Miskolc Mathematical Notes
Vol. 17 (2017), No. 2, pp. 947–955

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2017.

THE R-JACOBI-STIRLING NUMBERS OF THE SECOND KIND

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Received 28 December, 2014

Abstract. In this paper, we study the r-Jacobi-Stirling numbers of the second kind introduced by Gelineau in his Phd thesis. We give, upon using combinatorial and analytic arguments, the ordinary generating function of these numbers, two recurrence relations, their exact expressions and the log-concavity.

2010 *Mathematics Subject Classification:* 11B75; 03E05

Keywords: the r-Jacobi-Stirling of the second kind, generating function, recurrence relations, log-concavity

1. Introduction

Let α and β be real numbers such that $\alpha > -1$ and $\beta > -1$. The n-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is defined as the unique polynomial solution with degree n satisfies the following Jacobi ordinary differential equation of second order:

$$\left(1-x^2\right) y''(x) + (\beta - \alpha - (\alpha+\beta+2)x) y'(x) + n(n+\alpha+\beta+1) y(x) = 0 \tag{1.1}$$

with initial conditions

$$P_n^{(\alpha,\beta)}(1) = \binom{\alpha+1}{n} \text{ and } P_n^{(\alpha,\beta)}(-1) = (-1)^n \binom{\beta+1}{n}. \tag{1.2}$$

We introduce Jacobi differential operator $l_{(\alpha,\beta)}[y](x)$ to be

$$l_{(\alpha,\beta)}[y](x) = \frac{\left(-w_{(\alpha+1,\beta+1)}(x) y'(x)\right)'}{w_{(\alpha,\beta)}(x)}, \quad w_{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta. \tag{1.3}$$

Then, the polynomial solution $y = P_n^{(\alpha,\beta)}(x)$ of (1.1) satisfies

$$l_{(\alpha,\beta)}[y](x) = n(n+\alpha+\beta+1) y(x). \tag{1.4}$$

For the n-th composite powers of the Jacobi operator $l_{(\alpha,\beta)}$ given inductively by $l_{(\alpha,\beta)}^{(n)}[y]=l_{(\alpha,\beta)}\left(l_{(\alpha,\beta)}^{(n-1)}[y]\right)$, Everitt et al. [2, Thm. 4.2] proved that

$$l_{(\alpha,\beta)}^{(n)}[y](x)=\frac{1}{w_{(\alpha,\beta)}(x)}\sum_{k=0}^{n}(-1)^{k}JS(n,k;\alpha,\beta)\left(w_{(\alpha+k,\beta+k)}(x)\,y^{(k)}(x)\right)^{(k)}, \tag{1.5}$$

where the coefficients $JS(n,k;\alpha,\beta)$ are the Jacobi-Stirling numbers of the second kind. The same authors proved in [2, Eq. 4.4] that the Jacobi-Stirling number $JS(n,k;\alpha,\beta)$ has the following exact expression

$$JS(n,k;\alpha,\beta)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{(\alpha+\beta+2j+1)(j(j+\alpha+\beta+1))^{n}}{(\alpha+\beta+k+j+1)_{k+1}}, \tag{1.6}$$

where $(x)_n:=x(x-1)\cdots(x-n+1)$ if $n\geq 1$ and $(x)_0:=1$.
We remark that the last expression of $JS(n,k;\alpha,\beta)$ depends only on $\alpha+\beta$, so if we set $z:=\alpha+\beta+1$ and $JS(n,k;\alpha,\beta):=JS(n,k;z)$, the identity given in (1.6) becomes

$$JS(n,k;z)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{z+2j}{(z+k+j)_{k+1}}(j(j+z))^{n},\quad z>-1. \tag{1.7}$$

Using different methods, Everitt et al. [2, Thm. 4.1] and Gelineau et al. [4, Sec. 4.2] showed that

$$X^{n}=\sum_{k=0}^{n}JS(n,k;z)(X)_{k,z}, \tag{1.8}$$

where $(X)_{k,z}:=\prod_{i=0}^{k-1}(X-i(i+z))$ if $k\geq 1$ and $(X)_{0,z}:=1$.
The equation (1.8) shows the the Jacobi-Stirling numbers satisfy the following recurrence relation:

$$\begin{aligned}
&JS(0,0;z)=1,\\
&JS(n,0;z)=JS(0,k;z)=0\ \text{ if } n,k\geq 1,\\
&JS(n,k;z)=JS(n-1,k-1;z)+k(k+z)JS(n-1,k;z)\ \text{ if } n,k\geq 1.
\end{aligned}$$

To give a combinatorial interpretation of the Jacobi-Stirling number $JS(n,k;2\gamma-1)$, let $[\pm n]$ be the set $\{\pm 1,\pm 2,...,\pm n\}$ and we will use definition of the Jacobi-Stirling set partition [1, Def. 4.1] given as follows.

