Miskolc Mathematical Notes
Vol. 18 (2017), No. 2, pp. 731–738
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2017.1496

ON COFINITELY WEAK $\delta-$SUPPLEMENTED MODULES

FIGEN ERYILMAZ AND ŞENOL EREN

Received 15 January, 2015

Abstract. Let R be a ring and M be a left $R-$module. M is called *cofinitely weak $\delta-$supplemented* (or briefly $\delta-$ *CWS*-module) if every cofinite submodule of M has a weak $\delta-$supplement in M. In this paper, we give various properties of this kind of modules. It is shown that a module M is $\delta-CWS$-module if and only if every maximal submodule has a weak $\delta-$supplement in M. The class of cofinitely weak $\delta-$supplemented modules are closed under taking homomorphic images, arbitrary sums and short exact sequences. Also we give some conditions equivalent to being a $\delta-CWS$-module for a $\delta-$coatomic module.

2010 *Mathematics Subject Classification:* 16D10; 16E50; 16L30; 16D99

Keywords: cofinite submodule, $\delta-$small module, cofinitely weak $\delta-$supplemented module

1. INTRODUCTION & PRELIMINARIES

Throughout this paper, R will be an associative ring with identity and M will be an unitary left $R-$module and the symbol "$\leq$" will denote submodule property. Let M be an $R-$module. A submodule N of M is called *small* in M and denoted by $N \ll M$, if for every submodule K of M the equality $M = N + K$ implies $K = M$. A submodule N of M is said to be *essential* in M and denoted by $N \trianglelefteq M$, if $N \cap K \neq 0$ for every nonzero submodule K of M. A module M is said to be *singular* if $M \cong \frac{N}{K}$ for some module N and a submodule $K \leq N$ with $K \trianglelefteq N$. Let N, L be submodules of M. We call L as a *supplement* of N in M, if $M = N + L$ and $N \cap L$ is small in L [12]. Also L is called a *weak supplement* of N in M, if $M = N + L$ and $N \cap L \ll M$ [9, 15]. Clearly in this situation N is a weak supplement of K, too. A module M is called *(weakly) supplemented*, if every submodule of M has a *(weak) supplement*. By using this definition, Büyükaşık and Lomp showed that a ring R is left perfect if and only if every left $R-$module is weakly supplemented, if and only if R is semilocal and the radical of the countably infinite free left $R-$module has a weak supplement in [6]. Furthermore Alizade and Büyükaşık showed that a ring R is semilocal if and only if every direct product of simple modules is weakly supplemented in [4].

Following [14], recall that a submodule N of a module M is said to be $\delta-$*small* in M and written $N \ll_\delta M$, provided $M \neq N + X$ for any proper submodule X of

M with $\frac{M}{X}$ singular. The sum of $\delta-$small submodules of a module M is denoted by $\delta(M)$. Let M be an $R-$module. M is called $\delta-$*coatomic module* whenever $N \leq M$ and $\delta\left(\frac{M}{N}\right) = \frac{M}{N}$ implies $\frac{M}{N} = 0$. For more detailed discussion on $\delta-$ coatomic modules we refer to [7]. Let L be a submodule of a module M. A submodule K of M is called a $\delta-$*supplement* of L in M, if $M = L + K$ and $L \cap K \ll_\delta K$. The module M is called $\delta-$*supplemented* if every submodule of M has a $\delta-$supplement in M. On the other hand, the submodule L is said to be a *weak supplement* of N in M, if $M = L + N$ and $L \cap N \ll_\delta M$. Also, the module M is called *weakly* $\delta-$*supplemented* if every submodule of M has a weak $\delta-$supplement in M. For more discussion on $\delta-$small submodules, $\delta-$supplemented and weakly $\delta-$supplemented modules, we refer to [8, 13, 14].

