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NONEXISTENCE OF $2-(v,k,1)$ DESIGNS ADMITTING AUTOMORPHISM GROUPS WITH SOCLE $E_8(q)$

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Abstract. One of the outstanding problems in combinatorial design theory is concerning the existence of $2-(v,k,1)$ designs. In particular, the existence of $2-(v,k,1)$ designs admitting an interesting group of automorphisms is of great interest. Thirty years ago, a six-person team classified $2-(v,k,1)$ designs which have flag-transitive automorphism groups. Since then the effort has been to classify those $2-(v,k,1)$ designs which are block-transitive but not flag-transitive. This paper is a contribution to this program and we prove there is nonexistence of $2-(v,k,1)$ designs admitting a point-primitive block-transitive but not flag-transitive automorphism group G with socle $E_8(q)$.

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1. INTRODUCTION

This paper is part of a project to classify groups and $2-(v,k,1)$ designs where the group acts transitively on the blocks of the design. A $2-(v,k,1)$ design $\mathcal{D}=(\mathcal{P},\mathcal{B})$ is a pair consisting of a finite set $\mathcal{P}$ of points and a collection $\mathcal{B}$ of $k-$subsets of $\mathcal{P}$, called blocks, such that any 2-subsets of $\mathcal{P}$ is contained in exactly one block. Traditionally one defined $v=:|\mathcal{P}|$ and $b=:|\mathcal{B}|$. We will always assume that $2<k<v$.

One of the outstanding problems in combinatorial design theory is concerning the existence of $2-(v,k,1)$ designs. In particular, the existence of $2-(v,k,1)$ designs admitting an interesting group of automorphisms is of great interest. Thirty years ago, a six-person team [2] classified the pairs $(\mathcal{D},G)$ where $\mathcal{D}$ is a $2-(v,k,1)$ design and G is a flag-transitive automorphism group of $\mathcal{D}$, with the exception of those in which G is a one-dimensional affine group. Since then the effort has been to classify those $2-(v,k,1)$ designs which are block-transitive but not flag-transitive. These fall naturally into two classes, those where the action on points is primitive and those where the action on points is imprimitive. The primitive ones are now subdivided,

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according to the O'Nan-Scotte theorem and some further work by Camina, into the socles which are either elementary abelian or non-abelian simple. As a result of [6] it is known that the second only occur finitely times for a given line size. This paper contributes to the program for determining the pairs $(\mathcal{D}, G)$ in which $\mathcal{D}$ has a point-primitive block-transitive subgroup, G, of automorphisms. From the assumption that G is transitive on the set $\mathcal{B}$ of blocks, it follows that G is also transitive on the point set $\mathcal{P}$. This is a consequence of the theorem of Block in [1].

The classification of block-transitive $2-(v,3,1)$ designs was completed about thirty years ago (see [4]). In [3] Camina and Siemons classified $2-(v,4,1)$ designs with a block-transitive, solvable group of automorphisms. Li classified $2-(v,4,1)$ designs admitting a block-transitive, unsolvable group of automorphisms (see [11]). Tong and Li classified $2-(v,5,1)$ designs with a block-transitive, solvable group of automorphisms in [19]. Liu classified $2-(v,k,1)$ (where $k = 6,7,8,9,10$) designs with a block-transitive, solvable group of automorphisms in [16]. Ding [8] considered $2-(v,k,1)$ designs admitting block-transitive automorphism groups in $AGL(1,q)$ and prove the existence of $2-(v,6,1)$ designs which have block-transitive but not flag-transitive automorphism groups in $AGL(1,q)$ (see [7]). Dai and Zhao consider $2-(v,13,1)$ designs with point-primitive block-transitive unsolvable group of automorphisms whose socle is $Sz(2^{2n+1})$ in [5]. Recently, there have been a number contributions to this classification (see [13, 14]). Here we focus on the existence problem of $2-(v,k,1)$ ($k \leq 2793$) designs with a point-primitive block-transitive automorphism group of almost simple type and prove the following theorem:

Theorem 1. *Suppose that $E_8(q) \trianglelefteq G \leq Aut(E_8(q))$ for $q > 5$. Then there is nonexistence of $2-(v,k,1)$ ($k \leq 2793$) design $\mathcal{D}$ admitting a point-primitive block-transitive but not flag-transitive automorphism group G.*

We introduce some notation below. Let X and Y be arbitrary finite groups. The expression $X \cdot Y$ denotes an extension of X by Y and $X : Y$ denotes the split extension. If Y is a subgroup of X, then the symbol $|X : Y|$ denotes the index of Y in X. Let $\mathcal{D}$ be a $2-(v,k,1)$ design and G be an automorphism group of $\mathcal{D}$. If B is a block, then G_B denotes the setwise stabilizer of B in G and $G_{(B)}$ is the pointwise stabilizer of B in G. In addition, G^B denotes the permutation group induced by the action of G_B on the points of B. Then $G^B \cong G_B/G_{(B)}$. We will write α to be a point of $\mathcal{D}$ and G_α to be the stabilizer of α under the action of G. Other notation for group structure is standard.

