Miskolc Mathematical Notes
Vol. 16 (2015), No. 2, pp. 1041–1053
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2015.1456

THE EXPLICIT FORMULAS FOR REPRODUCING KERNEL OF SOME HILBERT SPACES

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Received 06 December, 2014

Abstract. This paper is devoted to finding explicit formulas for two kinds of reproducing kernels in some reproducing kernel spaces. Usually for calculating the reproducing kernel of space $W_2^m[a,b]$, one must solve a large system of ordinary differential equations with $4m$ equations in $4m$ unknowns. Hence, it is worthy to have explicit formulas to save time and computational efforts. The advantage of using these formulas is clearly seen in the plotted graphs.

2010 *Mathematics Subject Classification:* 46E22; 65L05

Keywords: system of ordinary differential equations, reproducing kernel, explicit formula

1. INTRODUCTION

In recent decades, the reproducing kernel Hilbert space method has been considered by many authors. Reproducing kernel Hilbert space method is an effective method for solving many problems, including ordinary, partial, integro-differential, integral equations, etc. [1–25].
To obtain reproducing kernel we have to perform complex calculations at high volumes. So we decided to introduce, in this paper, explicit formulas to calculate unknown coefficients of the reproducing kernel. That will save time and reduce the high cost of computing. First, we divide reproducing kernel spaces in terms of the types of problems, different initial and boundary conditions, the inner product, and the necessary norm on the environment. The first class (i.e. $W_2^m[a,b]$) is intended for problems without initial and boundary conditions and the second class (i.e. $W_{2,a}^{n+1}[a,b]$) is devoted to the problems with the initial conditions. The findings of this article for readers, especially for users of reproducing kernel method, is helpful in time saving.
The structure of this paper is as follows. In Section 2 we introduce some reproducing kernel spaces. We devote Section 3 to finding explicit formulas for reproducing kernels. We end the paper with conclusions.

2. Reproducing kernel Hilbert spaces

In this section, we first define the reproducing kernel and reproducing kernel space, and then we introduce some reproducing kernel spaces.

Definition 1 ([1,4]). Let

$$\mathcal{H} = \{f \mid f\ is\ a\ real\ valued\ function\ on\ \mathcal{X},\ \mathcal{X}\ is\ an\ abstract\ set\}$$

be a Hilbert space, with inner product

$$\langle f(.), g(.)\rangle_{\mathcal{H}}, \quad f,\ g \in \mathcal{H}.$$

If there exists a function $K(x,.) \in \mathcal{H}$ for each fixed $x \in \mathcal{X}$ and for any $f \in \mathcal{H}$

$$\langle f(.), K(x,.)\rangle_{\mathcal{H}} = f(x),$$

then $K(x,.)$ is called the reproducing kernel of $\mathcal{H}$ and Hilbert space $\mathcal{H}$ is called the reproducing kernel space.

For solving some functional equations such as integral and differential equations using the reproducing kernel Hilbert space method, we use the reproducing kernel space $W_2^m[a,b]$ defined as follows.

Definition 2 ([4]). For $m \in \mathbb{N}$,

$$W_2^m[a,b] = \{u | u^{(m-1)}\ is\ an\ absolutely\ continuous\ function, u^{(m)} \in L^2[a,b]\}.$$

The inner product and the norm in the function space $W_2^m[a,b]$ are defined as

$$\langle u, v\rangle_m = \sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}(\tau)v^{(m)}(\tau)\,d\tau, \quad u, v \in W_2^m[a,b]$$

$$||u||_m = \sqrt{\langle u, u\rangle_m}, \quad u \in W_2^m[a,b].$$

To solve initial value problems including ordinary differential equations, integro-differential equations and differential equations of fractional order, we define the reproducing kernel space $W_{2,a}^{n+1}[a,b]$ as follows.

Definition 3. For $n \in \mathbb{N}$,

$$W_{2,a}^{n+1}[a,b] = \{u \in W_2^{n+1}[a,b] | u^{(j)}(a) = 0,\ j = 0, \dots, n-1\}.$$

The inner product and the norm in the function space $W_{2,a}^{n+1}[a,b]$ are defined as

$$\langle u, v\rangle_{n,a} = u^{(n)}(a)v^{(n)}(a) + \int_a^b u^{(n+1)}(\tau)v^{(n+1)}(\tau)\,d\tau, \quad u, v \in W_{2,a}^{n+1}[a,b]$$

$$||u||_{n,a} = \sqrt{\langle u, u\rangle_{n,a}}, \quad u \in W_{2,a}^{n+1}[a,b].$$

3. EXPLICIT FORMULAS FOR REPRODUCING KERNELS

In this section we investigate explicit formula for each one of the reproducing kernel Hilbert spaces introduced in Definitions 2 and 3.

