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HARDY–SOBOLEV TYPE INEQUALITIES FOR GENERALIZED BAOUENDI–GRUSHIN OPERATORS

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Abstract. In this paper we establish a class of Hardy–Sobolev type inequalities related to generalized Baouendi–Grushin operators. Our results contain the well-known Hardy type inequality and Sobolev type inequality for the class of operators. Furthermore, some new inequalities are obtained.

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1. Introduction

In [6], a Hardy type inequality for the generalized Baouendi–Grushin vector fields

$$Z_i = \frac{\partial}{\partial x_i}, \quad Z_{n+j} = |x|^{\alpha} \frac{\partial}{\partial y_j}, \quad (1 \le i \le n, 1 \le j \le m)$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2 \dots, y_m) \in \mathbb{R}^m$, $\alpha > 0$, is given by

$$\int_{\mathbb{R}^{n+m}} \frac{u^2}{d^2} \psi_{2\alpha} dx dy \le \left(\frac{2}{Q-2} \right)^2 \int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 \, dx dy, \tag{1.1}$$

for $u \in L^2\left(\mathbb{R}^{n+m}, \psi_{2\alpha} dx dy\right)$ and $|\nabla_L u| \in L^2(\mathbb{R}^{n+m})$, with $\nabla_L = (Z_1, \dots, Z_n, Z_{n+1} \dots, Z_{n+m})$. Using (1.1), a unique continuation for the generalized Baouendi-Grushin operator

$$\mathfrak{L}_{\alpha} = \Delta_x + |x|^{2\alpha} \Delta_y = \sum_{i=1}^{n+m} Z_i = \nabla_L \cdot \nabla_L$$

was proved. The proof of (1.1) used representation formulae of functions by the fundamental solution of $\mathfrak{L}_{\alpha}$ at the origin.

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The general version of (1.1) was established with a different approach in [10]. For any open subset $\Omega \subset \mathbb{R}^{n+m}$, D'Ambrosio obtained also the following Hardy type inequalities in [3]:

Let $p > 1, n, m \geq 1$, and $\beta, \gamma \in \mathbb{R}$ such that $n + (1+\alpha)m > \gamma - \beta - p$ and $n > \alpha p - \beta$. Then, for every $u \in D^{1,p}\left(\Omega, |x|^{\beta-\alpha p} d^{(1+\alpha)p-\gamma}\right)$ it follows

$$c^p_{Q,p,\beta,\gamma} \int_\Omega |u|^p \frac{|x|^\beta}{d^\gamma} dxdy \leq \int_\Omega |\nabla_L u|^p |x|^{\beta-\alpha p} d^{(1+\alpha)p-\gamma} dxdy, \tag{1.2}$$

where $c_{Q,p,\beta,\gamma} = \frac{Q+\beta-\gamma}{p}$. If $(0,0) \in \Omega$, then the constant (1.2) $c^p_{Q,p,\beta,\gamma}$ is sharp.

Recently, a Sobolev type inequality for the vector fields $Z_1, \ldots, Z_{n+m}$ (see [7]) states

$$\left(\int_{\mathbb{R}^{n+m}} |u|^{2^*} dxdy\right)^{\frac{1}{2^*}} \leq S \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{1}{2}}, \tag{1.3}$$

where S is a positive constant, $2^* = \frac{2Q}{Q-2}$, $Q = n + (\alpha+1)m$ is the homogeneous dimension with respect to the dilations

$$\delta_\lambda(x,y) = \left(\lambda x, \lambda^{\alpha+1} y\right), \qquad \lambda > 0, (x,y) \in \mathbb{R}^{n+m}, \tag{1.4}$$

which is induced by $\mathcal{L}_\alpha$. Inequality (1.3) contains the result for $\alpha = 1$ in [2].

We define the following distance from the origin on $\mathbb{R}^{n+m}$

$$d(x,y) = \left(|x|^{2(\alpha+1)} + (\alpha+1)^2 |y|^2\right)^{\frac{1}{2(\alpha+1)}}.$$

It is easy to check that

$$\begin{aligned} \nabla_L d &= \frac{|x|^\alpha}{d^{2\alpha+1}} (|x|^\alpha x_1, |x|^\alpha x_2, \ldots, |x|^\alpha x_n, \\ &\qquad (\alpha+1) y_1, (\alpha+1) y_2, \ldots, (\alpha+1) y_m), \\ |\nabla_L d|^2 &= \frac{|x|^{2\alpha}}{d^{2\alpha}} = \psi_{2\alpha}. \end{aligned}$$

Let $C^k_0(\Omega)$ be the set of functions with compact in $C^k(\Omega)$ and $1 < p < \infty$. We denote by $D^{1,p}(\Omega)$ the closure of $C^\infty_0(\Omega)$ under the norm $\left(\int_\Omega |\nabla_L u|^p d\xi\right)^{1/p}$.