Alizade et al. studied certain modules whose maximal submodules have supplements, and introduced cofinitely supplemented modules in [3]. A submodule N of a module M is said to be *cofinite* if the factor module $\frac{M}{N}$ is finitely generated. M is called a *cofinitely (weak) supplemented* module if every cofinite submodule of M has a (weak) supplement in M (see [3, 5], respectively). Nevertheless, it is known by [3], Theorem 2.8 and [5], Theorem 2.11, an $R-$module M is cofinitely (weak) supplemented if and only if every maximal submodule of M has a (weak) supplement in M. Clearly, supplemented modules are cofinitely supplemented and weakly supplemented modules are cofinitely weak supplemented.

A module M is called *cofinitely* $\delta-$*supplemented*, if every submodule of M has a $\delta-$supplement in M. In [10], cofinitely $\delta-$supplemented modules are introduced as a generalization of cofinitely supplemented modules. On the other hand, some properties of these modules are given in [1, 2].

In this paper, we will call a module M is *cofinitely weak* $\delta-$*supplemented* (or briefly $\delta-CWS$-module) if every cofinite submodule of M has a weak $\delta-$supplement. We will introduce *cofinitely weak* $\delta-$*supplemented* modules and obtain some properties of these modules.

2. Cofinitely weak $\delta-$supplemented modules

Lemma 1. *Let M be a module and U be a cofinite (maximal) submodule of M. If V is a weak $\delta-$supplement of U in M, then U has a finitely generated (cyclic) weak $\delta-$supplement in M contained in V.*

Proof. If U is cofinite, then $\frac{M}{U} \cong \frac{V}{(V\cap U)}$ is finitely generated. Let $\frac{V}{(V\cap U)}$ be generated by elements $x_1 + V \cap U,\ \ x_2 + V \cap U,\ ....,x_n + V \cap U$ (for every $i = 1, 2, ...n\ x_i \in V$). Then for the finitely generated submodule $W = Rx_1 + Rx_2 + ... + Rx_n$ of V, we have $W + U = W + V \cap U + U = V + U = M$ and $W \cap U \leq V \cap U \ll_\delta M$. Therefore W is a finitely generated weak $\delta-$supplement of U in M contained in V. If U is maximal, then $\frac{V}{(V\cap U)}$ is a cyclic module generated by some element $x + (V \cap U)$ and $W = Rx$ is a weak $\delta-$supplement of U. □

Lemma 2. *Let M be a module. If, for every cofinite submodule U of M, there exists a submodule V of M such that $M = U + V$ and $U \cap V$ has a weak $\delta-$supplement in V, then M is a $\delta-$CWS-module.*

Proof. Let U be a cofinite submodule of M. By assumption, there is a submodule V in M such that $M = U + V$ and $U \cap V$ has a weak $\delta-$supplement X in V. Then $U \cap V + X = V$ and $(U \cap V) \cap X = U \cap X \ll_\delta V$. Note that $M = U + V = U + U \cap V + X = U + X$ and $U \cap X \ll_\delta M$. Hence X is a weak $\delta-$supplement of U in M. It follows that M is a $\delta-$CWS-module. □

Lemma 3. *Let M be a module and U be a cofinite submodule of M. If U has a weak $\delta-$supplement V in M and $\delta(K) = K \cap \delta(M)$ for every finitely generated submodule K of V, then U has a finitely generated $\delta-$supplement in M.*

Proof. V is a weak $\delta-$supplement of U in M, i.e. $U + V = M$ and $U \cap V \ll_\delta M$. Since $\frac{M}{U}$ is finitely generated, by Lemma 1 U has a finitely generated weak $\delta-$supplement $K \leq V$ in M, i.e. $M = U + K$ and $U \cap K \ll_\delta M$. Then $U \cap K \leq \delta(M)$. Therefore $U \cap K \leq K \cap \delta(M) = \delta(K)$ and so K is a $\delta-$supplement of U in M. □

Theorem 1. *Let M be a module such that for every finitely generated submodule K of M, $\delta(K) = K \cap \delta(M)$. Then M is cofinitely weak $\delta-$supplemented if and only if M is cofinitely $\delta-$supplemented.*

Proof. Let U be a cofinite submodule of M. Since M is a $\delta-CWS$-module, U has a weak $\delta-$supplement V in M and by Lemma 3, U has a $\delta-$supplement. Hence M is cofinitely $\delta-$supplemented.