3.1. *Reproducing kernel of the space* $W_2^m[a,b]$

Assume that function $K^{\{m\}}(x,t) \in W_2^m[a,b]$ satisfies the following generalized differential equation

$$\begin{cases} (-1)^m \frac{\partial^{2m} K^{\{m\}}(x,t)}{\partial t^{2m}} = \delta(t-x), \\ \frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}|_{t=a} = (-1)^{m-j-1} \frac{\partial^{2m-j-1} K^{\{m\}}(x,t)}{\partial t^{2m-j-1}}|_{t=a}, \\ \frac{\partial^{m+j} K^{\{m\}}(x,t)}{\partial t^{m+j}}|_{t=b} = 0, \ 0 \le j < m. \end{cases} \tag{3.1}$$

where δ is the Dirac delta function. The following theorem holds.

Theorem 1 ([4]). *Under the assumptions of Eq. (3.1), Hilbert space $W_2^m[a,b]$ is a reproducing kernel Hilbert space with the reproducing kernel function $K^{\{m\}}(x,.)$, namely for each $u \in W_2^m[a,b]$ and any fixed $x \in [a,b]$, it follows that*

$$\langle u(.), K^{\{m\}}(x,.)\rangle_m = u(x).$$

While $x \neq t$, function $K^{\{m\}}(x,t)$ is the solution of the following linear homogeneous differential equation of order $2m$,

$$(-1)^m \frac{\partial^{2m} K^{\{m\}}(x,t)}{\partial t^{2m}} = 0, \tag{3.2}$$

with the boundary conditions:

$$\begin{cases} \frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}|_{t=a} = (-1)^{m-j-1} \frac{\partial^{2m-j-1} K^{\{m\}}(x,t)}{\partial t^{2m-j-1}}|_{t=a}, \\ \frac{\partial^{m+j} K^{\{m\}}(x,t)}{\partial t^{m+j}}|_{t=b} = 0, \ 0 \le j < m. \end{cases} \tag{3.3}$$

We know that Eq. (3.2) has characteristic equation $\lambda^{2m} = 0$, and the eigenvalue $\lambda = 0$ is a root with multiplicity $2m$. Hence, the general solution of the system (3.1) is

$$K^{\{m\}}(x,t) = \begin{cases} \sum_{i=1}^{2m} c_i(x)(t-a)^{i-1}, & t \le x, \\ \sum_{i=1}^{2m} d_i(x)(t-a)^{i-1}, & t > x. \end{cases} \tag{3.4}$$

Now, we are ready to calculate the coefficients $c_i(x)$ and $d_i(x)$, $i = 1, \ldots, 2m$. Since

$$(-1)^m \frac{\partial^{2m} K^{\{m\}}(x,t)}{\partial t^{2m}} = \delta(t-x),$$

we have

$$\begin{cases} \frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}\Big|_{t=x^-}^{t=x^+} = 0, & j = 0, \ldots, 2m-2, \\ \frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}\Big|_{t=x^-}^{t=x^+} = (-1)^m, & j = 2m-1. \end{cases} \tag{3.5}$$

Therefore, using Eqs. (3.1) and (3.5), we have a system of $4m$ equations with $4m$ unknowns ($c_i(x)$ and $d_i(x)$, $i = 1,\dots,2m$). By solving this system the unknown coefficients of Eq. (3.4) are uniquely determined. Solving the above system needs high CPU time.

We have obtained the following formula for calculating the reproducing kernel of $W_2^m[a,b]$.

Theorem 2. *Let*

$$K^{\{m\}}(x,t) = \begin{cases} R^{\{m\}}(x,t), & t \le x, \\ L^{\{m\}}(x,t), & t > x, \end{cases}$$

be the reproducing kernel function of the reproducing kernel Hilbert space $W_2^m[a,b]$. *Then,* $R^{\{m\}}(x,t)$ *is given by the following formula*

$$R^{\{m\}}(x,t) = \sum_{i=0}^{m-1}\sum_{j=0}^{1}\left((-1)^{ij}\frac{(x-a)^{i+(m-2i-1)j}(t-a)^{mj+i}}{(i+(m-2i-1)j)!(mj+i)!}\right),$$

and $L^{\{m\}}(x,t) = R^{\{m\}}(t,x)$.

