MV2) $x \oplus y = y \oplus x$,
(MV3) $x \oplus 0 = x$,
(MV4) $x \oplus \neg 0 = \neg 0$,
(MV5) $\neg\neg x = x$,
(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

MV-algebras are known to be termwise equivalent to bounded commutative BCK-algebras (see [4]):

(a) Let $\mathcal{M} = (M; \oplus, \neg, 0)$ be an MV-algebra and define $x \to y = \neg x \oplus y$ and $1 = \neg 0$. Then $\mathcal{A}(\mathcal{M}) = (M; \to, 0, 1)$ is a bounded commutative BCK-algebra in which $x \oplus y = (x \to 0) \to y = (y \to 0) \to x$ and $\neg x = x \to 0$.

(b) Let $\mathcal{A} = (A; \to, 0, 1)$ be a bounded commutative BCK-algebra. Define $x \oplus y = (x \to 0) \to y$ and $\neg x = x \to 0$. Then $\mathcal{M}(\mathcal{A}) = (A; \oplus, \neg, 0)$ is an MV-algebra in which $x \to y = \neg x \oplus y$.

In what follows, we are concerned with algebras in the language $\{\oplus, \neg, 0\}$ which arise from bounded (non-commutative) BCK-algebras in the same manner as MV-algebras.

Let $\mathcal{A} = (A; \to, 0, 1)$ be a bounded BCK-algebra and define binary operation $\oplus$ and a unary operation $\neg$ on A by

$$x \oplus y = (x \to 0) \to y, \tag{14}$$

$$\neg x = x \to 0. \tag{15}$$

We refer to $\mathcal{M}(\mathcal{A}) = (A; \oplus, \neg, 0)$ as the *induced algebra* of a BCK-algebra $\mathcal{A}$. We also introduce a supplementary binary operation $\odot$ by

$$x \odot y = (x \to (y \to 0)) \to 0. \tag{16}$$

Lemma 8. *Given a bounded BCK-algebra $\mathcal{A}$, its induced algebra $\mathcal{M}(\mathcal{A})$ satisfies the following identities:*

(1) $0 \oplus x = x$, $x \oplus 0 = \neg\neg x$,
(2) $x \oplus \neg\neg y = y \oplus \neg\neg x$,
(3) $x \oplus 1 = 1 \oplus x = 1$,
(4) $\neg\neg\neg x = \neg x$,
(5) $\neg x \oplus 0 = \neg x$,
(6) $\neg\neg x \oplus y = x \oplus y$,
(7) $x \oplus \neg y = \neg y \oplus \neg\neg x$,
(8) $x \oplus (y \oplus z) = y \oplus (x \oplus z)$.

In addition, we have

(9) $x \odot 1 = 1 \odot x = \neg\neg x$,
(10) $x \odot y = y \odot x = \neg\neg x \odot y = \neg(\neg x \oplus \neg y)$,

(11) $x \odot 0 = 0$,
(12) $\neg x \odot 1 = \neg x$,
(13) $\neg(\neg x \odot \neg y) = \neg\neg(x \oplus \neg\neg y)$.

Proof. (1) $0 \oplus x = (0 \to 0) \to x = 1 \to x = x$ and $x \oplus 0 = (x \to 0) \to 0 = \neg\neg x$.

(2) $x \oplus \neg\neg y = (x \to 0) \to ((y \to 0) \to 0) = (y \to 0) \to ((x \to 0) \to 0) = y \oplus \neg\neg x$.

(3) $x \oplus 1 = (x \to 0) \to 1 = 1$ and $1 \oplus x = (1 \to 0) \to x = 0 \to x = 1$.

(4) $\neg\neg\neg x = ((x \to 0) \to 0) \to 0 = x \to 0 = \neg x$.

(5) This follows from (1) and (4).

(6) $\neg\neg x \oplus y = (((x \to 0) \to 0) \to 0) \to y = (x \to 0) \to y = x \oplus y$.

(7) This is a consequence of (2) and (4).

(8) $x \oplus (y \oplus z) = (x \to 0) \to ((y \to 0) \to z) = (y \to 0) \to ((x \to 0) \to z) = y \oplus (x \oplus z)$.

(9) $x \odot 1 = (x \to (1 \to 0)) \to 0 = (x \to 0) \to 0 = \neg\neg x$ and analogously $1 \odot x = (1 \to (x \to 0)) \to 0 = (x \to 0) \to 0 = \neg\neg x$.