The converse is obvious. □

Corollary 1. *Let M be a finitely generated module such that for every (finitely generated) submodule K of M, $\delta(K) = K \cap \delta(M)$. Then M is weakly $\delta-$supplemented if and only if M is $\delta-$supplemented.*

Proof. Follows from Theorem 1 as in a finitely generated module, every submodule is cofinite. □

Proposition 1. *A homomorphic image of a $\delta-$CWS-module is a $\delta-$CWS-module.*

Proof. Let $f : M \to N$ be a homomorphism and M be a $\delta - CWS$-module. Suppose that X is a cofinite submodule of $f(M)$. Then, we can easily get $\frac{M}{f^{-1}(X)} \cong \frac{(\frac{M}{Ker(f)})}{(\frac{f^{-1}(X)}{Ker(f)})} \cong \frac{f(M)}{X}$ which implies that $\frac{M}{f^{-1}(X)}$ is finitely generated. Since M is a $\delta - CWS$-module, $f^{-1}(X)$ has a weak $\delta-$supplement U in M, i.e. $f^{-1}(X) + U = M$ and $f^{-1}(X) \cap U \ll_\delta M$. So $f(f^{-1}(X) + f(U) = f(M)$ and since X is a submodule of $f(M)$, $f(f^{-1}(X)) = X$ and so $X + f(U) = f(M)$. Furthermore, $f(f^{-1}(X)) \cap f(U) \ll_\delta f(M)$ by Lemma 1.3(2) in [14]. Therefore $X \cap f(U) \ll_\delta f(M)$. □

Corollary 2. *Any factor module of a $\delta-$CWS-module is a $\delta-$CWS-module.*

To prove that an arbitrary sum of $\delta - CWS$-modules is a $\delta - CWS$-module, we use the following standard lemma.

Lemma 4. *Let M be a module, N and U be submodules of M with cofinitely weak $\delta-$suplemented N and cofinite U. If $N + U$ has a weak $\delta-$supplement in M, then U also has a weak $\delta-$supplement in M.*

Proof. Let X be a weak $\delta-$supplement of $N + U$ in module M. Then we have $\frac{N}{[N\cap(X+U)]} \cong \frac{N+(X+U)}{X+U} = \frac{M}{X+U} \cong \frac{(\frac{M}{U})}{\left(\frac{(X+U)}{U}\right)}$. The last module is a finitely generated module. Hence $N \cap (X + U)$ has a weak $\delta-$supplement Y in N, i.e. $Y + [N \cap (X + U)] = N$ and $Y \cap [N \cap (X + U)] = Y \cap (X + U) \ll_\delta N \leq M$. Since $M = U + X + N = U + X + Y + [N \cap (X + U)] = X + U + Y$, Y is a weak $\delta-$supplement of $X + U$ in M. Therefore $U \cap (X + Y) \leq [X \cap (Y + U)] + [Y \cap (X + U)] \ll_\delta M$ by Lemma 1.3(1) of [14]. This means that $X + Y$ is a weak $\delta-$supplement of U in M. □

Proposition 2. *Any arbitrary sum of $\delta-$CWS-modules is a $\delta-$CWS-module.*

