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UNIFORM APPROXIMATION BY GENERALIZED Q-BERNSTEIN OPERATORS

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Abstract. For a sequence of integral type operators involving q-integers we study its uniform convergence in $C[0,1]$ and estimate the rate of convergence with the aid of modulus of continuity. As applications we obtain quantitative estimates for old and new q-parametric operators.

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1. INTRODUCTION

The study of q-parametric operators was initiated by Lupaş [8] and Phillips [14]. The so-called q-Bernstein operators were introduced by Phillips in 1997 (see [14]) and they mean another generalization of the well-known Bernstein operators [4] based on q-integers. Nowadays, q-Bernstein operators form an area of an intensive research. A survey of the obtained main results and references in this area during the first decade of study can be found in [12]. Different type of q-integral operators, q-Bernstein type integral operators and q-summation-integral operators were studied in [3].

The goal of the paper is to define a sequence of general q-integral type operators involving q-integers which approximate each continuous function on $[0,1]$ in the uniform norm. The rate of convergence will be estimated by the modulus of continuity. As special cases we recover some q-Bernstein type operators and q-Bernstein type integral operators, respectively. For these operators we also obtain quantitative estimations.

To present our operators we recall some basic definitions and notations of quantum calculus (see [7]). For any $q>0$ and any non-negative integer n, the q-integers $[n]_q$ and the q-factorials $[n]_q!$ are defined by $[0]_q=0$,

$$[n]_q=\begin{cases} 1+q+\ldots+q^{n-1}, & \text{if} \quad q\neq 1 \\ n, & \text{if} \quad q=1 \end{cases}$$

and

$$[n]_q! = \begin{cases} [1]_q[2]_q \dots [n]_q, & \text{if} \quad n \geq 1 \\ 1, & \text{if} \quad n = 0. \end{cases}$$

For any integers n and k satisfying $0 \leq k \leq n$, the q-binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

For $x \in [0,1]$ and m non-negative integer, we set

$$(1-x)_q^m = \begin{cases} (1-x)(1-qx)\dots(1-q^{m-1}x), & \text{if} \quad m \geq 1 \\ 1, & \text{if} \quad m = 0. \end{cases}$$

Let $0 < b$, $0 < q < 1$ and f a real-valued function. The q-Jackson integral of f over the interval $[0,b]$ is defined by

$$\int_0^b f(t)\,d_q t = (1-q)b \sum_{j=0}^{\infty} f(bq^j)q^j. \tag{1.1}$$

On the general interval $[a,b]$ the q-Jackson integral is not appropriate to derive the q-analogues of some well-known integral inequalities. For this reason we consider the Riemann type q-integral defined as follows (see [6, 10]):

$$\int_a^b f(t)\,d_q^R t = (1-q)(b-a) \sum_{j=0}^{\infty} f(a+(b-a)q^j)q^j, \tag{1.2}$$

where $0 \leq a < b$ and $0 < q < 1$. If $a = 0$ in (1.2) then we recover the q-Jackson integral given by (1.1).

For $n \in \{1,2,\dots\}$, $0 < q < 1$, $r, r' \in \{0,1,2,\dots\}$, $f \in C[0,1+r']$ and $x \in [0,1]$, we consider the following q-integral type operators:

$$L_{n,q}^{r,r'}(f;x) = \sum_{k=0}^{n+r'} \begin{bmatrix} n+r' \\ k \end{bmatrix}_q x^k (1-x)_q^{n+r'-k} \tag{1.3}$$
$$\times \frac{1}{b(n,k,r,q)-a(n,k,r,q)} \int_{a(n,k,r,q)}^{b(n,k,r,q)} f(u(n,k,q)+tv(n,k,q))\,d_q^R t,$$

where $0 \leq a(n,k,r,q) < b(n,k,r,q) \leq 1+r'$, $u(n,k,q) \geq 0$, $v(n,k,q) \geq 0$ and $u(n,k,q)+b(n,k,r,q)v(n,k,q) \leq 1+r'$ for all $k \in \{0,1,\dots,n+r'\}$. With the notation $g_{n,k,r,q}(t) = u(n,k,q)+tv(n,k,q)$, $t \in [a(n,k,r,q), b(n,k,r,q)] \subseteq [0,1+r']$, we have for $f \in C[0,1+r']$ that there exists $M = M(f) > 0$ such that $|f(g_{n,k,r,q}(t))| \leq M$ for $t \in [0,1+r']$. Hence, in view of (1.3), (1.2) and

$0 \le a(n,k,r,q) + (b(n,k,r,q) - a(n,k,r,q))q^j \le 1 + r'$ for $j = 0,1,2,\ldots$, we get

$$
\begin{aligned}
|L_{n,q}^{r,r'}(f;x)| &\le \sum_{k=0}^{n+r'} \begin{bmatrix} n+r' \\ k \end{bmatrix}_q x^k (1-x)_q^{n+r'-k} \\
&\quad \times \frac{1}{b(n,k,r,q)-a(n,k,r,q)} \left| \int_{a(n,k,r,q)}^{b(n,k,r,q)} f(g_{n,k,r,q}(t))\, d_q^R t \right| \\
&\le \sum_{k=0}^{n+r'} \begin{bmatrix} n+r' \\ k \end{bmatrix}_q x^k (1-x)_q^{n+r'-k} \frac{1}{b(n,k,r,q)-a(n,k,r,q)} \\
&\quad \times (1-q)(b(n,k,r,q)-a(n,k,r,q)) \sum_{j=0}^{\infty} \{q^j \\
&\quad \times |f(g_{n,k,r,q}(a(n,k,r,q)+(b(n,k,r,q)-a(n,k,r,q))q^j))|\} \\
&\le M \sum_{k=0}^{n+r'} \begin{bmatrix} n+r' \\ k \end{bmatrix}_q x^k (1-x)_q^{n+r'-k} = M
\end{aligned}
$$

for all $x \in [0,1]$ (see [14, (13)]). In conclusion the operators (1.3) are well-defined for all continuous functions f on the interval $[0, 1+r']$.