Definition 1. For all positive integers n, k, γ, a Jacobi-Stirling set partition of $[\pm n]$ into γ zero blocks $A_1,\ldots,A_\gamma$ and k nonzero blocks $B_1,\ldots,B_k$ is an ordinary set partition of $[\pm n]$ into $k+\gamma$ blocks for which the following conditions hold:

(1) The blocks $A_1,\dots,A_\gamma$, called the zero blocks, are distinguishable, but the blocks $B_1,\dots,B_k$ are indistinguishable.
(2) The zero blocks may be empty, but all other blocks are nonempty.
(3) $\forall i\in[n]=\{1,2,\dots,n\}$, $\{-i,i\}\not\subseteq \bigcup_{j=1}^{\gamma} A_j$,
(4) $\forall j\in[k]$, $\forall i\in[n]$, we have $\{-i,i\}\subset B_j \Leftrightarrow i=\min B_j$.

Andrews et al. showed that the Jacobi-Stirling number $JS(n,k;2\gamma-1)$ is the number of Jacobi-Stirling set partitions of $[\pm n]$ into γ zero blocks and k nonzero blocks [1, Thm. 4.1].
In his Phd Thesis, Gelineau [3, Sec. 1.4] has introduced the r-Jacobi-Stirling numbers of the second kind $JS_r(n,k;z)$, $n\geq r\geq 0$, as follows.

Definition 2. The r-Jacobi-Stirling number $JS_r(n,k;2\gamma-1)$ is the number of Jacobi-Stirling set partitions of $[\pm n]$ into γ zero blocks and k nonzero blocks such that $\min B_j=j$ for $j=1,\dots,r$.

The author gives some interesting properties of these numbers. In this paper, we study the r-Jacobi-Stirling numbers of the second kind by giving, with combinatorial and analytic proofs, their ordinary generating function, two recurrence relations, their exact expressions and the log-concavity.

2. Main results

To start, we give in the following theorem the ordinary generating function.

Theorem 1. *For all positive integers n, k, the r-Jacobi-Stirling number $JS_r(n,k;z)$ has ordinary generating function to be*

$$\sum_{n\geq 0} JS_r(n+k,k;z)\,t^n=\left(\prod_{i=r}^{k}(1-i(i+z)t)\right)^{-1}.$$

Proof. Let γ be a positive integer and $z=2\gamma-1$. According to Definition 2, let $m_j=\min B_j$, $j=1,\dots,k$ be fixed and ordering the minimal elements of the nonzero blocks to get $m_1<m_2<\cdots<m_k$. We have $m_1=1,\dots,m_r=r$ and $m_j\geq j$ for $j=r+1,\dots,k$. For instance, let the sets $A_1,\dots,A_\gamma$ be empty and $B_j=\{m_j\}$, $j=1,\dots,k$. Hence, we count the number of ways to construct such blocks $A_1,\dots,A_\gamma$ and $B_1,\dots,B_k$ by insertion the elements of the set $[\pm n]-\{\pm m_1,\dots,\pm m_k\}$. Since the elements of the set $[n]-\{m_1,\dots,m_k\}$ are the integer elements of the set $]m_r,m_{r+1}[\cup\cdots\cup]m_k,n+1[$, we can proceed the proof by insertion the integer elements (and their opposites) of each interval.
Indeed, let $j\in\{r,\dots,k\}$ and $s\in\left]m_j,m_{j+1}\right[$ with convention $m_{k+1}=n+1$.

Case 1: If $-s\in\bigcup_{i=1}^{\gamma}A_i$ and $s\in\bigcup_{i=1}^{k}B_i$, there are γ ways to insert $-s$ in the zero blocks and since the blocks $B_{j+1},\dots,B_k$, when they exist, have minimal elements

$\geq m_{j+1} > s > m_j \geq j$, it follows that s can be inserted only in the blocks $B_1, \ldots, B_j$. So, we count in this case γj possibilities.