Proof. Let $M = \sum_{i\in I} M_i$ where each module M_i is a cofinitely weak $\delta-$supplemented and N be a cofinite submodule of M. Then $\frac{M}{N}$ is generated by some finite set $\{x_1 + N, x_2 + N, ..., x_n + N\}$ and therefore $M = Rx_1 + Rx_2 + ... + Rx_n + N$. Since each x_i is contained in the sum $\sum_{j\in J} M_j$ for some finite subset $J = \{1_1, ..., 1_{s(1)}, .., n_{s(n)}\}$ of I, $M = M_{1_1} + \sum_{j\in J-\{1_1\}} M_j + N$ has a trivial weak $\delta-$supplement 0 in M and since $M_{1_1}$is a $\delta - CWS$-module, $N + \sum_{j\in J} M_j$ has a weak $\delta-$supplement by Lemma 4. Continuing in this way, we will obtain (after we have used Lemma 4 $\sum_{i=1}^{n} s(i)$ times) N has a weak $\delta-$supplement in M. □

Let M and N be $R-$modules. If there is an epimorphism $f : M^{(\Lambda)} \longrightarrow N$ for some set Λ, then N is called an $M-$generated module. The following corollary follows from Corollary 2 and Proposition 2.

Corollary 3. *If M is a $\delta-$CWS-module, then any $M-$generated module is a $\delta-$CWS-module.*

Now we are going to prove that a module is cofinitely weak $\delta-$supplemented if and only if every maximal submodule has a weak $\delta-$supplement in M. Firstly we need the following lemma.

Lemma 5. *Let U, K be submodules of an $R-$module M. If K is a weak $\delta-$supplement of a maximal submodule N of M. If $K+U$ has a weak $\delta-$supplement in M, then U has a weak $\delta-$supplement in M.*

Proof. Let K be a weak $\delta-$supplement of a maximal submodule $N \leq M$, and X be a weak $\delta-$supplement of $K+U$ in M, i.e. $X+K+U=M$ and $X \cap (K+U) \ll_{\delta} M$. If $K \cap (X+U) \leq N$, then $(K+X) \cap U \leq [K \cap (X+U)] + [X \cap (K+U)] \ll_{\delta} M$. So, in this case $K+X$ is a weak $\delta-$supplement of U in M.

Now, suppose that $K \cap (X+U) \nsubseteq N$, i.e. $K \cap (X+U) \nsubseteq K \cap N$. Since $\frac{K}{(K \cap N)} \cong \left(\frac{(K+N)}{N}\right) = \frac{M}{N}$ and N is a maximal submodule of M, $K \cap N$ is a maximal submodule of K. Therefore $(K \cap N) + [K \cap (X+U)] = K$. Also, we get $M = U + K + X = U + (K \cap N) + [K \cap (X+U)] + X = U + (K \cap N) + X$ and $(U \cap [(K \cap N) + X]) \leq [(K \cap N) \cap (U+X)] + [((K \cap N) + U) \cap X] \leq (K \cap N) + [(K+U) \cap X] \ll_{\delta} M$ by Lemma 1.3(2) of [14]. So $((K \cap N) + X)$ is a weak $\delta-$supplement of U in M. Thus in both cases there is a weak $\delta-$supplement of U in M. □

For a module M, let E be the set of all submodules K such that K is a weak $\delta-$supplement for some maximal submodule of M and $CWS_{\delta}(M)$ denote the sum of all submodules from E.

Theorem 2. *Let M be a module. Then the following statements are equivalent:*

(i) *M is a $\delta-$CWS-module,*
(ii) *Every maximal submodule of M has a weak $\delta-$supplement,*
(iii) *$\frac{M}{CWS_{\delta}(M)}$ has no maximal submodules.*