In this paper we will establish a class of Hardy-Sobolev type inequalities related to generalized Baouendi-Grushin operators. Our results contain the well-known Hardy type inequality and Sobolev type inequality for the class of operators.

This paper is organized as follows. In the next section, we prove a Hardy-Sobolev type inequality for $\mathcal{L}_\alpha$. This generalizes the inequalities in the Euclidean space in [1]. In Section 3, we give some new Hardy type inequalities on bounded domains. Our results include those of [3].

2. HARDY-SOBOLEV INEQUALITIES

In this section, set $(x,y) = (x',y',x'',y'') = (z_1,z_2) \in \mathbb{R}^{n+m}$ with $z_1 = (x',y') \in \mathbb{R}^{k+l}$ $z_2 = (x'',y'') \in \mathbb{R}^{n-k+(m-l)}$, $1 \le k \le n, 1 \le l \le m$, and

$$d_1 = d\left(x',y'\right) = \left(\left|x'\right|^{2(\alpha+1)} + (\alpha+1)^2\left|y'\right|^2\right)^{\frac{1}{2(\alpha+1)}}.$$

We denote by $B'_R(R) = \{(x',y') \in \mathbb{R}^{k+l} \mid d_1 < R\}$ the open ball centered at $(0,0)$ with radius $R > 0$, and put $\nabla_{L'} = (Z_1,\ldots,Z_k,Z_{n+1},\ldots,Z_{n+l})$.

We note that the polar coordinate transformation defined in [5, 8] implies

$$dz_1 = dx'dy' = \rho^{k+(\alpha+1)l-1}d\rho d\sigma_1, \tag{2.1}$$

where $d\sigma_1 = \left(\frac{1}{\alpha+1}\right)^l |\sin\theta|^{\frac{k}{\alpha+1}-1}|\cos\theta|^{l-1}d\theta d\omega_k d\omega_l$, and ω_k and ω_l are the Lebesgue measures of the unitary Euclidean spheres in $\mathbb{R}^k$ and $\mathbb{R}^l$, respectively.

The main inequalities in this section are the following.

Theorem 1 (Hardy-Sobolev type inequalities)**.** *Let us assume that s satisfies the relations $0 \le s \le 2 < k + (\alpha+1)l \le Q$, and put*

$$2_*(s) = \frac{2(Q-s)}{Q-2}.$$

Then there exists a positive constant $C(s,\alpha,k,l)$ such that for every $u \in D^{1,2}(\mathbb{R}^{n+m})$

$$\int_{\mathbb{R}^{n+m}} \frac{|x'|^{s\alpha}}{d_1^{s\alpha}}\frac{|u|^{2_*(s)}}{d_1^s}dxdy \le C\left(\int_{\mathbb{R}^{n+m}}|\nabla_L u|^2dxdy\right)^{\frac{Q-s}{Q-2}}, \tag{2.2}$$

where $D^{1,2}(\mathbb{R}^{n+m})$ is the completion of $C_0^\infty(\mathbb{R}^{n+m})$ under the norm

$$\|u\| = \left(\int_{\mathbb{R}^{n+m}}|\nabla_L u|^2dxdy\right)^{\frac{1}{2}}.$$

Remark 1. If $s=0$, $k=n$, $l=m$, then (2.2) is (1.3); if $s=2$, $k=n$, $l=m$, then (2.2) is (1.1).

We prove first a lemma, which gives a representation formula of functions only depending on vector fields $(Z_1,\ldots,Z_{n+m})$ and dilation of $\mathcal{L}_\alpha$. It establishes the connection between the function u and its generalized gradient $\nabla_L u$.