Proof of Theorem 2. It is enough to prove that the function $K^{\{m\}}(x,t)$ satisfies Eqs. (3.2), (3.3) and (3.5). For this purpose we first rewrite $R^{\{m\}}(x,t)$ as

$$R^{\{m\}}(x,t) = \sum_{i=0}^{m-1}\frac{(x-a)^i}{i!}\frac{(t-a)^i}{i!} + \sum_{i=m}^{2m-1}(-1)^{i-m}\frac{(x-a)^{2m-i-1}}{(2m-i-1)!}\frac{(t-a)^i}{i!}.$$

For $j = 0,\dots,2m$, the j^{th}-derivative of the function $R^{\{m\}}(x,t)$ with respect to t is computed as

$$\frac{\partial^j R^{\{m\}}(x,t)}{\partial t^j} = \begin{cases} \sum_{i=j}^{m-1}\frac{(x-a)^i}{i!}\frac{(t-a)^{i-j}}{(i-j)!} + \sum_{i=m}^{2m-1}(-1)^{i-m}\frac{(x-a)^{2m-i-1}}{(2m-i-1)!}\frac{(t-a)^{i-j}}{(i-j)!}, & 0 \le j < m, \\ \sum_{i=j}^{2m-1}(-1)^{i-m}\frac{(x-a)^{2m-i-1}}{(2m-i-1)!}\frac{(t-a)^{i-j}}{(i-j)!}, & m \le j < 2m, \\ 0, & j = 2m. \end{cases} \tag{3.6}$$

Hence,

$$\left.\frac{\partial^j R^{\{m\}}(x,t)}{\partial t^j}\right|_{t=a} = \begin{cases} \frac{(x-a)^j}{j!}, & j = 0,\dots,m-1, \\ (-1)^{j-m}\frac{(x-a)^{2m-j-1}}{(2m-j-1)!}, & j = m,\dots,2m-1, \end{cases} \tag{3.7}$$

$$\left.\frac{\partial^j R^{\{m\}}(x,t)}{\partial t^j}\right|_{t=x} = \begin{cases} \sum_{i=j}^{m-1}\frac{(x-a)^{2i-j}}{i!(i-j)!} + \sum_{i=m}^{2m-1}(-1)^{i-m}\frac{(x-a)^{2m-j-1}}{(2m-i-1)!(i-j)!}, & 0 \le j < m, \\ \sum_{i=j}^{2m-1}(-1)^{i-m}\frac{(x-a)^{2m-j-1}}{(2m-i-1)!(i-j)!}, & m \le< 2m. \end{cases} \tag{3.8}$$

Also, we rewrite $L^{\{m\}}(x,t)$ as

$$L^{\{m\}}(x,t) = R^{\{m\}}(t,x) = \sum_{i=0}^{m-1}\left[\frac{(x-a)^i}{i!} + (-1)^{m-i-1}\frac{(x-a)^{2m-i-1}}{(2m-i-1)!}\right]\frac{(t-a)^i}{i!}.$$

Furthermore, for $j = 0,\dots,2m$, the j^{th}-derivative of the function $L^{\{m\}}(x,t)$ with respect to t is

$$\frac{\partial^j L^{\{m\}}(x,t)}{\partial t^j} = \begin{cases} \sum_{i=j}^{m-1}\left(\frac{(x-a)^i}{i!} + (-1)^{m-i-1}\frac{(x-a)^{2m-i-1}}{(2m-i-1)!}\right)\frac{(t-a)^{i-j}}{(i-j)!}, & 0 \le j < m, \\ 0, & m \le j \le 2m. \end{cases} \tag{3.9}$$

Therefore,

$$\left.\frac{\partial^j L^{\{m\}}(x,t)}{\partial t^j}\right|_{t=b} = \begin{cases} \sum_{i=j}^{m-1}\left(\frac{(x-a)^i}{i!} + (-1)^{m-i-1}\frac{(x-a)^{2m-i-1}}{(2m-i-1)!}\right)\frac{(b-a)^{i-j}}{(i-j)!}, & 0 \le j < m, \\ 0, & m \le j < 2m. \end{cases} \tag{3.10}$$

$$\left.\frac{\partial^j L^{\{m\}}(x,t)}{\partial t^j}\right|_{t=x} = \begin{cases} \sum_{i=j}^{m-1}\left(\frac{(x-a)^{2i-j}}{i!(i-j)!} + (-1)^{m-i-1}\frac{(x-a)^{2m-j-1}}{(2m-i-1)!(i-j)!}\right), & 0 \le j < m, \\ 0, & m \le j < 2m. \end{cases} \tag{3.11}$$