In what follows we study the uniform convergence of $L_{n,q}^{r,r'}(f;x)$ to $f(x)$ in $x \in [0,1]$ as $n \to \infty$. The rate of convergence will be estimated with the aid of the modulus of continuity of $f \in C[0,1+r']$ defined by

$$\omega(f;\delta) = \sup\{|f(x) - f(y)| : x, y \in [0,1+r'], |x-y| \le \delta\},\ \delta > 0. \tag{1.4}$$

For particular cases of $a(n,k,r,q)$, $b(n,k,r,q)$, $u(n,k,q)$ and $v(n,k,q)$ we recover some known operators (see [1, 2, 5, 9, 11, 13–15]), and we may introduce some new integral type operators. As applications we give quantitative estimations for these operators using (1.4).

2. Main results

Theorem 1. *Let $L_{n,q_n}^{r,r'}(f;x)$ be defined by (1.3), where $n \ge 2$ and $q_n \in (0,1)$ such that $q_n \to 1$ as $n \to \infty$. If there exist $C_1 = C_1(r,r') > 0$ and $C_2 = C_2(r,r') > 0$ such that*

(i) $\Big| u(n,k,q_n) + v(n,k,q_n)\{a(n,k,r,q_n) + \frac{1}{1+q_n}(b(n,k,r,q_n) - a(n,k,r,q_n))\} - \frac{[k]_{q_n}}{[n+r']_{q_n}} \Big| \le \frac{C_1}{[n]_{q_n}}$,

(ii) $\Big| u^2(n,k,q_n) + 2u(n,k,q_n)v(n,k,q_n)\{a(n,k,r,q_n) + \frac{1}{1+q_n}(b(n,k,r,q_n) - a(n,k,r,q_n))\} + v^2(n,k,q_n)\{a^2(n,k,r,q_n) + 2a(n,k,r,q_n)(b(n,k,r,q_n)$

$$-a(n,k,r,q_n))\frac{1}{1+q_n}+(b(n,k,r,q_n)-a(n,k,r,q_n))^2\frac{1}{1+q_n+q_n^2}\}$$
$$-\frac{[k]_{q_n}}{[n+r']_{q_n}}\frac{[k-1]_{q_n}}{[n+r'-1]_{q_n}}\Big|\leq\frac{C_2}{[n]_{q_n}}$$

hold true for all $k\in\{0,1,\ldots,n+r'\}$, *then* $L_{n,q_n}^{r,r'}(f;x)$ *converges uniformly to* $f(x)$ *with respect* $x\in[0,1]$ *as* $n\to\infty$.

Proof. In view of Korovkin's theorem we have to prove that $L_{n,q_n}^{r,r'}(e_i;x)$ converges uniformly to $e_i(x)$ in $x\in[0,1]$ as $n\to\infty$, where $e_i(x)=x^i$, $x\in[0,1]$ and $i\in\{0,1,2\}$.
We denote the n-th order q-Bernstein polynomial associated to f by

$$B_{n,q}(f;x)=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q x^k(1-x)_q^{n-k}f\left(\frac{[k]_q}{[n]_q}\right).$$

Then, (1.3), (1.2) and [14, (13)] imply that

$$L_{n,q_n}^{r,r'}(e_0;x)=\sum_{k=0}^{n+r'}\left[\begin{array}{c}n+r'\\k\end{array}\right]_{q_n}x^k(1-x)_{q_n}^{n+r'-k}$$
$$=B_{n+r',q_n}(e_0;x)=e_0(x). \tag{2.1}$$

Further, using (1.2), we have

$$\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}e_1(u(n,k,q_n)+tv(n,k,q_n))\,d_{q_n}^R t$$
$$=\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}\{u(n,k,q_n)+tv(n,k,q_n)\}\,d_{q_n}^R t=u(n,k,q_n)\{b(n,k,r,q_n)$$
$$-a(n,k,r,q_n)\}+v(n,k,q_n)a(n,k,r,q_n)\{b(n,k,r,q_n)-a(n,k,r,q_n)\}$$
$$+v(n,k,q_n)\{b(n,k,r,q_n)-a(n,k,r,q_n)\}^2\frac{1}{1+q_n}. \tag{2.2}$$

But

$$B_{n+r',q_n}(e_1;x)=\sum_{k=0}^{n+r'}\left[\begin{array}{c}n+r'\\k\end{array}\right]_{q_n}x^k(1-x)_{q_n}^{n+r'-k}\frac{[k]_{q_n}}{[n+r']_{q_n}}=x \tag{2.3}$$

(see [14, (14)]), therefore, by (1.3), (2.2), (i) and (2.1), we have

$$|L_{n,q_n}^{r,r'}(e_1;x)-e_1(x)|$$
$$\leq\sum_{k=0}^{n+r'}\left[\begin{array}{c}n+r'\\k\end{array}\right]_{q_n}x^k(1-x)_{q_n}^{n+r'-k}\Big|u(n,k,q_n)+v(n,k,q_n)$$
$$\times\left\{a(n,k,r,q_n)+\frac{1}{1+q_n}(b(n,k,r,q_n)-a(n,k,r,q_n))\right\}-\frac{[k]_{q_n}}{[n+r']_{q_n}}\Big|$$

$$\leq \frac{C_1}{[n]_{q_n}}. \tag{2.4}$$

If $q_n \to 1$ as $n \to \infty$ then $[n]_{q_n} \to \infty$ as $n \to \infty$. Thus, by (2.4), $L_{n,q_n}^{r,r'}(e_1;x)$ converges uniformly to $e_1(x)$ in $x \in [0,1]$ as $n \to \infty$.

