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A NOTE ABOUT ITERATED ARITHMETIC FUNCTIONS

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Abstract. Let $f\colon \mathbb{N} \to \mathbb{N}_0$ be a multiplicative arithmetic function such that for all primes p and positive integers α, $f(p^\alpha) < p^\alpha$ and $f(p) | f(p^\alpha)$. Suppose also that any prime that divides $f(p^\alpha)$ also divides $pf(p)$. Define $f(0) = 0$, and let $H(n) = \lim_{m\to\infty} f^m(n)$, where f^m denotes the m^{th} iterate of f. We prove that the function H is completely multiplicative.

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1. Introduction

The study of iterated arithmetic functions, especially functions related to the Euler totient function φ, has burgeoned over the past century. In 1943, H. Shapiro's monumental work on a function $C(n)$, which counts the number of iterations of φ needed for n to reach 2, paved the way for subsequent number-theoretic research [5]. In this paper, we study a problem concerning the limiting behavior of iterations of functions related to the Euler totient function.

Throughout this paper, we let $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{P}$ denote the set of positive integers, the set of nonnegative integers, and the set of prime numbers, respectively. We will let $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ be a multiplicative arithmetic function which has the following properties for all primes p and positive integers α.

I. $f(p^\alpha) < p^\alpha$.
II. $f(p) | f(p^\alpha)$.
III. If q is prime and $q | f(p^\alpha)$, then $q | pf(p)$.
IV. $f(0) = 0$.

First, note that property IV does not effectively restrict the choice of f. Indeed, we may let f be any multiplicative arithmetic function that satisfies properties I, II, and III and then simply define $f(0) = 0$. One class of arithmetic functions which satisfy I, II, and III are the Schemmel totient functions. For each positive integer r, the

Schemmel totient function S_r is a multiplicative arithmetic function which satisfies

$$S_r(p^\alpha) = \begin{cases} 0, & \text{if } p \le r; \\ p^{\alpha-1}(p-r), & \text{if } p > r \end{cases}$$

for all primes p and positive integers α [4]. These interesting generalizations of the Euler totient function have applications in the study of magic squares [3, page 184] and in the enumeration of cliques in certain graphs [1].

Because f is multiplicative, properties I and II of f are equivalent to the following properties, which we will later reference.

A. For all integers $n > 1$, $f(n) < n$.

B. If p is a prime divisor of a positive integer n, then $f(p) \mid f(n)$.

Let $f^0(n) = n$ and $f^{k+1}(n) = f(f^k(n))$ for all nonnegative integers k and n. Observe that, for any $n \in \mathbb{N}$, $f^n(n) \in \{0, 1\}$. Furthermore, $f^n(n) = \lim_{m\to\infty} f^m(n)$, so we will define $H(n) = \lim_{m\to\infty} f^m(n)$. The author has shown that the function $H : \mathbb{N} \to \{0, 1\}$ is completely multiplicative for the case in which f is a Schemmel totient function [2]. Our purpose is to prove that H is completely multiplicative for any choice of a multiplicative arithmetic function f that satisfies properties I, II, III, and IV. To help do so, we define the following sets.

$$P = \{p \in \mathbb{P} : H(p) = 1\}$$

$$Q = \{q \in \mathbb{P} : H(q) = 0\}$$

$$S = \{n \in \mathbb{N} : q \nmid n \ \forall\, q \in Q\}$$

We define T to be the unique set of positive integers defined by the following criteria:

- $1 \in T$.
- If p is prime, then $p \in T$ if and only if $f(p) \in T$.
- If x is composite, then $x \in T$ if and only if there exist $x_1, x_2 \in T$ such that $x_1, x_2 > 1$ and $x_1 x_2 = x$.

Note that T is a set of *positive* integers; in particular, $0 \notin T$. We may now establish a couple of lemmas that should make the proof of the desired theorem relatively painless.

Lemma 1. *Let $k \in \mathbb{N}$. If all the prime divisors of k are in T, then all the positive divisors of k (including k) are in T. Conversely, if $k \in T$, then every positive divisor of k is an element of T.*

Proof. First, suppose that all the prime divisors of k are in T, and let d be a positive divisor of k. Then all the prime divisors of d are in T. Let $d = \prod_{i=1}^{r} p_i^{\alpha_i}$ be the canonical prime factorization of d. As $p_1 \in T$, the third defining criterion of T tells us that $p_1^2 \in T$. Then, by the same token, $p_1^3 \in T$. Eventually, we find that

$p_1^{\alpha_1} \in T$. As $p_1^{\alpha_1}, p_2 \in T$, we have $p_1^{\alpha_1} p_2 \in T$. Repeatedly using the third criterion, we can keep multiplying by primes until we find that $d \in T$. This completes the first part of the proof. Now we will prove that if $k \in T$, then every positive divisor of k is an element of T. The proof is trivial if k is prime, so suppose k is composite. We will induct on $\Omega(k)$, the number of prime divisors (counting multiplicities) of k. If $\Omega(k) = 2$, then, by the third defining criterion of T, the prime divisors of k must be elements of T. Therefore, if $\Omega(k) = 2$, we are done. Now, suppose the result holds whenever $\Omega(k) \leq h$, where $h > 1$ is an integer. Consider the case in which $\Omega(k) = h + 1$. By the third defining criterion of T, we can write $k = k_1 k_2$, where $1 < k_1, k_2 < k$ and $k_1, k_2 \in T$. By the induction hypothesis, all of the positive divisors of k_1 and all of the positive divisors of k_2 are in T. Therefore, all of the prime divisors of k are in T. By the first part of the proof, we conclude that all of the positive divisors of k are in T. □