Thus, by Eqs. (3.6) and (3.9), $\frac{\partial^{2m} K^{\{m\}}(x,t)}{\partial t^{2m}} = 0$. Also, Eq. (3.7) implies

$$\begin{aligned} &\left.\frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}\right|_{t=a} - (-1)^{m-j-1}\left.\frac{\partial^{2m-j-1} K^{\{m\}}(x,t)}{\partial t^{2m-j-1}}\right|_{t=a} \\ &\quad= \left.\frac{\partial^j R^{\{m\}}(x,t)}{\partial t^j}\right|_{t=a} - (-1)^{m-j-1}\left.\frac{\partial^{2m-j-1} R^{\{m\}}(x,t)}{\partial t^{2m-j-1}}\right|_{t=a} \\ &\quad= \frac{(x-a)^j}{j!} - (-1)^{m-j-1}(-1)^{(2m-j-1)-m}\frac{(x-a)^{2m-(2m-j-1)-1}}{(2m-(2m-j-1)-1)!} \\ &\quad= \frac{(x-a)^j}{j!} - (-1)^{m-j-1}(-1)^{m-j-1}\frac{(x-a)^j}{j!} = 0, \quad j = 0,\ldots,m-1, \end{aligned} \tag{3.12}$$

and Eq. (3.10) implies

$$\frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}|_{t=b} = \frac{\partial^j L^{\{m\}}(x,t)}{\partial t^j}|_{t=b} = 0, j = m,\ldots,2m-1. \tag{3.13}$$

According to Eqs. (3.12) and (3.13), Eq. (3.3) is proved. Also, using Eqs. (3.8) and (3.11) we have

$$\begin{aligned} \frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}|_{t=x^+} &= \frac{\partial^j L^{\{m\}}(x,t)}{\partial t^j}|_{t=x} \\ &= \begin{cases} 0, & m \le j < 2m, \\ \sum_{i=j}^{m-1}\left(\frac{(x-a)^{2i-j}}{i!(i-j)!} + (-1)^{m-i-1}\frac{(x-a)^{2m-j-1}}{(2m-i-1)!(i-j)!}\right), & 0 \le j < m, \end{cases} \end{aligned} \tag{3.14}$$

$$\begin{aligned} \frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}|_{t=x^-} &= \frac{\partial^j R^{\{m\}}(x,t)}{\partial t^j}|_{t=x} \\ &= \begin{cases} \sum_{i=j}^{m-1}\frac{(x-a)^{2i-j}}{i!(i-j)!} + \sum_{i=m}^{2m-1}(-1)^{i-m}\frac{(x-a)^{2m-j-1}}{(2m-j-1)!(i-j)!}, & 0 \le j < m, \\ \sum_{i=j}^{2m-1}(-1)^{i-m}\frac{(x-a)^{2m-j-1}}{(2m-j-1)!(i-j)!}, & m \le j < 2m. \end{cases} \end{aligned} \tag{3.15}$$

TABLE 1. Explicit expressions of $R^{\{m\}}(x,t)$ for $m=1,\ldots,5$

m	$R^{\{m\}}(x,t)$
1	$1-a+t$
2	$1+(x-a)(t-a)+(x-a)(t-a)^2/2-(t-a)^3/6$
3	$1+(x-a)(t-a)+(x-a)^2(t-a)^2/4+(x-a)^2(t-a)^3/12$ $-(x-a)(t-a)^4/24+(t-a)^5/120$
4	$1+(x-a)(t-a)+(x-a)^2(t-a)^2/4+(x-a)^3(t-a)^3/36+(x-a)^3(t-a)^4/144$ $-(x-a)^2(t-a)^5/240+(x-a)(t-a)^6/720-(t-a)^7/5040$
5	$1+(x-a)(t-a)+(x-a)^2(t-a)^2/4+(x-a)^3(t-a)^3/36+(x-a)^4(t-a)^4/576$ $+(x-a)^4(t-a)^5/2880-(x-a)^3(t-a)^6/4320+(x-a)^2(t-a)^7/10080$ $-(x-a)(t-a)^8/40320+(t-a)^9/362880$

So, Eqs. (3.14) and (3.15) imply that

$$\frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}\Big|_{t=x+}-\frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}\Big|_{t=x-}$$
$$=\begin{cases}\sum_{i=j}^{m-1}\frac{(-1)^{m+i-1}(x-a)^{2m-j-1}}{(2m-i-1)!(i-j)!}+\sum_{i=m}^{2m-1}\frac{(-1)^{m+i-1}(x-a)^{2m-j-1}}{(2m-i-1)!(i-j)!}, & 0\le j<m,\\ (x-a)^{2m-j-1}\sum_{i=j}^{2m-1}\frac{(-1)^{m+i-1}}{(2m-i-1)!(i-j)!}, & m\le j<2m-1,\\ (-1)^m, & j=2m-1.\end{cases}$$
$$=\begin{cases}(-1)^{m-1}(x-a)^{2m-j-1}\sum_{i=j}^{2m-1}\frac{(-1)^i}{(2m-i-1)!(i-j)!}, & 0\le j\le 2m-2,\\ (-1)^m, & j=2m-1.\end{cases}$$
$$=\begin{cases}\frac{(-1)^{m+j-1}(x-a)^{2m-j-1}}{(2m-j-1)!}\sum_{i=0}^{2m-j-1}(-1)^i\binom{2m-j-1}{i}, & 0\le j\le 2m-2,\\ (-1)^m, & j=2m-1.\end{cases}$$
$$=\begin{cases}(-1)^{m+j-1}(x-a)^{2m-j-1}(1-1)^{2m-j-1}, & 0\le j\le 2m-2,\\ (-1)^m, & j=2m-1.\end{cases}$$
$$=\begin{cases}0, & 0\le j\le 2m-2,\\ (-1)^m, & j=2m-1.\end{cases}$$