Analogously, using (1.2), we have

$$\begin{aligned}
&\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)} e_2(u(n,k,q_n)+tv(n,k,q_n))\, d_{q_n}^R t \\
&= \int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)} \{u^2(n,k,q_n)+2u(n,k,q_n)v(n,k,q_n)t+v^2(n,k,q_n)t^2\}\, d_{q_n}^R t \\
&= u^2(n,k,q_n)\{b(n,k,r,q_n)-a(n,k,r,q_n)\}+2u(n,k,q_n)v(n,k,q_n)(1-q) \\
&\quad \times \{b(n,k,r,q_n)-a(n,k,r,q_n)\}\sum_{j=0}^{\infty} q^j \{a(n,k,r,q_n)+(b(n,k,r,q_n) \\
&\quad -a(n,k,r,q_n))q^j\}+v^2(n,k,q_n)(1-q_n)\{b(n,k,r,q_n)-a(n,k,r,q_n)\} \\
&\quad \times \sum_{j=0}^{\infty} q^j \{a(n,k,r,q_n)+2a(n,k,r,q_n)(b(n,k,r,q_n)-a(n,k,r,q_n))q^j \\
&\quad +(b(n,k,r,q_n)-a(n,k,r,q_n))^2 q^{2j}\} \\
&= u^2(n,k,q_n)\{b(n,k,r,q_n)-a(n,k,r,q_n)\}+2u(n,k,q_n)v(n,k,q_n) \\
&\quad \times \{b(n,k,r,q_n)-a(n,k,r,q_n)\}\left\{a(n,k,r,q_n)+\frac{1}{1+q_n}(b(n,k,r,q_n)\right. \\
&\quad \left. -a(n,k,r,q_n))\right\}+v^2(n,k,q_n)\{b(n,k,r,q_n)-a(n,k,r,q_n)\} \\
&\quad \times \left\{a^2(n,k,r,q_n)+2a(n,k,r,q_n)\frac{1}{1+q_n}(b(n,k,r,q_n)-a(n,k,r,q_n))\right. \\
&\quad \left. +\frac{1}{1+q_n+q_n^2}(b(n,k,r,q_n)-a(n,k,r,q_n))^2\right\}.
\end{aligned} \tag{2.5}$$

Because

$$\sum_{k=0}^{n+r'} \begin{bmatrix} n+r' \\ k \end{bmatrix}_{q_n} x^k (1-x)_{q_n}^{n+r'-k} \frac{[k]_{q_n}}{[n+r']_{q_n}} \frac{[k-1]_{q_n}}{[n+r'-1]_{q_n}} = x^2, \tag{2.6}$$

we get, in view of (1.3), (2.5), (ii) and (2.6), that

$$|L_{n,q_n}^{r,r'}(e_2;x)-e_2(x)| \leq \sum_{k=0}^{n+r'} \begin{bmatrix} n+r' \\ k \end{bmatrix}_{q_n} x^k (1-x)_{q_n}^{n+r'-k} \Bigg| u^2(n,k,q_n)$$

$$+2u(n,k,q_n)v(n,k,q_n)\left\{a(n,k,r,q_n)+\frac{1}{1+q_n}(b(n,k,r,q_n)\right.$$
$$\left.-a(n,k,r,q_n))\right\}+v^2(n,k,q_n)\left\{a^2(n,k,r,q_n)+2a(n,k,r,q_n)(b(n,k,r,q_n)\right.$$
$$\left.-a(n,k,r,q_n))\frac{1}{1+q_n}+(b(n,k,r,q_n)-a(n,k,r,q_n))^2\frac{1}{1+q_n+q_n^2}\right\}$$
$$\left.-\frac{[k]_{q_n}}{[n+r']_{q_n}}\frac{[k-1]_{q_n}}{[n+r'-1]_{q_n}}\right|\le\frac{C_2}{[n]_{q_n}}.$$

Hence, because of $[n]_{q_n}\to\infty$ as $n\to\infty$, we find that $L_{n,q_n}^{r,r'}(e_2;x)$ converges uniformly to $e_2(x)$ in $x\in[0,1]$ as $n\to\infty$. This completes the proof of the theorem. □

Theorem 2. *Let $L_{n,q_n}^{r,r'}(f;x)$ and q_n, $n\ge 2$, be defined as in Theorem 1 satisfying the conditions (i) and (ii). Then*

$$|L_{n,q_n}^{r,r'}(f;x)-f(x)|\le\{1+\sqrt{2C_1+C_2}\}\omega(f;[n]_{q_n}^{-1/2})$$

for all $f\in C[0,1+r']$ and $x\in[0,1]$.

Proof. Taking into account (1.4), we have

$$\begin{aligned}|f(u(n,k,q_n)+tv(n,k,q_n))-f(x)|&\\ &\le\omega(f;|u(n,k,q_n)+tv(n,k,q_n)-x|)\\ &\le\{1+\delta^{-1}|u(n,k,q_n)+tv(n,k,q_n)-x|\}\omega(f;\delta).\end{aligned}$$

Hence, by (2.1) and Hölder's inequality,

$$\begin{aligned}&|L_{n,q_n}^{r,r'}(f;x)-f(x)|\\ &\le\sum_{k=0}^{n+r'}\begin{bmatrix}n+r'\\ k\end{bmatrix}_{q_n}x^k(1-x)_{q_n}^{n+r'-k}\frac{1}{b(n,k,r,q_n)-a(n,k,r,q_n)}\\ &\quad\times\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}|f(u(n,k,q_n)+tv(n,k,q_n))-f(x)|\,d_{q_n}^Rt\\ &\le\omega(f;\delta)\sum_{k=0}^{n+r'}\begin{bmatrix}n+r'\\ k\end{bmatrix}_{q_n}x^k(1-x)_{q_n}^{n+r'-k}\frac{1}{b(n,k,r,q_n)-a(n,k,r,q_n)}\\ &\quad\times\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}\{1+\delta^{-1}|u(n,k,q_n)+tv(n,k,q_n)-x|\}\,d_{q_n}^Rt\\ &\le\omega(f;\delta)\left\{1+\delta^{-1}\sum_{k=0}^{n+r'}\begin{bmatrix}n+r'\\ k\end{bmatrix}_{q_n}x^k(1-x)_{q_n}^{n+r'-k}\right.\end{aligned}$$