(10) We have $x \odot y = (x \to (y \to 0)) \to 0 = (y \to (x \to 0)) \to 0 = y \odot x$ and $\neg\neg x \odot y = (((x \to 0) \to 0) \to (y \to 0)) \to 0 = (y \to (((x \to 0) \to 0) \to 0)) \to 0 = (y \to (x \to 0)) \to 0 = y \odot x$, and finally $\neg\neg x \odot y = (((x \to 0) \to 0) \to (y \to 0)) \to 0 = \neg(\neg x \oplus \neg y)$.

(11) $x \odot 0 = (x \to (0 \to 0)) \to 0 = (x \to 1) \to 0 = 1 \to 0 = 0$.

(12) This is a consequence of (9) and (4).

(13) By (10) and (7) we have $\neg(\neg x \odot \neg y) = \neg(\neg y \odot \neg x) = \neg\neg(\neg\neg y \oplus \neg\neg x) = \neg\neg(x \oplus \neg\neg y)$. ☐

Lemma 9. *Let $\mathcal{M}(\mathcal{A})$ be the induced algebra of a BCK-algebra $\mathcal{A}$. Then*

(a) *$x \leq y$ implies $x \oplus z \leq y \oplus z$ and $z \oplus x \leq z \oplus y$, and the same is true for $\odot$,*
(b) *$x \leq y \to \neg z$ iff $x \odot y \leq \neg z$,*

where $\leq$ is the partial order of $\mathcal{A}$ defined by (1).

Proof. (a) From $x \leq y$ it follows $x \to 0 \geq y \to 0$ and hence $x \oplus z = (x \to 0) \to z \leq (y \to 0) \to z = y \oplus z$. Similarly, $x \leq y$ yields $z \oplus x = (z \to 0) \to x \leq (z \to 0) \to y = z \oplus y$.

Now, let $x \leq y$. Then $\neg x \geq \neg y$ which implies $\neg x \oplus \neg z \geq \neg y \oplus \neg z$ and hence $x \odot z = \neg(\neg x \oplus \neg y) \leq \neg(\neg y \oplus \neg z) = y \odot z$.

(b) If $x \leq y \to \neg z$, then $x \odot y \leq (y \to \neg z) \odot y = ((y \to (z \to 0)) \to (y \to 0)) \to 0 \leq ((z \to 0) \to 0) \to 0 = z \to 0 = \neg z$ since $(z \to 0) \to 0 \leq (y \to (z \to 0)) \to (y \to 0)$. Conversely, if $x \odot y \leq \neg z$, then $y \to \neg z \geq y \to (x \odot y) = y \to ((x \to (y \to 0)) \to 0) = (x \to (y \to 0)) \to (y \to 0) = (y \to (x \to 0)) \to (y \to 0) \geq (x \to 0) \to 0 \geq x$. ☐

From now on, we will assume that a given bounded BCK-algebra $\mathcal{A}$ satisfies the *law of double negation*

$$x = \neg\neg x. \tag{17}$$

It should be underlined that (17) is not enough for a BCK-algebra $\mathcal{A}$ to be commutative. Indeed, for instance, the following bounded BCK-algebra obeys the law of double negation but it is not commutative as $(a \to b) \to b = b \neq 1 = (b \to a) \to a$:

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	a	1	1
1	0	a	b	1

Recall that a structure $(P;\leq,\cdot,e)$ is a *partially ordered monoid* or briefly a *po-monoid* if $(P;\cdot,e)$ is a monoid, $(P;\leq)$ is a poset and, for all $x,y,z \in P$, $x \leq y$ implies $x\cdot z \leq y\cdot z$ and $z\cdot x \leq z\cdot y$.

A *partially ordered commutative residuated integral monoid*, a *pocrim* for short, is a structure $(P;\leq,\cdot,\to,1)$ such that $(P;\leq,\cdot,1)$ is a commutative po-monoid with the greatest element 1, and for all $x,y,z \in P$,

$$x\cdot y \leq z \quad \Leftrightarrow \quad x \leq y \to z. \tag{18}$$

Lemma 10. *Let $\mathcal{A} = (A;\to,0,1)$ be a bounded BCK-algebra satisfying the law of double negation* (17)*. Then both $(A;\leq,\oplus,0)$ and $(A;\leq,\odot,1)$ are commutative po-monoids. Moreover, $(A;\leq,\odot,\to,1)$ is a pocrim.*

Proof. First, we show that $\oplus$ is commutative:

$$\begin{aligned} x \oplus y &= (x \to 0) \to y \\ &= (x \to 0) \to ((y \to 0) \to 0) \\ &= (y \to 0) \to ((x \to 0) \to 0) \\ &= (y \to 0) \to x \\ &= y \oplus x. \end{aligned}$$