□

Explicit expressions of the reproducing kernel functions in $W_2^m[a,b]$ for $1\le m\le 5$ are shown in Table 1.

3.2. *Reproducing kernel of the space* $W_{2,a}^{n+1}[a,b]$

Suppose that function $K_a^{\{n+1\}}(x,t)\in W_{2,a}^{n+1}[a,b]$ satisfies the following generalized differential equations

$$\begin{cases}(-1)^{n+1}\frac{\partial^{2n+2}K_a^{\{n+1\}}(x,t)}{\partial t^{2n+2}}=\delta(t-x),\\ \frac{\partial^{n+1}K_a^{\{n+1\}}(x,t)}{\partial t^{n+1}}\Big|_{t=a}=\frac{\partial^{n}K^{\{n+1\}}(x,t)}{\partial t^{n}}\Big|_{t=a}, & 0\le j\le n,\\ \frac{\partial^{j}K_a^{\{n+1\}}(x,t)}{\partial t^{j}}\Big|_{t=a}=0, & 0\le j\le n,\\ \frac{\partial^{j}K_a^{\{n+1\}}(x,t)}{\partial t^{j}}\Big|_{t=b}=0, & n<j<2n,\end{cases}\tag{3.16}$$

where δ is the Dirac delta function. Similar to Theorem 1, we have the following theorem.

Theorem 3. *Under the assumptions of Eq. (3.16), Hilbert space $W_{2,a}^{n+1}[a,b]$ is a reproducing kernel Hilbert space with the reproducing kernel function $K_a^{\{n+1\}}(x,.)$, namely for each $u \in W_{2,a}^{n+1}[a,b]$ and any fixed $x \in [a,b]$, it follows that*

$$\langle u(.), K_a^{\{n+1\}}(x,.)\rangle_{n,a} = u(x).$$

While $x \neq t$, function $K_a^{\{n+1\}}(x,t)$ is the solution of the following linear homogeneous differential equation of order $2n+2$,

$$(-1)^{n+1}\frac{\partial^{2n+2}K_a^{\{n+1\}}(x,t)}{\partial t^{2n+2}} = 0, \tag{3.17}$$

with the boundary conditions:

$$\begin{cases} \frac{\partial^{n+1}K_a^{\{n+1\}}(x,t)}{\partial t^{n+1}}\Big|_{t=a} = \frac{\partial^{n}K^{\{n+1\}}(x,t)}{\partial t^{n}}\Big|_{t=a}, & 0 \le j \le n, \\ \frac{\partial^{j}K_a^{\{n+1\}}(x,t)}{\partial t^{j}}\Big|_{t=a} = 0, & 0 \le j \le n, \\ \frac{\partial^{j}K_a^{\{n+1\}}(x,t)}{\partial t^{j}}\Big|_{t=b} = 0, & n < j < 2n. \end{cases} \tag{3.18}$$

We know that Eq. (3.17) has characteristic equation $\lambda^{2n+2} = 0$, and the eigenvalue $\lambda = 0$ is a root with multiplicity $2n+2$. Hence, the general solution of the system (3.16) is

$$K_a^{\{n+1\}}(x,t) = \begin{cases} \sum_{i=1}^{2n+2}\alpha_i(x)(t-a)^{i-1}, & t \le x, \\ \sum_{i=1}^{2n+2}\beta_i(x)(t-a)^{i-1}, & t > x. \end{cases} \tag{3.19}$$

Now, we are ready to calculate the coefficients $\alpha_i(x)$ and $\beta_i(x)$, $i = 1,\ldots,2n+2$. Since

$$(-1)^{n+1}\frac{\partial^{2n+2}K_a^{\{n+1\}}(x,t)}{\partial t^{2n+2}} = \delta(t-x),$$

we have

$$\begin{cases} \frac{\partial^{j}K_a^{\{n+1\}}(x,t)}{\partial t^{j}}\Big|_{t=x^-}^{t=x^+} = 0, & j = 0,\ldots,2n, \\ \frac{\partial^{j}K_a^{\{n+1\}}(x,t)}{\partial t^{j}}\Big|_{t=x^-}^{t=x^+} = (-1)^{n+1}, & j = 2n+1. \end{cases} \tag{3.20}$$

Therefore, using Eqs. (3.16) and (3.20), we have a system of $4n+4$ equations with $4n+4$ unknowns ($\alpha_i(x)$ and $\beta_i(x)$, $i = 1,\ldots,2n+2$). By solving this system the unknown coefficients of Eq. (3.19) are uniquely obtained. Solving the above system needs high CPU time.