Proof.
(i) $\Rightarrow$ (ii): Since every maximal submodule is cofinite, the proof is obvious.
(ii) $\Rightarrow$ (iii): Suppose that there is a maximal submodule of $\frac{N}{CWS_{\delta}(M)}$ of $\frac{M}{CWS_{\delta}(M)}$ and $CWS_{\delta}(M) \leq N$. Then N is a maximal submodule of M. By hypothesis, there is a weak $\delta-$supplement K of N. Then $K \in E$ and so $K \leq CWS_{\delta}(M) \leq N \leq M$. Hence $N = M$. This contradiction shows that $\frac{M}{CWS_{\delta}(M)}$ has no maximal submodules.
(iii) $\Rightarrow$ (i): Let U be a cofinite submodule of M. Since $\frac{\left(\frac{M}{U}\right)}{\left(\frac{(U+CWS_{\delta}(M))}{U}\right)} \cong \frac{M}{(U+CWS_{\delta}(M))}$, $U + CWS_{\delta}(M)$ is a cofinite submodule of M. If $\frac{M}{[U+CWS_{\delta}(M)]} \neq 0$ i.e. $U + CWS_{\delta}(M) \neq M$, then there is a maximal submodule $\frac{N}{[U+CWS_{\delta}(M)]}$ of the finitely generated $\frac{M}{[U+CWS_{\delta}(M)]}$. It follows that N is a maximal submodule of M and $\frac{N}{CWS_{\delta}(M)}$ is a maximal submodule of $\frac{M}{CWS_{\delta}(M)}$. This contradicts hypothesis. So $M = U + CWS_{\delta}(M)$. Now $\frac{M}{U}$ is finitely generated, say by elements $x_1+U, x_2+U, \ldots, x_m+U$, we have $M = Rx_1 + Rx_2 + \ldots + Rx_m + U$.

Each element x_i $(i = 1,2,..,m)$ can be written as $x_i = u_i + c_i$, where $u_i \in U, c_i \in CWS_\delta(M)$. Since each c_i is contained in the sum of finite number of submodules from E, $M = U + K_1 + K_2 + ... + K_n$ for some submodules $K_1, K_2, ..., K_n$ of M from E. Now $M = (U + K_1 + ... + K_{n-1}) + K_n$ has a weak $\delta-$supplement, namely 0. By Lemma 5, $U + K_1 + K_2 + ... + K_{n-1}$ has a weak $\delta-$supplement. Continuing in this way we obtain that U has a weak $\delta-$supplement in M. Hence M is a $\delta - CWS$-module. □

Proposition 3. *Let M be a module and $\frac{M}{\delta(M)}$ be a cofinitely weak $\delta-$supplemented. Then every cofinite submodule of $\frac{M}{\delta(M)}$ is a direct summand.*

Proof. Let $\frac{K}{\delta(M)}$ be a cofinite submodule of $\frac{M}{\delta(M)}$. By hypothesis, $\frac{K}{\delta(M)}$ has a weak $\delta-$supplement $\frac{L}{\delta(M)}$, i.e. $\left(\frac{K}{\delta(M)}\right) + \left(\frac{L}{\delta(M)}\right) = \frac{M}{\delta(M)}$ and $\left(\frac{K}{\delta(M)}\right) \cap \left(\frac{L}{\delta(M)}\right) \ll_\delta \frac{M}{\delta(M)}$. Since $\delta\left(\frac{M}{\delta(M)}\right) = 0$, $\left(\frac{K}{\delta(M)}\right) \cap \left(\frac{L}{\delta(M)}\right) = 0_{\frac{M}{\delta(M)}}$. Hence $\frac{K}{\delta(M)}$ is a direct summand. □

Theorem 3. *Let M be $\delta-$coatomic module. Then the following statements are equivalent:*

(i) *M is a $\delta-$CWS-module,*
(ii) *$\frac{M}{\delta(M)}$ is a $\delta-$CWS-module,*
(iii) *Every cofinite submodule of $\frac{M}{\delta(M)}$ is a direct summand,*
(iv) *Every maximal submodule of $\frac{M}{\delta(M)}$ is a direct summand,*
(v) *Every maximal submodule of M has a weak $\delta-$supplement.*

Proof.
(i)⇒(ii) By Corollary 2.
(ii)⇒(iii) By Proposition 3.
(iii)⇒(iv) Maximal submodules are cofinite so by the assumption they are direct summand.
(iv)⇒(v) If N is a maximal submodule of M, then $\frac{N}{\delta(M)}$ is a maximal submodule of $\frac{M}{\delta(M)}$. So there is a submodule $\frac{K}{\delta(M)}$ of $\frac{M}{\delta(M)}$ such that $\frac{M}{\delta(M)} = \left(\frac{K}{\delta(M)}\right) \oplus \left(\frac{N}{\delta(M)}\right)$. Therefore $K \cap N \leq \delta(M) \ll_\delta M$. Hence K is a weak $\delta-$supplement in M.