Case 2: If $s \in \bigcup_{i=1}^{\gamma} A_i$ and $-s \in \bigcup_{i=1}^{k} B_i$, by symmetry, there are γj ways.

Case 3: If $\pm s \in \bigcup_{i=1}^{k} B_i$, since $-s$ and s can be inserted only in the blocks $B_1, \ldots, B_j$ and can't be inserted in the same block, there are $j\,(j-1)$ ways.

So, there are $2\gamma j + j\,(j-1) = j\,(j+z)$ ways to insert $\pm s$ and since the interval $\left]m_j, m_{j+1}\right[$ contains $n_j := m_{j+1} - m_j - 1$ integer elements, then the number of ways to insert the elements (and their opposites) of the interval $\left]m_j, m_{j+1}\right[$ is $(j\,(j+z))^{n_j}$. It follows that the number to insert the elements (and their opposites) of the set $]m_r, m_{r+1}[\, \cup \cdots \cup \,]m_k, m_{k+1}[$ is $\prod_{j=r}^{k} (j\,(j+z))^{n_j}$. So, the number $JS_r\,(n,k;z)$ must be equal to

$$\sum_{r=m_r<\cdots<m_k<n+1=m_{k+1}} \prod_{j=r}^{k} (j\,(j+z))^{n_j} = \sum_{n_r+\cdots+n_k=n-k} \prod_{j=r}^{k} (j\,(j+z))^{n_j}$$

which gives the desired result. □

Proposition 1. *For all positive integers n, k, r, the r-Jacobi-Stirling numbers $JS_r\,(n,k;z)$ satisfy*

$$JS_r\,(n,k;z) = JS_{r-1}\,(n,k;z) - (r-1)\,(r-1+z)\,JS_{r-1}\,(n-1,k;z)\,.$$

Proof. The number of the set partitions of $[\pm n]$ into γ zero blocks and k nonzero blocks such that $\min B_j = j$ for $j = 1, \ldots, r-1$ but $\min B_r \neq r$ is $JS_{r-1}\,(n,k;2\gamma-1) - JS_r\,(n,k;2\gamma-1)$.

This number is exactly $(r-1)\,(r-2+2\gamma)\,JS_{r-1}\,(n-1,k;2\gamma-1)$ since the number of set partitions of the set $[\pm n] - \{\pm r\}$ is exactly the number of the set partitions of the set $[\pm\,(n-1)]$ and the elements $\pm r$ can be inserted only in the zero blocks or in $B_1 \cup \ldots \cup B_{r-1}$ in $(r-1)\,(r-2+2\gamma)$ ways, i.e.:

Case 1: If r is in one of the zero blocks (there are γ ways to insert it), then the element $-r$ can be inserted in the nonzero blocks in $r-1$ ways, i.e. $-r$ can't be inserted in the blocks $B_r, \ldots, B_k$ since these blocks have minimal elements $\geq r+1$. So this case gives $\gamma\,(r-1)$ possibilities.

Case 2: By symmetry, if $-r$ is in one of the zero blocks, we get $\gamma\,(r-1)$ possibilities.

Case 3: If $-r$ and r are both in the nonzero blocks, we get $(r-1)\,(r-2)$ possibilities since they can't both be in the same block. So $JS_{r-1}\,(n,k;2\gamma-1) - JS_r\,(n,k;2\gamma-1) = (r-1)\,(r-2+2\gamma)\,JS_{r-1}\,(n-1,k;2\gamma-1)$. □

Proposition 2. *For all non-negative integers* n, k, r *such that* $0 \leq r \leq k \leq n$ *and* $r < n$, *the* r*-Jacobi-Stirling numbers* $JS_r(n,k;z)$ *satisfy*

$$JS_r(n,k;z) = JS_r(n-1,k-1;z) + k(k+z)\, JS_r(n-1,k;z).$$

Proof. The number of the set partitions of $[\pm n]$ into γ zero blocks and k nonzero blocks such that $\min B_j = j$ for $j = 1,\ldots,r$ is $JS_r(n,k;2\gamma-1)$. This number can be obtained in another manner by considering n as a minimal element or not of a nonzero block. Indeed, if n is a minimal of such nonzero block, then this block contains only the two elements $\pm n$, in this case we have $JS_r(n-1,k-1;2\gamma-1)$ ways, otherwise, $[\pm(n-1)]$ can be partitioned in $k+\gamma$ blocks in $JS_r(n-1,k;2\gamma-1)$ ways, and, similar to the above proofs, $\pm n$ can be inserted in the $k+\gamma$ blocks in $k(k+2\gamma-1)$ ways. So $JS_r(n,k;z) = JS_r(n-1,k-1;z) + k(k+z)$ $JS_r(n,k-1;z)$, where $z = 2\gamma - 1$. □