The paper is organized as follows. Section 2 describes several preliminary results concerning the group $E_8(q)$ and $2-(v,k,1)$ designs. Section 3 gives the proof of the theorem.

2. PRELIMINARY RESULTS

Suppose that G is a block-transitive automorphism group of a $2-(v,k,1)$ design. It is well-known that:

$$v = r(k-1)+1; \tag{2.1}$$

$$v(v-1) = bk(k-1). \tag{2.2}$$

Then we have $r = (v-1)/(k-1)$. We can show that $b \geq v$ and so $k \leq r$. If $k = r$ then $v = k^2 - k + 1$; if $r \geq k+1$, then $v \geq k^2$.

We use a result of W. Fang and H. Li [9]. Define the following constants:

$$b_1 = (b,v),\ b_2 = (b,v-1),\ k_1 = (k,v),\ \text{and}\ k_2 = (k,v-1).$$

Using the basic equalities 2.1 and 2.2, we get the Fang-Li Equations:

$$k = k_1k_2,\ b = b_1b_2,\ r = b_2k_2,\ \text{and}\ v = b_1k_1.$$

We shall state a number of basic results which will be used repeatedly throughout the paper. Liebeck and Saxl have determined the maximal subgroups of $Soc(G) = E_8(q)$ in [15].

Lemma 1 ([15]). *Suppose that $T = E_8(q) \trianglelefteq G \leq Aut(T)$. Let M be a maximal subgroup of G not containing T. Then one of the following holds*

(1) $|M| < q^{110}|G:T|$;
(2) *$M \cap T$ is a parabolic group;*
(3) *$M \cap T$ is isomorphic to $(SL_2(q) \circ E_7(q)).d$, $D_8(q).d$, or $E_8(q^{\frac{1}{2}})$ with q square, where $d = (2, q-1)$.*

Lemma 2 ([18]). *Let $G = T:\langle x \rangle$ and act block-transitively on a $2-(v,k,1)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, where $x \in Out(T)$. Then T acts transitively on $\mathcal{P}$.*

Lemma 3 ([17]). *Let G be a solvable block-transitive automorphism group of a $2-(v,k,1)$ design. If G is point-primitive, then*

(1) *there exists a prime number p and a positive integer n such that $v = p^n$;*
(2) *if there exists a p-primitive prime divisor e of $p^n - 1$, such that $e||G|$, then either $G \leq A\Gamma L(1, p^n)$ or $k|v$.*

Lemma 4 ([10]). *Let $\mathcal{D}$ be a $2-(v,k,1)$ design admitting a block-transitive and point-primitive but not flag-transitive automorphism group G. Assume that $T = Soc(G)$ and $T_\alpha = T \cap G_\alpha$ where $\alpha \in \mathcal{P}$. Then the following hold:*

(1) $\frac{v}{z} < (k_2k - k_2 + 1)|G:T|$, *where z is the size of a T_α–orbit in $\mathcal{P} \setminus \{\alpha\}$;*
(2) *if $(v-1, q) = 1$, then there exists a T_α-orbit with size y in $\mathcal{P} \setminus \{\alpha\}$ such that $y||T_\alpha|_{p'}$.*

Lemma 5. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design admitting a block-transitive automorphism group G. Assume that $T = Soc(G)$ and $T_\alpha = T \cap G_\alpha$ where $\alpha \in \mathcal{P}$. Then*

(1) $v = k_2(k-1)b_2 + 1$;

(2) *$b_2||T_\alpha|_{v'}|G:T|$ and $v \le 1+k(k-1)|T_\alpha|_{v'}|G:T|$;*
(3) *If G is not flag-transitive and non-solvable, then $\frac{|T|}{|T_\alpha|^2} \le \frac{k(k-1)+1}{2}|G:T|$.*

Proof. (1) Since $k(k-1)b = v(v-1)$ and $k = k_1k_2, b = b_1b_2, v = b_1k_1$, we obtain $k_2(k-1)b_2 = v-1$ and hence $v = 1+k_2(k-1)b_2$.