Now, we present the following formula for calculating the reproducing kernel of $W_{2,a}^{n+1}[a,b]$.

Theorem 4. *Let*

$$K_a^{\{n+1\}}(x,t) = \begin{cases} R_a^{\{n+1\}}(x,t), & t \le x, \\ L_a^{\{n+1\}}(x,t), & t > x, \end{cases}$$

be the reproducing kernel function of the reproducing kernel Hilbert space $W_{2,a}^{n+1}[a,b]$. *Then,* $R_a^{\{n+1\}}(x,t)$ *is given by the following formula.*

$$R_a^{\{n+1\}}(x,t) = \frac{(x-a)^n(t-a)^n}{n!n!} + \sum_{i=n+1}^{2n+1}\left((-1)^{i-n-1}\frac{(x-a)^{2n-i+1}(t-a)^i}{(2n-i+1)!i!}\right),$$

and $L_a^{\{n+1\}}(x,t) = R_a^{\{n+1\}}(t,x)$.

Proof of Theorem 4. It is enough to prove that the function $K_a^{\{n+1\}}(x,t)$ satisfies in Eqs. (3.17), (3.18) and (3.20). For $j = 0,\dots,2n+2$, the j^{th}-derivative of the function $R_a^{\{n+1\}}(x,t)$ with respect to t is computed as follows:

$$\frac{\partial^j R_a^{\{n+1\}}(x,t)}{\partial t^j} = \begin{cases} \frac{(x-a)^n}{n!}\frac{(t-a)^{n-j}}{(n-j)!} + \\ \sum_{i=n+1}^{2n+1}(-1)^{i-n-1}\frac{(x-a)^{2n-i+1}}{(2n-i+1)!}\frac{(t-a)^{i-j}}{(i-j)!}, & 0 \le j < n+1, \\ \sum_{i=j}^{2n+1}(-1)^{i-n-1}\frac{(x-a)^{2n-i+1}}{(2n-i+1)!}\frac{(t-a)^{i-j}}{(i-j)!}, & n < j < 2n+2, \\ 0, & j = 2n+2. \end{cases} \tag{3.21}$$

Then,

$$\frac{\partial^j R_a^{\{n+1\}}(x,t)}{\partial t^j}\Big|_{t=a} = \begin{cases} \frac{(x-a)^j}{j!}, & j = n, \\ (-1)^{j-n-1}\frac{(x-a)^{2n-j+1}}{(2n-j+1)!}, & n < j < 2n, \end{cases} \tag{3.22}$$

$$\frac{\partial^j R_a^{\{n+1\}}(x,t)}{\partial t^j}\Big|_{t=x} = \begin{cases} \frac{(x-a)^{2n-j}}{n!(n-j)!} + \\ \sum_{i=n+1}^{2n+1}(-1)^{i-n-1}\frac{(x-a)^{2n-j+1}}{(2n-i+1)!(i-j)!}, & 0 \le j \le n, \\ \sum_{i=j}^{2n+1}(-1)^{i-n-1}\frac{(x-a)^{2n-j+1}}{(2n-i+1)!(i-j)!}, & n < j < 2n. \end{cases} \tag{3.23}$$

For $L_a^{\{n+1\}}(x,t)$ we have

$$L_a^{\{n+1\}}(x,t) = R_a^{\{n+1\}}(t,x) = \frac{(x-a)^n}{n!}\frac{(t-a)^n}{n!} + \sum_{i=0}^{n}(-1)^{n-i}\frac{(x-a)^{2n-i+1}}{(2n-i+1)!}\frac{(t-a)^i}{i!}.$$

Furthermore, for $j = 0,\dots,2n+2$, the j^{th}-derivative of the function $L_a^{\{n+1\}}(x,t)$ with respect to t is computed as follows:

$$\frac{\partial^j L_a^{\{n+1\}}(x,t)}{\partial t^j} = \begin{cases} \frac{(x-a)^n}{n!}\frac{(t-a)^{n-j}}{(n-j)!} + \\ \sum_{i=j}^{n}(-1)^{n-i}\frac{(x-a)^{2n-i+1}}{(2n-i+1)!}\frac{(t-a)^{i-j}}{(i-j)!}, & 0 \le j < n+1, \\ 0, & n < j \le 2n+2. \end{cases} \tag{3.24}$$