Theorem 2. *For non-negative integers* n, k, r, *the* r*-Jacobi-Stirling numbers have the following expression:*

$$JS_r(n+r,k+r;z) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{(2j+2r+z)\left((j+r)(j+r+z)\right)^n}{(j+k+2r+z)_{k+1}}.$$

In particular, for $r = 0$ *or* $r = 1$, *we get*

$$JS(n,k;z) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{2j+z}{(j+k+z)_{k+1}}\left(j(j+z)\right)^n.$$

Proof. From Theorem 1, there exist numbers $\lambda_r,\ldots,\lambda_k$ such that

$$\sum_{n\geq 0} JS_r(n+k,k;z)\,t^n = \left(\prod_{i=r}^{k}\left(1-i(i+z)t\right)\right)^{-1} = \sum_{j=r}^{k}\frac{\lambda_j}{1-j(j+z)t}.$$

One can verify that we have

$$\lambda_j = \frac{1}{\prod\limits_{i=r,i\neq j}^{k}\left(1-\frac{i(i+z)}{j(j+z)}\right)} = \frac{\left(j(j+z)\right)^{k-r}}{\prod\limits_{i=r,i\neq j}^{k}\left((j-i)(j+i+z)\right)}$$

which can be simplified as

$$\lambda_j = \frac{(-1)^{k-j}}{(k-r)!}\binom{k-r}{j-r}\frac{2j+z}{(j+k+z)_{k-r+1}}\left(j(j+z)\right)^{k-r}.$$

In other words, we have

$$\sum_{n\geq 0} JS_r(n+k,k;z)\,t^n = \sum_{j=r}^{k}\frac{\lambda_j}{1-j(j+z)t} = \sum_{j=r}^{k}\lambda_j\sum_{n\geq 0}(j(j+z))^n t^n$$

which shows with the expressions of $\lambda_r,\ldots,\lambda_k$ that

$$\begin{aligned} JS_r(n,k;z) &= \sum_{j=r}^{k}\lambda_j\,(j(j+z))^{n-k} \\ &= \frac{1}{(k-r)!}\sum_{j=r}^{k}(-1)^{k-j}\binom{k-r}{j-r}\frac{2j+z}{(j+k+z)_{k-r+1}}(j(j+z))^{n-r}. \end{aligned}$$

and this completes the proof. □

Remark 1. We have $JS_r(n+r,n+r;z) = 1$ which gives with Theorem 2 the following identity

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{2j+2r+z}{(j+n+2r+z)_{n+1}}((j+r)(j+r+z))^n = n!.$$

Corollary 1. *For non-negative integers n, k, r, the r-Jacobi-Stirling numbers can be expressed in terms of the Jacobi-Stirling numbers as follows:*

$$JS_r(n+r,k+r;z) = \sum_{i=0}^{n-k}\binom{n}{i}(r(r+z))^i\,JS(n-i,k;z+2r).$$

Proof. From Theorem 2 we may state:

$$\begin{aligned} &JS_r(n+r,k+r;z) \\ &= \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{2j+2r+z}{(j+2r+k+z)_{k+1}}(j(j+z+2r)+r(r+z))^n \\ &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^{k-j}(2j+2r+z)}{(j+2r+k+z)_{k+1}}\sum_{i=0}^{n}\binom{n}{i}(j(j+z+2r))^{n-i}(r(r+z))^i \\ &= \sum_{i=0}^{n}\binom{n}{i}(r(r+z))^i\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^{k-j}(2j+2r+z)}{(j+2r+k+z)_{k+1}}(j(j+z+2r))^{n-i} \\ &= \sum_{i=0}^{n}\binom{n}{i}(r(r+z))^i\,JS(n-i,k;z+2r). \end{aligned}$$

□

Remark 2. For $r = 1$ and $z = -1$ in Corollary 1 we obtain the following identity
$$JS(n+1,k+1;-1) = JS(n,k;1).$$