Let N be a maximal submodule of M which does not contain $\delta(M)$. In this case, we have $\delta(M) + N = M$. So $\delta(M)$ is a $\delta-$supplement of N in M.
(v)⇒(i) By Theorem 2 this proof holds for every module M. □

Theorem 4. *Let M be an $R-$module with $\delta(M) \ll_\delta M$ and $\frac{M}{\delta(M)}$ be a $\delta-$CWS-module. Then M is a $\delta-$CWS-module.*

Proof. Let U be a cofinite submodule of M. Then $\frac{M}{(U+\delta(M))} \cong \frac{(\frac{M}{U})}{\left(\frac{(U+\delta(M))}{U}\right)}$ is finitely generated, i.e. $U+\delta(M)$ is cofinite. On the other hand

$$\frac{\left(\frac{M}{\delta(M)}\right)}{\left[\frac{(U+\delta(M))}{\delta(M)}\right]} \cong \frac{M}{(U+\delta(M))}$$

is finitely generated and so $\frac{(U+\delta(M))}{\delta(M)}$ is a cofinite submodule of $\frac{M}{\delta(M)}$. By assumption, there exists a submodule $\frac{V}{\delta(M)}$ of $\frac{M}{\delta(M)}$ such that $\left[\frac{(U+\delta(M))}{\delta(M)}\right]+\left(\frac{V}{\delta(M)}\right)=\frac{M}{\delta(M)}$ and $\left[\frac{(U+\delta(M))}{\delta(M)}\right]\cap\left(\frac{V}{\delta(M)}\right)=\frac{[(U\cap V)+\delta(M)]}{\delta(M)} \ll_\delta \frac{M}{\delta(M)}$. Now we get $M=U+\delta(M)+V=U+V$. Since $\delta\left(\frac{M}{\delta(M)}\right)=0_{\frac{M}{\delta(M)}}$, we obtain that $(U\cap V)+\delta(M)=\delta(M)$, that is $U\cap V\leq\delta(M)$ and since $\delta(M)\ll_\delta M$, $U\cap V$ is also $\delta-$small in M. Therefore M is a $\delta-CWS$-module. □

Let M and N be $R-$modules. We call an epimorphism $f: M\to N$ is a $\delta-$cover in case $Kerf \ll_\delta M$ [11].

Corollary 4. *A $\delta-$cover of a $\delta-$CWS-module is a $\delta-$CWS-module.*

Theorem 5. *Let $0\to L\to M\to N\to 0$ be a short exact sequence. If L and N are $\delta-$CWS-modules and L has a weak $\delta-$supplement in M, then M is a $\delta-$CWS-module.*

Proof. Without restriction of generality, we will assume that $L\leq M$. Let S be weak $\delta-$supplement of L in M, i.e. $L+S=M$ and $L\cap S\ll_\delta M$. Then we have, $\frac{M}{L\cap S}\cong\frac{L}{L\cap S}\oplus\frac{S}{L\cap S}$. $\frac{L}{L\cap S}$ is cofinitely weak $\delta-$supplemented as a factor module of L which is cofinitely weak $\delta-$supplemented. On the other hand, $\frac{S}{L\cap S}\cong\frac{M}{L}\cong N$ is cofinitely weak $\delta-$supplemented. Then $\frac{M}{L\cap S}$ is cofinitely weak $\delta-$supplemented as a sum of cofinitely weak $\delta-$ supplemented. If we take Theorem 4 into consideration, then M became a $\delta-CWS$-module since $f: M\to\frac{M}{L\cap S}$ is a $\delta-$cover. □

REFERENCES

[1] K. Al-Takhman, “Cofinitely δ-supplemented and cofinitely δ-semiperfect modules.” *Int. J. Algebra*, vol. 1, no. 9-12, pp. 601–613, 2007.