Lemma 1. *For any $u(x,y) \in C_0^\infty(\mathbb{R}^{n+m})$, we have*

$$u(x,y) = -\int_1^\infty\left[\frac{1}{\lambda|x|^{2\alpha}}\left\langle\nabla_L u, \nabla_L\left(\frac{d^{2(\alpha+1)}}{2(\alpha+1)}\right)\right\rangle\right]\circ\delta_\lambda d\lambda. \tag{2.3}$$

Proof. Clearly,

$$\nabla_L\left(d^{2(\alpha+1)}\right) = 2(\alpha+1)d^{2\alpha+1}\nabla_L d = 2(\alpha+1)|x|^\alpha \\ \left(|x|^\alpha x_1, |x|^\alpha x_2,\ldots,|x|^\alpha x_n,(\alpha+1)y_1,(\alpha+1)y_2,\ldots,(\alpha+1)y_m\right). \tag{2.4}$$

By (1.4) and (2.4), one has

$$\begin{aligned}
\frac{d}{d\lambda}u\circ\delta_\lambda &= \frac{du\left(\delta_\lambda\left(x,y\right)\right)}{d\lambda}\\
&= \left[\sum_{i=1}^{n} x_i\frac{\partial u}{\partial x_i}\circ\delta_\lambda\left(x,y\right)+\left(\alpha+1\right)\sum_{j=1}^{m}\lambda^\alpha y_j\frac{\partial u}{\partial y_j}\circ\delta_\lambda\left(x,y\right)\right]\\
&= \left[\sum_{i=1}^{n}\frac{x_i}{\lambda}\frac{\partial u}{\partial x_i}+\left(\alpha+1\right)\sum_{j=1}^{m}\frac{y_j}{\lambda}\frac{\partial u}{\partial y_j}\right]\circ\delta_\lambda\left(x,y\right)\\
&= \left[\frac{1}{\lambda\left|x\right|^{2\alpha}}\left(\sum_{i=1}^{n}\left|x\right|^{2\alpha}x_i\frac{\partial u}{\partial x_i}+\left(\alpha+1\right)\sum_{j=1}^{m}\left|x\right|^{2\alpha}y_j\frac{\partial u}{\partial y_j}\right)\right]\circ\delta_\lambda\left(x,y\right)\\
&= \left[\frac{1}{\lambda\left|x\right|^{2\alpha}}\left\langle\nabla_L u,\nabla_L\left(\frac{d^{2(\alpha+1)}}{2\left(\alpha+1\right)}\right)\right\rangle\right]\circ\delta_\lambda\left(x,y\right). \qquad (2.5)
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
u\left(x,y\right) &= -\int_1^\infty\frac{d}{d\lambda}\left(u\left(\delta_\lambda\left(x,y\right)\right)\right)d\lambda\\
&= -\int_1^\infty\left[\frac{1}{\lambda\left|x\right|^{2\alpha}}\left\langle\nabla_L u,\nabla_L\left(\frac{d^{2(\alpha+1)}}{2\left(\alpha+1\right)}\right)\right\rangle\right]\circ\delta_\lambda d\lambda,
\end{aligned}$$

as required. □

Remark 2. We note that $\nabla_L\left(d^{2(\alpha+1)}\right)=2\left(\alpha+1\right)d^{2\alpha+1}\nabla_L d$ in the proof above, and so

$$\begin{aligned}
\left|\nabla_L\left(d^{2(\alpha+1)}\right)\right|^2 &= 2\left[\left(\alpha+1\right)d^{2\alpha+1}\right]^2\left|\nabla_L d\right|^2\\
&= \left[2\left(\alpha+1\right)\right]^2\left|x\right|^{2\alpha}d^{2(\alpha+1)}. \qquad (2.6)
\end{aligned}$$

Proof of Theorem 1. We need only to consider the cases where $u\geq 0$ and $u\in C_0^\infty(\mathbb{R}^{n+m})$. Introduce the notation $S_1=\partial B_1'=\left\{(x',y')\in\mathbb{R}^{k+l}\ (x',y')=1\right\}$ and $\vartheta=(\tau_1,\tau_2)=(\tau_{11},\dots,\tau_{1k},\tau_{21},\dots,\tau_{2k})\in S_1$. We introduce the transformation

$$z_1=\left(x',y'\right)=\left(\rho,\vartheta\right),$$

where $\rho = d_1$, $\vartheta = (\tau_1, \tau_2) = \delta_{\frac{1}{\rho}}(x', y')$. By Lemma 1, we get

$$\begin{aligned}
&\int_{\mathbb{R}^{n+m}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha}} \frac{|u|^{2_*}}{\rho^s} dx dy = \int_{\mathbb{R}^{n-k+(m-l)}} dz_2 \int_{\mathbb{R}^{k+l}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha}} \frac{|u|^{2_*}}{\rho^s} dz_1 \\
&\quad = -\int_1^\infty d\lambda \int_{\mathbb{R}^{n-k+(m-l)}} dz_2 \\
&\quad\quad \times \int_{\mathbb{R}^{k+l}} \left[\frac{|x'|^{s\alpha}}{\rho^{s\alpha+s}} \frac{1}{\lambda |x'|^{2\alpha}} \left\langle \nabla_{L'} u^{2_*}, \nabla_{L'} \left(\frac{\rho^{2(\alpha+1)}}{2(\alpha+1)} \right) \right\rangle \right] \circ \delta_\lambda (x', y') \, dz_1. \quad (2.7)
\end{aligned}$$