Lemma 2. *The sets S and T are equal.*

Proof. First, note that $1 \in S \cap T$. Let $m > 1$ be an integer such that, for all $k \in \{1, 2, \ldots, m-1\}$, either $k \in S \cap T$ or $k \notin S \cup T$. We will show that $m \in S$ if and only if $m \in T$. First, we must show that if $k \in \{1, 2, \ldots, m-1\}$, then $k \in S$ if and only if $f(k) \in S$. Suppose, by way of contradiction, that $f(k) \in S$ and $k \notin S$. As $k \notin S$, we have that $k > 1$ and $k \notin T$. Lemma 1 then guarantees that there exists a prime q such that $q | k$ and $q \notin T$. As $q \notin T$, the second defining criterion of T implies that $f(q) \notin T$. As $f(k) \in S$, $f(k) \neq 0$. By property B of f, $f(q) | f(k)$, so $f(q) \neq 0$. Therefore, $f(q) \in \{1, 2, \ldots, m-1\}$, and $f(q) \notin T$. By the induction hypothesis, $f(q) \notin S$. Therefore, there exists some $q_0 \in Q$ such that $q_0 | f(q)$. Thus, $q_0 | f(q) | f(k)$, which contradicts the assumption that $f(k) \in S$.

Conversely, suppose that $f(k) \notin S$ and $k \in S$. The fact that $f(k) \notin S$ implies that $k > 1$, and the fact that $k \in S$ implies (by the induction hypothesis) that $k \in T$. By Lemma 1, all prime divisors of k are elements of T. The second criterion defining T then implies that $f(p) \in T$ for all prime divisors p of k. Using Lemma 1 again, we conclude that, for any prime divisor p of k, all prime divisors of $f(p)$ are in T. By property III of f, all prime divisors of $f(k)$ are elements of T. Therefore, Lemma 1 guarantees that $f(k) \in T$. From property A of f and the fact that $0 \notin T$, we see that $f(k) \in \{1, 2, \ldots, m-1\}$. The induction hypothesis then implies that $f(k) \in S$, which is a contradiction. Thus, we have established that if $k \in \{1, 2, \ldots, m-1\}$, then $k \in S$ if and only if $f(k) \in S$.

We are now ready to establish that $m \in S$ if and only if $m \in T$. Assume, first, that m is prime. By the second criterion defining T, $m \in T$ if and only if $f(m) \in T$. By the induction hypothesis and property A of f, $f(m) \in T$ if and only if $f(m) \in S$. From the preceding argument, we see that $f(m) \in S$ if and only if $f^2(m) \in S$. Similarly, $f^2(m) \in S$ if and only if $f^3(m) \in S$. Continuing this pattern, we eventually find that $m \in T$ if and only if $f^m(m) \in S$. Observe that $f^m(m) = H(m)$ and that $0 \notin S$ and $1 \in S$. Hence, $m \in T$ if and only if $H(m) = 1$. Because m is prime, $H(m) = 1$ if and only if $m \notin Q$. Finally, it follows from the definition of S that $m \notin Q$ if and only if $m \in S$. This completes the proof of the case in which m is prime.

Assume, now, that m is composite. By Lemma 1, $m \in T$ if and only if all the prime divisors of m are in T. Because m is composite, all the prime divisors of m are elements of $\{1, 2, \ldots, m-1\}$. Therefore, by the induction hypothesis, all the prime divisors of m are in T if and only if all the prime divisors of m are in S. It should be clear from the definition of S that all the prime divisors of m are in S if and only if $m \in S$. Hence, $m \in T$ if and only if $m \in S$. □

We may now use the sets S and T interchangeably. In addition, part of the above proof gives rise to the following corollary.

Corollary 1. *Let $k, r \in \mathbb{N}$. Then $f^r(k) \in S$ if and only if $k \in S$.*

Proof. The proof follows from the argument in the above proof that $f(k) \in S$ if and only if $k \in S$ whenever $k \in \{1, 2, \ldots, m-1\}$. As we now know that we can make m as large as we need, it follows that $f(k) \in S$ if and only if $k \in S$. Repeating this argument, we see that $f^2(k) \in S$ if and only if $f(k) \in S$. The proof then follows from repeated application of the same argument. □

Corollary 2. *For any positive integer k, $H(k) = 1$ if and only if $k \in S$.*

Proof. It is clear that $H(k) = 1$ if and only if $H(k) \in S$. Therefore, the proof follows immediately from setting $r = k$ in Corollary 1. □

Notice that Corollary 2, Lemma 2, and the defining criteria of T provide a simple recursive means of constructing the set of positive integers x that satisfy $H(x) = 1$. We also have the following theorem.

Theorem 1. *The function $H : \mathbb{N} \to \{0, 1\}$ is completely multiplicative.*

Proof. Corollary 2 tells us that H is the characteristic function of the set S of positive integers that are not divisible by primes in Q. If $x, y \in \mathbb{N}$, then it is clear that $xy \in S$ if and only if $x \in S$ and $y \in S$. The proof follows immediately. □

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