$$
\begin{aligned}
&\times\left(\frac{1}{b(n,k,r,q_n)-a(n,k,r,q_n)}\right.\\
&\left.\left.\times\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}(u(n,k,q_n)+tv(n,k,q_n)-x)^2\,d_{q_n}^R t\right)^{1/2}\right\}\\
&\leq\omega(f;\delta)\left\{1+\delta^{-1}\left(\sum_{k=0}^{n+r'}\left[\begin{array}{c}n+r'\\k\end{array}\right]_{q_n}x^k(1-x)_{q_n}^{n+r'-k}\right.\right.\\
&\times\frac{1}{b(n,k,r,q_n)-a(n,k,r,q_n)}\\
&\left.\left.\times\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}(u(n,k,q_n)+tv(n,k,q_n)-x)^2\,d_{q_n}^R t\right)^{1/2}\right\}. \qquad (2.7)
\end{aligned}
$$

On the other hand, by (2.2), (i), (2.5) and (ii), we have

$$
\begin{aligned}
&\frac{1}{b(n,k,r,q_n)-a(n,k,r,q_n)}\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}\{u(n,k,q_n)+tv(n,k,q_n)-x\}^2\,d_{q_n}^R t\\
&=\frac{1}{b(n,k,r,q_n)-a(n,k,r,q_n)}\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}\{u(n,k,q_n)+tv(n,k,q_n)\}^2\,d_{q_n}^R t\\
&-\frac{[k]_{q_n}}{[n+r']_{q_n}}\frac{[k-1]_{q_n}}{[n+r'-1]_{q_n}}-\frac{2x}{b(n,k,r,q_n)-a(n,k,r,q_n)}\\
&\times\int_{a(n,k,r,q_n)}^{b(n,k,r,q_n)}\{u(n,k,q_n)+tv(n,k,q_n)\}\,d_{q_n}^R t+2x\frac{[k]_{q_n}}{[n+r']_{q_n}}\\
&+\frac{[k]_{q_n}}{[n+r']_{q_n}}\frac{[k-1]_{q_n}}{[n+r'-1]_{q_n}}-2x\frac{[k]_{q_n}}{[n+r']_{q_n}}+x^2\\
&\leq\frac{C_2}{[n]_{q_n}}+\frac{2C_1}{[n]_{q_n}}+\frac{[k]_{q_n}}{[n+r']_{q_n}}\frac{[k-1]_{q_n}}{[n+r'-1]_{q_n}}-2x\frac{[k]_{q_n}}{[n+r']_{q_n}}+x^2.
\end{aligned}
$$

Hence, in view of (2.7), (2.3), (2.6) and (2.1),

$$
\begin{aligned}
&|L_{n,q_n}^{r,r'}(f;x)-f(x)|\\
&\leq\omega(f;\delta)\left\{1+\delta^{-1}\left(\frac{C_2}{[n]_{q_n}}+\frac{2C_1}{[n]_{q_n}}+\sum_{k=0}^{n+r'}\left[\begin{array}{c}n+r'\\k\end{array}\right]_{q_n}x^k(1-x)_{q_n}^{n+r'-k}\right.\right.\\
&\left.\left.\times\left(\frac{[k]_{q_n}}{[n+r']_{q_n}}\frac{[k-1]_{q_n}}{[n+r'-1]_{q_n}}-2x\frac{[k]_{q_n}}{[n+r']_{q_n}}+x^2\right)\right)^{1/2}\right\}
\end{aligned}
$$

$$= \omega(f;\delta)\left\{1+\delta^{-1}\left(\frac{2C_1+C_2}{[n]_{q_n}}\right)^{1/2}\right\}.$$

Choosing $\delta = [n]_{q_n}^{-1/2}$, we arrive at the required estimate. □

3. Applications

In this section we apply the main results for some q-Bernstein type operators and q-Bernstein type integral operators. This means that we have to verify the conditions of Theorem 1, and apply Theorem 2 to obtain quantitative estimates for these operators.

1) In [2] the following operators are introduced:

$$S_{n,p}^{(\alpha,\beta)}(f;q,x) = \sum_{k=0}^{n+p}\left[\begin{array}{c} n+p \\ k \end{array}\right]_q x^k(1-x)_q^{n+p-k} f\left(\frac{[k]_q+\alpha}{[n]_q+\beta}\right),$$

where $f \in C[0,1+p]$, $0<q<1$, $0\le\alpha\le\beta$, $x\in[0,1]$. Using the notations $r'=p$, $a(n,k,r,q)=0$, $b(n,k,r,q)=1$, $u(n,k,q)=\frac{[k]_q+\alpha}{[n]_q+\beta}$, $v(n,k,q)=0$ for $k=0,1,\ldots,n+p$, we obtain $L_{n,q}^{r,r'}(f;x)=S_{n,p}^{(\alpha,\beta)}(f;q,x)$. Moreover, if $\alpha=\beta=0$ then we recover the q-Bernstein-Schurer operators

$$\tilde{B}_{n,p}(f;q,x) = \sum_{k=0}^{n+p}\left[\begin{array}{c} n+p \\ k \end{array}\right]_q x^k(1-x)_q^{n+p-k} f\left(\frac{[k]_q}{[n]_q}\right)$$

(see [11]), while for $\alpha=\beta=p=0$ we recover the q-Bernstein operators introduced by Phillips [14].