Further, $\oplus$ is associative:

$$\begin{aligned} (x \oplus y) \oplus z &= (((x \to 0) \to y) \to 0) \to z \\ &= (((x \to 0) \to y) \to 0) \to ((z \to 0) \to 0) \\ &= (z \to 0) \to ((((x \to 0) \to y) \to 0) \to 0) \\ &= (z \to 0) \to ((x \to 0) \to y) \\ &= (x \to 0) \to ((z \to 0) \to y) \\ &= x \oplus (z \oplus y) \\ &= x \oplus (y \oplus z). \end{aligned}$$

Commutativity and associativity of the operation $\odot$ is a direct consequence. Now, by Lemma 8 (1), (9) and Lemma 9 (a), $(A;\leq,\oplus,1)$ as well as $(A;\leq,\odot,1)$ is a commutative po-monoid.

In order to prove the latter statement, it suffices to note that by (b) of Lemma 9 we have $x\leq y\to\neg\neg z=y\to z$ if and only if $x\odot y\leq\neg\neg z=z$ verifying (18). □

In the next theorem we characterize "MV-like" algebras arising from bounded BCK-algebras satisfying the law of double negation:

Theorem 11. *Given a bounded BCK-algebra $\mathcal{A}=(A;\to,0,1)$ satisfying the law of double negation* (17)*, the induced algebra $\mathcal{M}(\mathcal{A})=(A;\oplus,\neg,0)$ fulfils the identities* (MV1)–(MV5) *and the axioms*

(A1) $\neg(\neg x\oplus y)\oplus\neg(\neg y\oplus z)\oplus\neg x\oplus z=1$,
(A2) $(\neg x\oplus y=1\ \&\ \neg y\oplus x=1)\ \Rightarrow\ x=y$;

moreover, we have $x\to y=\neg x\oplus y$.

Conversely, let $\mathcal{M}=(M;\oplus,\neg,0)$ be an algebra of type $(2,1,0)$ and put $1=\neg0$. If $\mathcal{M}$ satisfies (MV1)–(MV5), (A1) *and* (A2)*, then upon defining $x\to y=\neg x\oplus y$, the algebra $\mathcal{A}(\mathcal{M})=(M;\to,0,1)$ is a bounded BCK-algebra satisfying* (17). *In addition, $x\oplus y=(x\to0)\to y$ and $\neg x=x\to0$.*

Proof. Let $\mathcal{A}$ be a bounded BCK-algebra with (17). Due to Lemma 8 and the identity (17) $\mathcal{M}(\mathcal{A})$ satisfies (MV1)–(MV5). Further, $\neg x\oplus y=((x\to0)\to0)\to y=x\to y$, and hence (A1) can be written as

$$(x\to y)\to((y\to z)\to(x\to z))=1$$

which is just axiom (I). Similarly, (A2) can be rewritten in the form

$$(x\to y=1\ \&\ y\to x=1)\ \Rightarrow\ x=y$$

which is (V), the fifth axiom of BCK-algebras.

Conversely, let $\mathcal{M}=(M;\oplus,\neg,0)$ be an algebra having the required properties. By (A1) we have $(x\to y)\to((y\to z)\to(x\to z))=\neg(\neg x\oplus y)\oplus(\neg(\neg y\oplus z)\oplus(\neg x\oplus z))=1$ proving (I). When we put $y=z=0$ in (A1), we obtain $x\to x=x\oplus\neg x=\neg\neg x\oplus\neg x=\neg(\neg x\oplus0)\oplus\neg(\neg0\oplus0)\oplus\neg x\oplus0=1$, i. e., $\mathcal{A}(\mathcal{M})$ fulfils (III). Now, $x\to((x\to y)\to y)=\neg x\oplus\neg(\neg x\oplus y)\oplus y=1$ which is (II). The axiom (IV) is clear: $x\to1=\neg x\oplus1=1$. Finally, according to (A2), $x\to y=1$ and $y\to x=1$ imply $x=y$. Altogether, $\mathcal{A}(\mathcal{M})$ is a BCK-algebra. Obviously, it is bounded and obeys (17).

In addition, $\neg x=\neg x\oplus0=x\to0$ and $x\oplus y=\neg(\neg x\oplus0)\oplus y=(x\to0)\to y$. □

Corollary 12. *Bounded BCK-algebras satisfying the law of double negation are termwise equivalent to the class of all algebras $\mathcal{M}=(M;\oplus,\neg,0)$ that satisfy* (MV1)–(MV5), (A1) *and* (A2).

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