Corollary 2. *For non-negative integers n, r, the r-Jacobi-Stirling numbers satisfy*
$$(X + r(r+z))^n = \sum_{k=0}^{n} JS_r(n+r,k+r;z)\,(X)_{k,z+2r}.$$

Proof. From the first identity of Corollary 1 and identity (1.8) we obtain
$$\begin{aligned}
&\sum_{k=0}^{n} JS_r(n+r,k+r;z)\,(X)_{k,z+2r} \\
&= \sum_{k=0}^{n}\left(\sum_{i=0}^{n}\binom{n}{i}(r(r+z))^i\, JS(n-i,k;z+2r)\right)(X)_{k,z+2r} \\
&= \sum_{i=0}^{n}\binom{n}{i}(r(r+z))^i\left(\sum_{k=0}^{n} JS(n-i,k;z+2r)\,(X)_{k,z+2r}\right) \\
&= \sum_{i=0}^{n}\binom{n}{i}(r(r+z))^i\, X^{n-i} \\
&= (X + r(r+z))^n.
\end{aligned}$$

□

Remark 3. 1) Since the Jacobi-Stirling number $JS(n,k;z)$ is a polynomial in z of degree $n-k$ whose leading coefficient is the Stirling number of the second kind $S(n,k)$, it follows from Corollary 2 that the r-Jacobi-Stirling number $JS_r(n+r,k+r;z)$ is also a polynomial in z of degree $n-k$ with leader coefficient the r-Stirling number of the second kind $S_r(n,k)$.
2) Corollary 2 can also be obtained by replacing x_i with $(i+r)(i+r+z)$ and X with $X + r(r+z)$ in the Newton interpolation formula,
$$X^n = \sum_{k=0}^{n}\left(\sum_{j=0}^{k} x_j^n\left(\prod_{l=0,l\neq j}^{k}(x_j - x_l)\right)^{-1}\right)\prod_{i=0}^{k-1}(X - x_i).$$

Theorem 3. *For every positive integer n and every real number $z \geq -1$, the polynomial*
$$P_n(x;r) = \sum_{k=0}^{n} JS_r(n+r,k+r;z)\,x^k$$
has only real simple non-positive zeros, and then, the r-Jacobi-Stirling numbers of the second kind are strictly log-concave (thus unimodal).

Proof. For $r=0$ the theorem is Theorem 4 given in [6]. For $r\geq 1$, since the coefficient of x^{n} in $P_{n}(x)$ is $JS_{r}(n+r,n+r;z)=1$, P_{n} is a polynomial of degree n. For $n=1$, $P_{1}(x)=1+x$ has a real negative root. Assume $P_{n}(x)$ has only real negative simple roots, $n\geq 2$. Then, from Proposition 2, we get

$$\begin{aligned}
&P_{n+1}(x;r)\\
&=\sum_{k=0}^{n+1}JS_{r}(n+1+r,k+r;z)\,x^{k}\\
&=\sum_{k=0}^{n+1}\left(JS_{r}(n+r,k+r-1;z)+(k+r)(k+r+z)\,JS_{r}(n+r,k+r;z)\right)x^{k}\\
&=xP_{n}(x;r)+x^{1-r}D\left(x^{1-z}D\left(x^{r+z}\sum_{k=0}^{n}JS_{r}(n+r,k+r;z)\,x^{k}\right)\right)\\
&=xP_{n}(x;r)+x^{1-r}D\left(x^{1-z}D\left(x^{r+z}P_{n}(x;r)\right)\right),
\end{aligned}$$

and by setting $Q_{n}(x)=x^{r}P_{n}(x;r)$, the last identity becomes

$$Q_{n+1}(x)=x\left(Q_{n}(x)+D\left(x^{1-z}D\left(x^{z}Q_{n}(x)\right)\right)\right),\quad r\geq 1.$$

By induction, the polynomial $Q_{n}(x)$ satisfies all the assumptions of Lemma 2 given in [6], so $Q_{n+1}(x)$ has only real negative simple zeros and the root $x=0$ with multiplicity r. This shows that $P_{n+1}(x;r)$ has only real negative simple zeros. On using Newton's inequality [5, p. 52], it follows that the sequence $(JS_{r}(n+r,k+r;z),\ 0\leq k\leq n)$ is strongly log-concave. □

Acknowledgement

The authors thank the anonymous referee for his/her careful reading and valuable suggestions that led to an improved version of this paper.

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