Therefore,

$$\frac{\partial^j L_a^{\{n+1\}}(x,t)}{\partial t^j}\Big|_{t=b} = \begin{cases} \frac{(x-a)^n}{n!}\frac{(b-a)^{n-j}}{(n-j)!} + \\ \sum_{i=j}^{n}(-1)^{n-i}\frac{(x-a)^{2n-i+1}}{(2n-i+1)!}\frac{(b-a)^{i-j}}{(i-j)!}, & 0 \le j < n+1, \\ 0, & n < j \le 2n+2, \end{cases} \tag{3.25}$$

$$\frac{\partial^j L_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=x} = \begin{cases} \frac{(x-a)^{2n-j}}{n!(n-j)!} + \\ \sum_{i=j}^{n}(-1)^{n-i}\frac{(x-a)^{2n-j+1}}{(2n-i+1)!(i-j)!}, & 0 \le j < n+1, \\ 0, & n < j \le 2n+2. \end{cases} \tag{3.26}$$

Thus, Eqs. (3.21) and (3.24), it is clear that $\frac{\partial^{2n+2} K_a^{\{n+1\}}(x,t)}{\partial t^{2n+2}} = 0$. Also, Eq. (3.22) implies that

$$\begin{aligned} &\frac{\partial^j K_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=a} = \frac{\partial^j L_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=a} = 0, \quad j = 0,\dots,n-1, \\ &\frac{\partial^{n+1} K_a^{\{n+1\}}(x,t)}{\partial t^{n+1}}|_{t=a} - \frac{\partial^n K_a^{\{n+1\}}(x,t)}{\partial t^n}|_{t=a} = \\ &\frac{\partial^{n+1} L_a^{\{n+1\}}(x,t)}{\partial t^{n+1}}|_{t=a} - \frac{\partial^n L_a^{\{n+1\}}(x,t)}{\partial t^n}|_{t=a} = \frac{(x-a)^n}{n!} - \frac{(x-a)^n}{n!} = 0, \end{aligned} \tag{3.27}$$

and Eq. (3.25) implies that

$$\frac{\partial^j K_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=b} = \frac{\partial^j L_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=b} = 0, n < j < 2n. \tag{3.28}$$

According to Eqs. (3.27) and (3.28), Eq. (3.18) is proved. Also, using Eqs. (3.23) and (3.26) we have

$$\begin{aligned} \frac{\partial^j K_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=x^+} &= \frac{\partial^j L_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=x} \\ &= \begin{cases} 0, & n < j < 2n, \\ \frac{(x-a)^{2n-j}}{n!(n-j)!} + \sum_{i=j}^{n}(-1)^{n-i}\frac{(x-a)^{2n-j+1}}{(2n-i+1)!(i-j)!}, & 0 \le j \le n, \end{cases} \end{aligned} \tag{3.29}$$

$$\begin{aligned} \frac{\partial^j K_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=x^-} &= \frac{\partial^j R_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=x} \\ &= \begin{cases} \sum_{i=j}^{2n+1}(-1)^{i-n-1}\frac{(x-a)^{2n-j+1}}{(2n-i+1)!(i-j)!}, & n < j < 2n, \\ \frac{(x-a)^{2n-j}}{n!(n-j)!} + \sum_{i=n+1}^{2n+1}(-1)^{i-n-1}\frac{(x-a)^{2n-j+1}}{(2n-i+1)!(i-j)!}, & 0 \le j \le n. \end{cases} \end{aligned} \tag{3.30}$$

So, Eqs. (3.29) and (3.30) imply that

$$\begin{aligned} &\frac{\partial^j K_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=x^+} - \frac{\partial^j K_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=x^-} \\ &= \begin{cases} \sum_{i=j}^{n}\frac{(-1)^{n+i}(x-a)^{2n-j+1}}{(2n-i+1)!(i-j)!} + \sum_{i=n+1}^{2n+1}\frac{(-1)^{n+i}(x-a)^{2n-j+1}}{(2n-i+1)!(i-j)!}, & 0 \le j < n+1, \\ (x-a)^{2n-j+1}\sum_{i=j}^{2n+1}\frac{(-1)^{n+i}}{(2n-i+1)!(i-j)!}, & n < j < 2n+1, \\ (-1)^{n+1}, & j = 2n+1. \end{cases} \\ &= \begin{cases} (-1)^n(x-a)^{2n-j+1}\sum_{i=j}^{2n+1}\frac{(-1)^i}{(2n-i+1)!(i-j)!}, & 0 \le j \le 2n, \\ (-1)^{n+1}, & j = 2n+1. \end{cases} \\ &= \begin{cases} \frac{(-1)^{n+j}(x-a)^{2n-j+1}}{(2n-j+1)!}\sum_{i=0}^{2n-j+1}(-1)^i\binom{2n-j+1}{i}, & 0 \le j \le 2n, \\ (-1)^{n+1}, & j = 2n+1. \end{cases} \\ &= \begin{cases} (-1)^{n+j}(x-a)^{2n-j+1}(1-1)^{2n-j+1}, & 0 \le j \le 2n, \\ (-1)^{n+1}, & j = 2n+1. \end{cases} \\ &= \begin{cases} 0, & 0 \le j \le 2n, \\ (-1)^{n+1}, & j = 2n+1. \end{cases} \end{aligned}$$