Now, we verify the conditions of Theorem 1:

(i) for $k=0,1,\ldots,n+p$ we have

$$\begin{aligned}\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}-\frac{[k]_{q_n}}{[n+p]_{q_n}}\right| &= \frac{|[k]_{q_n}([n+p]_{q_n}-[n]_{q_n})+\alpha[n+p]_{q_n}-\beta[k]_{q_n}|}{[n+p]_{q_n}([n]_{q_n}+\beta)}\\ &\le \frac{[n+p]_{q_n}q_n^n[p]_{q_n}+(\alpha+\beta)[n+p]_{q_n}}{[n+p]_{q_n}([n]_{q_n}+\beta)}\\ &\le \frac{\alpha+\beta+p}{[n]_{q_n}};\end{aligned}$$

(ii) by (i) and

$$\frac{[n]_{q_n}}{[n-1]_{q_n}} = \frac{1+q_n[n-1]_{q_n}}{[n-1]_{q_n}} = \frac{1}{[n-1]_{q_n}}+q_n \le 2 \tag{3.1}$$

for $n\ge 2$, we get for all $k=0,1,\ldots,n+p$ that

$$\left|\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}\right)^2-\frac{[k]_{q_n}}{[n+p]_{q_n}}\frac{[k-1]_{q_n}}{[n+p-1]_{q_n}}\right|$$

$$\leq \left| \left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta} \right)^2 - \left(\frac{[k]_{q_n}}{[n+p]_{q_n}} \right)^2 \right|$$
$$+ \left| \left(\frac{[k]_{q_n}}{[n+p]_{q_n}} \right)^2 - \frac{[k]_{q_n}}{[n+p]_{q_n}} \frac{[k-1]_{q_n}}{[n+p-1]_{q_n}} \right|$$
$$\leq \left\{ \frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta} + \frac{[k]_{q_n}}{[n+p]_{q_n}} \right\} \left| \frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta} - \frac{[k]_{q_n}}{[n+p]_{q_n}} \right|$$
$$+ \left| \frac{[k]_{q_n}+\alpha}{[n+p]_{q_n}+\beta} - \frac{[k-1]_{q_n}}{[n+p-1]_{q_n}} \right|$$
$$\leq \left\{ \frac{[n+p]_{q_n}+\alpha}{[n]_{q_n}+\beta} + 1 \right\} \frac{\alpha+\beta+p}{[n]_{q_n}} + \frac{q_n^{k-1}[n+p-k]_{q_n}}{[n+p]_{q_n}[n+p-1]_{q_n}}$$
$$\leq \left\{ \frac{[n]_{q_n}+q_n^n[p]_{q_n}+\alpha}{[n]_{q_n}+\beta} + 1 \right\} \frac{\alpha+\beta+p}{[n]_{q_n}} + \frac{1}{[n+p]_{q_n}}$$
$$\leq \frac{(\alpha+p+2)(\alpha+\beta+p)}{[n]_{q_n}} + \frac{1}{[n]_{q_n}} \frac{[n]_{q_n}}{[n+p-1]_{q_n}}$$
$$\leq \frac{(\alpha+p+2)(\alpha+\beta+p)}{[n]_{q_n}} + \frac{1}{[n]_{q_n}} \frac{[n]_{q_n}}{[n-1]_{q_n}}$$
$$\leq \{(\alpha+p+2)(\alpha+\beta+p)+2\} \frac{1}{[n]_{q_n}}.$$

Then, by Theorem 2 with $C_1 = \alpha+\beta+p$ and $C_2 = (\alpha+p+2)(\alpha+\beta+p)+2$, we have

$$|S_{n,p}^{(\alpha,\beta)}(f;q,x) - f(x)| \leq \{1+\sqrt{(\alpha+\beta+p)(\alpha+p+4)+2}\}\,\omega(f;[n]_{q_n}^{-1/2}).$$

2) The following q-Kantorovich type operators are introduced in [13]:

$$K_n^p(f;q,x)$$
$$= \sum_{k=0}^{n+p} \left[\begin{array}{c} n+p \\ k \end{array} \right]_q x^k (1-x)_q^{n+p-k} \int_0^1 f\left(\frac{[k]_q}{[n+1]_q} + \frac{1+(q-1)[k]_q}{[n+1]_q} t \right) d_q t,$$

where $f \in C[0,1+p]$, $0<q<1$, $x \in [0,1]$. Using the notations $r' = p$, $a(n,k,r,q) = 0$, $b(n,k,r,q) = 1$, $u(n,k,q) = \frac{[k]_q}{[n+1]_q}$, $v(n,k,q) = \frac{1+(q-1)[k]_q}{[n+1]_q}$ for $k = 0,1,\ldots,n+p$, we obtain $L_{n,q}^{r,r'}(f;x) = K_n^p(f;q,x)$. For $p=0$ we recover the operators $B_{n,q}^*(f,x)$ of [9].

We verify the conditions of Theorem 1:

(i) for $k = 0, 1, \ldots, n+p$ we have

$$\left| \frac{[k]_{q_n}}{[n+1]_{q_n}} + \frac{1+(q_n-1)[k]_{q_n}}{[n+1]_{q_n}} \frac{1}{1+q_n} - \frac{[k]_{q_n}}{[n+p]_{q_n}} \right|$$
$$\leq \frac{[k]_{q_n} |[n+p]_{q_n} - [n+1]_{q_n}|}{[n+p]_{q_n}[n+1]_{q_n}} + \frac{1}{[n]_{q_n}}$$
$$\leq \frac{|[n]_{q_n} + q_n^n [p]_{q_n} - [n]_{q_n} - q_n^n|}{[n+1]_{q_n}} + \frac{1}{[n]_{q_n}} \leq \frac{p+2}{[n]_{q_n}};$$