(2) Since $rv = bk$, it follows that $r|G:G_\alpha| = k|G:G_B|$, where $\alpha \in \mathcal{P}, B \in \mathcal{B}$. Recall that $k = k_1k_2, r = b_2k_2$. It is clear that $b_2|G_B| = k_1|G_\alpha|$. Note that $(b_2,k_1) = 1$ and hence b_2 divides $|G_\alpha|$. Since $(b_2,v) = 1$, then $b_2||G_\alpha|_{v'}$. Since G is block-transitive, by Lemma 2, T is point-transitive. We conclude that $v = |G:G_\alpha| = |T:T_\alpha|$. Hence $|G_\alpha| = |T_\alpha||G:T|$ and so $b_2||T_\alpha|_{v'}|G:T|$. Together with (1), it deduces that $v \le 1+k_2(k-1)|T_\alpha|_{v'}|G:T|$ and hence $v \le 1+k(k-1)|T_\alpha|_{v'}|G:T|$.

(3) Let B be a block of $\mathcal{D}$. Since G is non-solvable, the following possibility for the structure of G^B, the rank and subdegree of G does not occur:

Type of G^B	Rank of G	Subdegree of G
$\langle 1\rangle$	$1+k_2(k-1)$	$1, \overbrace{b_2, b_2, \cdots, b_2}^{k_2(k-1)}$

Otherwise, $|G^B|$ is odd, whence $|G|$ is odd and so G is solvable, which contradicts the fact that G is non-solvable. Then by the proof of Proposition 3.1 in [10] the conclusion holds. □

Lemma 6 ([12]). *Suppose that $\mathcal{D}$ is a $2-(v,k,1)$ design and G is an almost simple group acting on $\mathcal{D}$ block-transitively. Let G_α be the stabilizer in G of a point α of $\mathcal{D}$ and suppose the socle T of G is a simple group of Lie type. If the intersection of G_α and T is a parabolic subgroup of T, then G acts on $\mathcal{D}$ flag-transitively.*

3. Proof of Theorem 1

Suppose that there exists a $2-(v,k,1)$ $(k \le 2793)$ design $\mathcal{D}$ satisfying the conditions of the Main Theorem. We will derive contradictions to prove the Main Theorem.

Since $T = E_8(q) \trianglelefteq G \le Aut(E_8(q))$, then $G = T:\langle x\rangle$ and $|Out(T)| = a$, where $x \in Out(T)$. Let $o(x) = m$. Then we obtain that $m|a$ and $|G| = q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)m$. Since G is point-primitive, G_α is the maximal subgroup of G for any $\alpha \in \mathcal{P}$. Then $M = G_\alpha$ satisfies one of the three cases in Lemma 1. If $G_\alpha \cap T$ is a parabolic subgroup of T, then by Lemma 6 we see that G is flag-transitive, which is a contradiction. Therefore, the case (2) in Lemma 1 does not occur and it suffices to consider the following two cases.

Case 3.1: $|G_\alpha| < q^{110}|G:T|$.

Since G is block-transitive, by Lemma 2, T is point-transitive. Hence $|G_\alpha| = |T_\alpha||G:T|$ and so $|T_\alpha| < q^{110}$. Then $v = |T:T_\alpha|$ is not a prime power and by Lemma 3 we have that G is non-solvable. Note that $m = |G:T|$. It follows by

Lemma 5 (3) that

$$|T| \le \frac{k(k-1)+1}{2}|T_\alpha|^2|G:T| \le \frac{7798057}{2}q^{220}m.$$

This gives,

$$\frac{|T|}{q^{220}} = \frac{(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)}{q^{100}} < \frac{7798057}{2}m.$$

Since

$$(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1) > \frac{7}{10}q^{128},$$

it implies that

$$\frac{7}{10}q^8 < \frac{7798057}{2}m.$$

Recall that $m|a$, $q=p^a$, $p\ge 2$. We can conclude therefore that

$$\frac{7}{10}\cdot 2^{8a} \le \frac{7}{10}\cdot p^{8a} = \frac{7}{10}q^8 < \frac{7798057}{2}a, \tag{3.1}$$

which forces $a \le 2$. We calculate to obtain all possibilities for the values of p and a satisfying the inequality 3.1: (1) $a=1$, $p\le 5$, a prime; (2) $a=2$, $p=2$. This contradicts $q>5$.

Case 3.2: $G_\alpha \cap T$ is case (3) in Lemma 1.

Now we consider three cases.

Subcase 3.2.1: $T_\alpha = (SL_2(q)\circ E_7(q)).d$ where $d=(2,q-1)$.