TABLE 2. Explicit expressions of $R_a^{\{n+1\}}(x,t)$ for $n = 1,\dots,4$

n	$R_a^{\{n+1\}}(x,t)$
1	$(x-a)(t-a)+(x-a)(t-a)^2/2-(t-a)^3/6$
2	$(x-a)^2(t-a)^2/4+(x-a)^2(t-a)^3/12-(x-a)(t-a)^4/24+(t-a)^5/120$
3	$(x-a)^3(t-a)^3/36+(x-a)^3(t-a)^4/144-(x-a)^2(t-a)^5/240$ $+(x-a)(t-a)^6/720-(t-a)^7/5040$
4	$(x-a)^4(t-a)^4/576+(x-a)^4(t-a)^5/2880-(x-a)^3(t-a)^6/4320$ $+(x-a)^2(t-a)^7/10080-(x-a)(t-a)^8/40320+(t-a)^9/362880$

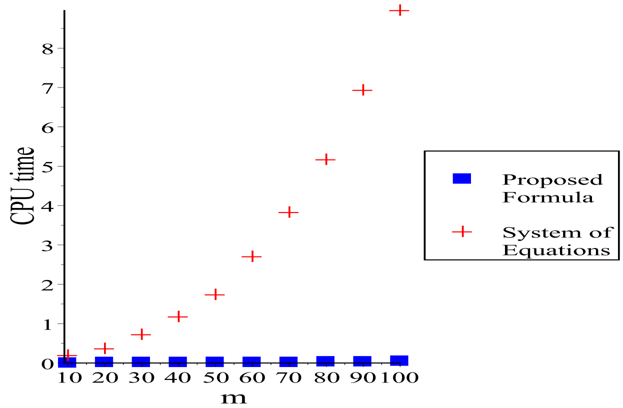

FIGURE 1. Comparing CPU time to compute the reproducing kernel for space $W_2^m[a,b]$

□

Explicit expressions of reproducing kernel functions in $W_{2,a}^{n+1}[a,b]$ for $n = 1,\dots,4$, are shown in Table 2.

Remark 1. For some values of m (or n), we have calculated the reproducing kernel, using two methods: the proposed explicit formula, and also by solving the system of $4m$ (or $4n+4$) equations in $4m$ (or $4n+4$) unknowns. Figures 1 and 2, represent the advantage of the explicit formulas.

4. CONCLUSIONS

In this paper we examined some reproducing kernel spaces and we obtained explicit formulas for the reproducing kernel of these spaces. Usually finding the reproducing kernel for solving large systems of differential equations is required (especially for high-order differential equations). For an nth-order differential equation, we must solve a system of $4(n+1)$ equations in $4(n+1)$ unknowns which is very time-consuming and therefore an explicit formula for these reproducing kernels is

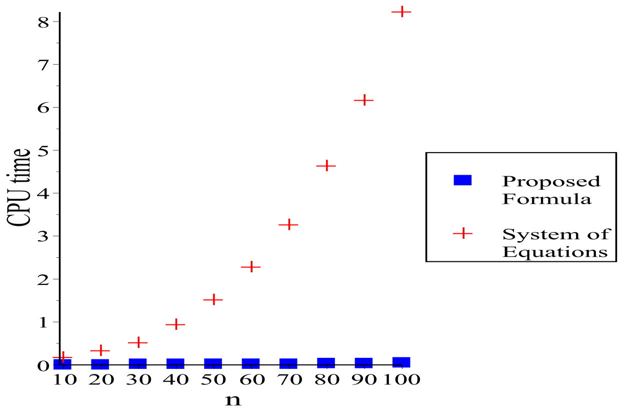

FIGURE 2. Comparing CPU time to compute the reproducing kernel for space $W_2^{n+1}[a,b]$

very useful (see Figures 1 and 2).

ACKNOWLEDGEMENT

The authors thank the anonymous referees of this paper for their careful reading, constructive comments and suggestions.

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