(ii) as in (ii) of case **1)**, we may write for $k = 0, 1, \ldots, n+p$ that

$$\left| \left(\frac{[k]_{q_n}}{[n+1]_{q_n}} \right)^2 + 2\frac{[k]_{q_n}}{[n+1]_{q_n}} \frac{1+(q_n-1)[k]_{q_n}}{[n+1]_{q_n}} \frac{1}{1+q_n} + \left(\frac{1+(q_n-1)[k]_{q_n}}{[n+1]_{q_n}} \right)^2 \right.$$
$$\left. \times \frac{1}{1+q_n+q_n^2} - \frac{[k]_{q_n}}{[n+p]_{q_n}} \frac{[k-1]_{q_n}}{[n+p-1]_{q_n}} \right|$$
$$\leq \left| \left(\frac{[k]_{q_n}}{[n+1]_{q_n}} \right)^2 - \left(\frac{[k]_{q_n}}{[n+p]_{q_n}} \right)^2 + \left(\frac{[k]_{q_n}}{[n+p]_{q_n}} \right)^2 - \frac{[k]_{q_n}}{[n+p]_{q_n}} \frac{[k-1]_{q_n}}{[n+p-1]_{q_n}} \right|$$
$$+ \frac{2}{[n]_{q_n}} \frac{[n+p]_{q_n}}{[n]_{q_n}} + \frac{1}{[n]_{q_n}}$$
$$\leq \left\{ \frac{[k]_{q_n}}{[n+1]_{q_n}} + \frac{[k]_{q_n}}{[n+p]_{q_n}} \right\} \frac{[k]_{q_n} |[n+p]_{q_n} - [n+1]_{q_n}|}{[n+p]_{q_n}[n+1]_{q_n}}$$
$$+ \left| \frac{[k]_{q_n}}{[n+p]_{q_n}} - \frac{[k-1]_{q_n}}{[n+p-1]_{q_n}} \right| + \frac{2}{[n]_{q_n}} \frac{[n]_{q_n} + q_n^n [p]_{q_n}}{[n]_{q_n}} + \frac{1}{[n]_{q_n}}$$
$$\leq \left\{ \frac{[n+p]_{q_n}}{[n+1]_{q_n}} + 1 \right\} \frac{p+1}{[n]_{q_n}} + \frac{2}{[n]_{q_n}} + \frac{2(p+1)}{[n]_{q_n}} + \frac{1}{[n]_{q_n}}$$
$$\leq \left\{ \frac{[n]_{q_n} + q_n^n [p]_{q_n}}{[n]_{q_n}} + 1 \right\} \frac{p+1}{[n]_{q_n}} + \frac{2p+5}{[n]_{q_n}}$$
$$\leq \frac{(p+2)(p+1)}{[n]_{q_n}} + \frac{2p+5}{[n]_{q_n}} = \frac{p^2+5p+7}{[n]_{q_n}}.$$

For $C_1 = p+2$ and $C_2 = p^2+5p+7$, we have, by Theorem 2,

$$|K_n^p(f; q, x) - f(x)| \leq \{1 + \sqrt{p^2+7p+11}\}\omega(f; [n]_{q_n}^{-1/2}).$$

3) Recently, in [1] the following operators were introduced:

$$K_{n,p}(f;q,x) = [n+1]_q \sum_{k=0}^{n+p} \left[\begin{array}{c} n+p \\ k \end{array} \right]_q x^k (1-x)_q^{n+p-k} q^{-k} \int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} f(t)\, d_q^R t,$$

where $f \in C[0,1+p]$, $0<q<1$, $x \in [0,1]$. When $p=0$ we recover the operators $B_n^*(f;q,x)$ studied in [5]. Using the notations $r'=p$, $a(n,k,r,q)=\frac{[k]_q}{[n+1]_q}$, $b(n,k,r,q)=\frac{[k+1]_q}{[n+1]_q}$, $u(n,k,q)=0$, $v(n,k,q)=1$ for $k=0,1,\ldots,n+p$, we obtain $L_{n,q}^{r,r'}(f;x)=K_{n,p}(f;q,x)$.

Analogously to the previous two cases, and taking into account Theorem 2, we find that

$$|K_{n,p}(f;q,x)-f(x)| \le \{1+\sqrt{p^2+7p+11}\}\,\omega(f;[n]_{q_n}^{-1/2}).$$

In what follows we introduce some new q-Bernstein type integral operators.

4) For $f \in C[0,1]$, $0<q<1$, $x \in [0,1]$, $p \in \{0,1,2,\ldots\}$, we consider the operators

$$U_n^p(f;q,x) = [n+p+1]_q \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q x^k (1-x)_q^{n-k} q^{-p-k} \int_{[k+p]_q/[n+p+1]_q}^{[k+p+1]_q/[n+p+1]_q} f(t)\, d_q^R t,$$

where $f \in C[0,1+p]$, $0<q<1$, $x \in [0,1]$. Using the notations $r=p$, $r'=0$, $a(n,k,r,q)=\frac{[k+p]_q}{[n+p+1]_q}$, $b(n,k,r,q)=\frac{[k+p+1]_q}{[n+p+1]_q}$, $u(n,k,q)=0$, $v(n,k,q)=1$ for $k=0,1,\ldots,n$, we have $L_{n,q}^{r,r'}(f;x)=U_n^p(f;q,x)$.