[2] A. O. Alattass, “Cofinitely δ_M-supplemented and cofinitely δ_M-semiperfect modules.” *Int. J. Algebra*, vol. 5, no. 29-32, pp. 1575–1588, 2011.

[3] R. Alizade, G. Bilhan, and P. Smith, “Modules whose maximal submodules have supplements.” *Commun. Algebra*, vol. 29, no. 6, pp. 2389–2405, 2001, doi: 10.1081/AGB-100002396.

[4] R. Alizade and E. Büyükaşik, “Extensions of weakly supplemented modules.” *Math. Scand.*, vol. 103, no. 2, pp. 161–168, 2008.

[5] R. Alizade and E. Büyükaşık, “Cofinitely weak supplemented modules.” *Commun. Algebra*, vol. 31, no. 11, pp. 5377–5390, 2003, doi: 10.1081/AGB-120023962.

[6] E. Büyükaşık and C. Lomp, "Rings whose modules are weakly supplemented are perfect. Applications to certain ring extensions." *Math. Scand.*, vol. 105, no. 1, pp. 25–30, 2009.
[7] M. Koşan and A. Harmanci, "Generalizations of coatomic modules." *Cent. Eur. J. Math.*, vol. 3, no. 2, pp. 273–281, 2005, doi: 10.2478/BF02479203.
[8] M. T. Koşan, "δ-lifting and δ-supplemented modules." *Algebra Colloq.*, vol. 14, no. 1, pp. 53–60, 2007, doi: 10.1142/S1005386707000065.
[9] C. Lomp, "On semilocal modules and rings." *Commun. Algebra*, vol. 27, no. 4, pp. 1921–1935, 1999, doi: 10.1080/00927879908826539.
[10] L. Thuyet, M. Koşan, and T. Quynh, "On cofinitely δ-semiperfect modules." *Acta Math. Vietnam.*, vol. 33, no. 2, pp. 197–207, 2008.
[11] Y. Wang, "δ-small submodules and δ-supplemented modules." *Int. J. Math. Math. Sci.*, vol. 2007, p. 8, 2007, doi: 10.1155/2007/58132.
[12] R. Wisbauer, *Foundations of module and ring theory. A handbook for study and research. Revised and updated Engl. ed.*, revised and updated engl. ed. ed. Philadelphia etc.: Gordon and Breach Science Publishers, 1991.
[13] M. F. Yousif and Y. Zhou, "Semiregular, semiperfect and perfect rings relative to an ideal." *Rocky Mt. J. Math.*, vol. 32, no. 4, pp. 1651–1671, 2002, doi: 10.1216/rmjm/1181070046.
[14] Y. Zhou, "Generalizations of perfect, semiperfect, and semiregular rings." *Algebra Colloq.*, vol. 7, no. 3, pp. 305–318, 2000, doi: 10.1007/s10011-000-0305-9.
[15] H. Zöschinger, "Invarianten wesentlicher Überdeckungen." *Math. Ann.*, vol. 237, pp. 193–202, 1978, doi: 10.1007/BF01420175.

Authors' addresses

Figen Eryılmaz
Ondokuz Mayıs University, Faculty of Education, Department of Mathematics Education, 55200 Kurupelit, Samsun, TURKEY
E-mail address: `fyuzbasi@omu.edu.tr`

Şenol Eren
Ondokuz Mayıs University, Faculty of Sciences and Arts, Deparment of Mathematics, 55200 Kurupelit, Samsun, TURKEY
E-mail address: `seren@omu.edu.tr`