We observe that

$$|T_\alpha| = q^{64}(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)(q^8-1)(q^6-1)(q^2-1)^2$$

and

$$v = \frac{q^{56}(q^{30}-1)(q^{24}-1)(q^{20}-1)}{(q^{10}-1)(q^6-1)(q^2-1)}.$$

Hence

$$|T_\alpha|_{v'} \le (q^2-1)^8(q^{12}+q^6+1)(1+q^2+q^4+q^6+q^8+q^{10}+q^{12}) < \frac{7}{5}q^{40}.$$

Since

$$v = \frac{q^{56}(q^{30}-1)(q^{24}-1)(q^{20}-1)}{(q^{10}-1)(q^6-1)(q^2-1)} > \frac{1}{50}q^{112},$$

we can appeal to Lemma 5 (2) to observe that

$$\frac{1}{50}q^{112} < v \le 1+k(k-1)|T_\alpha|_{v'}|G:T| < 1+7798056\cdot\frac{7}{5}\cdot q^{40}a.$$

This implies the following inequality

$$\frac{1}{50}\cdot 2^{72a} \le \frac{1}{50}\cdot q^{72} < \frac{1}{2^{40a}} + 7798056\cdot\frac{7}{5}\cdot a < \frac{4}{5}\cdot 2^{24}a,$$

which is impossible.

Subcase 3.2.2: $T_\alpha = D_8(q).d$, where $d = (2, q-1)$.

We calculate that

$$|T_\alpha| = \frac{dq^{56}(q^8-1)\prod_{i=1}^{7}(q^{2i}-1)}{d_1}$$

and

$$v = \frac{d_1 q^{64}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)}{d(q^{10}-1)(q^8-1)(q^6-1)(q^4-1)},$$

where $d_1 = (4, q^8-1)$. Since $(v-1, q) = 1$, by Lemma 4 (2), there exists in $\mathcal{P}\setminus\{\alpha\}$ a T_α−orbit of size y such that $y \,||\, |T_\alpha|_{p'}$. Hence

$$y \le |T_\alpha|_{p'} \le 2(q^8-1)\prod_{i=1}^{7}(q^{2i}-1).$$

Thus

$$\begin{aligned}\frac{v}{y} &\ge \frac{d_1 q^{64}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)}{2d(q^{14}-1)(q^{12}-1)(q^{10}-1)^2(q^8-1)^3(q^6-1)^2(q^4-1)^2(q^2-1)}\\ &> \frac{\frac{1}{10}\cdot q^{108}}{4\cdot\frac{15}{2}\cdot q^{44}} = \frac{1}{300}q^{64}.\end{aligned}$$

Note that $k_2 \le k$. We now apply Lemma 4 (1) to conclude that

$$\frac{1}{300}\cdot 2^{64a} \le \frac{1}{300}q^{64} < \frac{v}{y} < (k(k-1)+1)|G:T| \le 7798057a < \frac{19}{20}\cdot 2^{23}a,$$

which is a contradiction.

Subcase 3.2.3: $T_\alpha = E_8(q^{\frac{1}{2}})$.

We obtain that

$$|T_\alpha| = q^{60}(q^{15}-1)(q^{12}-1)(q^{10}-1)(q^9-1)(q^7-1)(q^6-1)(q^4-1)(q-1)$$

and

$$v = q^{60}(q^{15}+1)(q^{12}+1)(q^{10}+1)(q^9+1)(q^7+1)(q^6+1)(q^4+1)(q+1).$$

Then it deduces that

$$\begin{aligned}|T_\alpha|_{v'} &\le (q-1)^8(q^2+q+1)^4(q^6+q^3+1)(1+q+q^2+q^3+q^4)^2\\ &\cdot(1+q+q^2+q^3+q^4+q^5+q^6)(1-q+q^3-q^4+q^5-q^7+q^8) < 48q^{44}.\end{aligned}$$

Since

$$v = q^{60}(q^{15}+1)(q^{12}+1)(q^{10}+1)(q^9+1)(q^7+1)(q^6+1)(q^4+1)(q+1) > q^{124},$$

by Lemma 5 (2) this implies that

$$q^{124} < v \leq 1 + k(k-1)|T_\alpha|_{v'}|G:T| < 1 + 7798056 \cdot 48 \cdot q^{44} \cdot a.$$

This leads to the following result

$$2^{80a} \leq q^{80} < \frac{1}{2^{44a}} + 7798056 \cdot 48a < \frac{4}{5} \cdot 2^{29} a,$$

which gives a contradiction.

This completes the proof of **Theorem 1**. □

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