We verify the conditions of Theorem 1 as follows:

(i) for $k=0,1,\ldots,n$ we have

$$\begin{aligned}
&\left| \frac{[k+p]_{q_n}}{[n+p+1]_{q_n}} + \frac{1}{1+q_n}\frac{[k+p+1]_{q_n}-[k+p]_{q_n}}{[n+p+1]_{q_n}} - \frac{[k]_{q_n}}{[n]_{q_n}} \right| \\
&\le \left| \frac{[k+p]_{q_n}}{[n+p+1]_{q_n}} - \frac{[k]_{q_n}}{[n]_{q_n}} \right| + \frac{1}{[n]_{q_n}} = \left| \frac{[k]_{q_n}+q_n^k[p]_{q_n}}{[n]_{q_n}+q_n^n[p+1]_{q_n}} - \frac{[k]_{q_n}}{[n]_{q_n}} \right| + \frac{1}{[n]_{q_n}} \\
&= \frac{|q_n^k[p]_{q_n}[n]_{q_n} - q_n^n[p+1]_{q_n}[k]_{q_n}|}{[n]_{q_n}[n+p+1]_{q_n}} + \frac{1}{[n]_{q_n}} \\
&\le \frac{p[n]_{q_n}+(p+1)[n]_{q_n}}{[n]_{q_n}[n+p+1]_{q_n}} + \frac{1}{[n]_{q_n}} \le \frac{2p+2}{[n]_{q_n}};
\end{aligned}$$

(ii) using (3.1), we have for $k=0,1,\dots,n$ that

$$\Bigg|\left(\frac{[k+p]_{q_n}}{[n+p+1]_{q_n}}\right)^2+2\frac{[k+p]_{q_n}}{[n+p+1]_{q_n}}\frac{[k+p+1]_{q_n}-[k+p]_{q_n}}{[n+p+1]_{q_n}}\frac{1}{1+q_n}$$
$$+\left(\frac{[k+p+1]_{q_n}-[k+p]_{q_n}}{[n+p+1]_{q_n}}\right)^2\frac{1}{1+q_n+q_n^2}-\frac{[k]_{q_n}}{[n]_{q_n}}\frac{[k-1]_{q_n}}{[n-1]_{q_n}}\Bigg|$$
$$\le\left|\left(\frac{[k+p]_{q_n}}{[n+p+1]_{q_n}}\right)^2-\left(\frac{[k]_{q_n}}{[n]_{q_n}}\right)^2\right|+\left|\left(\frac{[k]_{q_n}}{[n]_{q_n}}\right)^2-\frac{[k]_{q_n}}{[n]_{q_n}}\frac{[k-1]_{q_n}}{[n-1]_{q_n}}\right|$$
$$+\frac{2}{[n]_{q_n}}+\frac{1}{[n]_{q_n}}$$
$$\le\left\{\frac{[k+p]_{q_n}}{[n+p+1]_{q_n}}+\frac{[k]_{q_n}}{[n]_{q_n}}\right\}+\left|\frac{[k+p]_{q_n}}{[n]_{q_n}}-\frac{[k-1]_{q_n}}{[n-1]_{q_n}}\right|+\frac{3}{[n]_{q_n}}$$
$$\le 2\frac{2p+1}{[n]_{q_n}}+\frac{q_n^{k-1}[n-k]_{q_n}}{[n]_{q_n}[n-1]_{q_n}}+\frac{3}{[n]_{q_n}}\le\frac{4p+2}{[n]_{q_n}}+\frac{[n]_{q_n}}{[n-1]_{q_n}}\frac{1}{[n]_{q_n}}+\frac{3}{[n]_{q_n}}$$
$$\le\frac{4p+7}{[n]_{q_n}}.$$

Hence, by Theorem 2,

$$|U_n^p(f;q,x)-f(x)|\le\{1+\sqrt{8p+11}\}\omega(f;[n]_{q_n}^{-1/2}).$$

5) For $f\in C[0,1+p]$, $p\in\{0,1,2,\dots\}$, $0<q<1$, $x\in[0,1]$, we introduce the operators

$$V_n^p(f;q,x)=(1-x)_q^{n+p}f(0)+\frac{1}{2}[n]_q^2\sum_{k=1}^{n+p-1}\left[\begin{array}{c}n+p\\k\end{array}\right]_q x^k(1-x)_q^{n+p-k}$$
$$\times\int_{\frac{[k]_q}{[n]_q}-\frac{1}{[n]_q^2}}^{\frac{[k]_q}{[n]_q}+\frac{1}{[n]_q^2}}f(t)\,d_q^Rt+x^{n+p}f\left(\frac{[n+p]_q}{[n]_q}\right).$$

We set $r'=p$,

$$a(n,0,r,q)=0,\quad b(n,0,r,q)=1,\quad u(n,0,q)=0,\quad v(n,0,q)=0;$$
$$a(n,k,r,q)=\frac{[k]_q}{[n]_q}-\frac{1}{[n]_q^2},\quad b(n,k,r,q)=\frac{[k]_q}{[n]_q}+\frac{1}{[n]_q^2},$$
$$u(n,k,q)=0,\quad v(n,k,q)=1,\text{for } k=1,2,\dots,n+p-1;$$
$$a(n,n+p,r,q)=0,\quad b(n,n+p,r,q)=1,$$
$$u(n,n+p,q)=\frac{[n+p]_q}{[n]_q},\quad v(n,n+p,q)=0.$$

Then $L_{n,q}^{r,r'}(f;x) = V_n^p(f;q,x)$. In what follows, we verify the conditions of Theorem 1.

(i) for $k = 0$ it is obvious; for $k = 1,2,\ldots,n+p-1$ we have

$$\left| \frac{[k]_{q_n}}{[n]_{q_n}} - \frac{1}{[n]_q^2} + \frac{2}{[n]_q^2}\frac{1}{1+q_n} - \frac{[k]_{q_n}}{[n+p]_{q_n}} \right| \\ \leq \frac{[k]_{q_n}([n+p]_{q_n} - [n]_{q_n})}{[n]_{q_n}[n+p]_{q_n}} + \frac{3}{[n]_{q_n}} \leq \frac{p+3}{[n]_{q_n}},$$

and for $k = n+p$ we have

$$\left| \frac{[n+p]_{q_n}}{[n]_{q_n}} - \frac{[n+p]_{q_n}}{[n+p]_{q_n}} \right| = \frac{[n+p]_{q_n} - [n]_{q_n}}{[n]_{q_n}} = \frac{q_n^n[p]_{q_n}}{[n]_{q_n}} \leq \frac{p}{[n]_{q_n}};$$

(ii) for $k = 0$ it is obvious; for $k = 1,2,\ldots,n+p-1$ we have

$$\begin{aligned}
&\left| \left(\frac{[k]_{q_n}}{[n]_{q_n}} - \frac{1}{[n]_q^2} \right)^2 + 2\left(\frac{[k]_{q_n}}{[n]_{q_n}} - \frac{1}{[n]_q^2} \right) \frac{2}{[n]_q^2}\frac{1}{1+q_n} \right. \\
&\quad \left. + \frac{4}{[n]_q^2}\frac{1}{1+q_n+q_n^2} - \frac{[k]_{q_n}}{[n+p]_{q_n}}\frac{[k-1]_{q_n}}{[n+p-1]_{q_n}} \right| \\
&\leq \left| \left(\frac{[k]_{q_n}}{[n]_{q_n}} - \frac{1}{[n]_q^2} \right)^2 - \left(\frac{[k]_{q_n}}{[n+p]_{q_n}} \right)^2 \right| \\
&\quad + \left| \left(\frac{[k]_{q_n}}{[n+p]_{q_n}} \right)^2 - \frac{[k]_{q_n}}{[n+p]_{q_n}}\frac{[k-1]_{q_n}}{[n+p-1]_{q_n}} \right| \\
&\quad + \frac{4}{[n]_{q_n}^2}\frac{[n+p]_{q_n}}{[n]_{q_n}} + \frac{4}{[n]_{q_n}^2} \left| \frac{1}{1+q_n+q_n^2} - \frac{1}{1+q_n} \right| \\
&\leq \left| \frac{[k]_{q_n}}{[n]_{q_n}} - \frac{1}{[n]_q^2} + \frac{[k]_{q_n}}{[n+p]_{q_n}} \right| \left| \frac{[k]_{q_n}}{[n]_{q_n}} - \frac{1}{[n]_q^2} - \frac{[k]_{q_n}}{[n+p]_{q_n}} \right| \\
&\quad + \left| \frac{[k]_{q_n}}{[n+p]_{q_n}} - \frac{[k-1]_{q_n}}{[n+p-1]_{q_n}} \right| + \frac{4(p+1)}{[n]_{q_n}} + \frac{4}{[n]_{q_n}} \\
&\leq \left\{ \frac{[n+p]_{q_n}}{[n]_{q_n}} + 1 + 1 \right\} \left\{ \frac{[k]_{q_n}([n+p]_{q_n} - [n]_{q_n})}{[n+p]_{q_n}[n]_{q_n}} + \frac{1}{[n]_{q_n}} \right\} \\
&\quad + \frac{q_n^{k-1}[n+p-k]_{q_n}}{[n+p]_{q_n}[n+p-1]_{q_n}} + \frac{4(p+2)}{[n]_{q_n}} \\
&\leq (p+2)\frac{p+1}{[n]_{q_n}} + \frac{[n]_{q_n}}{[n-1]_{q_n}}\frac{1}{[n]_{q_n}} + \frac{4(p+2)}{[n]_{q_n}} \leq \frac{p^2+7p+12}{[n]_{q_n}}
\end{aligned}$$

(see (3.1)); for $k = n + p$ we have

$$\begin{aligned}
&\left| \left(\frac{[n+p]_{q_n}}{[n]_{q_n}} \right)^2 - \frac{[n+p]_{q_n}}{[n+p]_{q_n}} \frac{[n+p-1]_{q_n}}{[n+p-1]_{q_n}} \right| \\
&= \frac{[n+p]_{q_n} + [n]_{q_n}}{[n]_{q_n}} \frac{[n+p]_{q_n} - [n]_{q_n}}{[n]_{q_n}} \\
&= \frac{2[n]_{q_n} + q_n^n [p]_{q_n}}{[n]_{q_n}} \frac{q_n^n [p]_{q_n}}{[n]_{q_n}} \le (p+2) \frac{p}{[n]_{q_n}} = \frac{p^2+2p}{[n]_{q_n}}.
\end{aligned}$$

Choosing $C_1 = p+3$ and $C_2 = p^2 + 7p + 12$, we have, in view of Theorem 2, that

$$|V_n^p(f;q,x) - f(x)| \le \{1 + \sqrt{p^2 + 9p + 18}\} \omega(f;[n]_{q_n}^{-1/2}).$$

6) Analogously, for

$$\begin{aligned}
&W_n^p(f;q,x) \\
&= ([n]_q + \beta) \sum_{k=0}^{n+p} \left[\begin{array}{c} n+p \\ k \end{array} \right]_q x^k (1-x)_q^{n+p-k} q^{-k} \int_{\frac{[k]_q+\alpha}{[n]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n]_q+\beta}} f(t)\, d_q^R t,
\end{aligned}$$

where $f \in C[0, 1+p]$, $p \in \{0, 1, 2, \dots\}$, $0 < q < 1$, $x \in [0, 1]$, $0 \le \alpha \le \beta$, we have

$$|W_n^p(f;q,x) - f(x)| \le \{1 + \sqrt{c(p,\alpha,\beta)}\} \omega(f;[n]_{q_n}^{-1/2}),$$

where $c(p,\alpha,\beta) = (\alpha + p + 1)(\alpha + 1) + (\alpha + \beta + p + 4)(\beta + 1)(p + 1) + 3\alpha + 2$.

7) Finally, we consider the operators

$$\bar{K}_n^p(f;q,x) = [n+1]_q \sum_{k=0}^{n+p} \left[\begin{array}{c} n+p \\ k \end{array} \right]_q x^k (1-x)_q^{n+p-k} \int_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t)\, d_q^R t,$$

where $f \in C[0,1]$, $0 < q < 1$, $x \in [0,1]$. For $p = 0$ we recover the operators studied in [15]. Then, by Theorem 2, we have

$$|\bar{K}_n^p(f;q,x) - f(x)| \le \{1 + \sqrt{p^2 + 7p + 11}\} \omega(f;[n]_{q_n}^{-1/2}).$$

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