Putting

$$F = \frac{1}{|x'|^{2\alpha}} \left\langle \nabla_{L'} u^{2_*}, \nabla_{L'} \left(\frac{\rho^{2(\alpha+1)}}{2(\alpha+1)} \right) \right\rangle,$$

we obtain from (2.1) that

$$\begin{aligned}
&\int_{\mathbb{R}^{n+m}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha}} \frac{|u|^{2_*}}{\rho^s} dx dy \quad (2.8) \\
&\quad = -\int_1^\infty d\lambda \int_{\mathbb{R}^{n-k+(m-l)}} dz_2 \int_{\mathbb{R}^{k+l}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha+s}} \frac{F}{\lambda} \circ \delta_\lambda (x', y') dz_1 \\
&\quad = -\int_1^\infty d\lambda \int_{\mathbb{R}^{n-k+(m-l)}} dz_2 \int_{S_1} d\sigma_1 \int_0^\infty \frac{\rho^{s\alpha} |\tau_1|^{s\alpha}}{\rho^{s\alpha+s}} \frac{F}{\lambda} \circ \delta_{\lambda\rho}(\vartheta) \, \rho^{k+(\alpha+1)l-1} d\rho \\
&\quad = -\int_1^\infty \lambda^{-(k+(\alpha+1)l-1-s)-2} d\lambda \int_{\mathbb{R}^{n-k+(m-l)}} dz_2 \\
&\quad\quad \times \int_{S_1} d\sigma_1 \int_0^\infty |\tau_1|^{s\alpha} F \circ \delta_r(\vartheta) \, r^{k+(\alpha+1)l-1-s} dr \quad (\lambda\rho = r) \\
&\quad = -\frac{1}{k+(\alpha+1)l-s} \int_{\mathbb{R}^{n-k+(m-l)}} dz_2 \int_{\mathbb{R}^{k+l}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha+s}} F dz_1 \\
&\quad = -\frac{1}{k+(\alpha+1)l-s} \int_{\mathbb{R}^{n-k+(m-l)}} dz_2 \\
&\quad\quad \times \int_{\mathbb{R}^{k+l}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha+s}} \left[\frac{1}{|x'|^{2\alpha}} \left\langle \nabla_{L'} u^{2_*}, \nabla_{L'} \left(\frac{\rho^{2(\alpha+1)}}{2(\alpha+1)} \right) \right\rangle \right] dz_1. \quad (2.9)
\end{aligned}$$

Now we consider the cases $1 \le s \le 2$ and $0 < s < 1$, respectively.

Case 1: $1 \le s \le 2$. If $1 < s < 2$, then (2.6), (2.9) and Hölder's inequality yield

$$\begin{aligned}
&\int_{\mathbb{R}^{n+m}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha}} \frac{|u|^{2_*}}{\rho^s} dxdy \\
&\le \frac{1}{k+(\alpha+1)l-s} \int_{\mathbb{R}^{n-k+(m-l)}} dz_2 \int_{\mathbb{R}^{k+l}} \frac{|x'|^{s\alpha-2\alpha}}{\rho^{s\alpha+s}} |\nabla_{L'} u^{2_*}| \frac{|\nabla_{L'}\left(\rho^{2(\alpha+1)}\right)|}{2(\alpha+1)} dz_1 \\
&= \frac{2_*}{k+(\alpha+1)l-s} \int_{\mathbb{R}^{n+m}} \frac{|x'|^{s\alpha-2\alpha}}{\rho^{s\alpha+s}} u^{2_*-1} |\nabla_{L'} u| \left|x'\right|^{\alpha} \rho^{\alpha+1} dxdy \\
&= \frac{2_*}{k+(\alpha+1)l-s} \int_{\mathbb{R}^{n+m}} \frac{|x'|^{\alpha(s-1)}}{\rho^{\alpha(s-1)+s-1}} u^{2_*-1} |\nabla_{L'} u| dxdy \\
&\le \frac{2_*}{k+(\alpha+1)l-s} \int_{\mathbb{R}^{n+m}} \frac{|x'|^{\alpha(s-1)}}{\rho^{\alpha(s-1)}} \frac{u^{2_* \frac{s-1}{s}}}{\rho^{s-1}} u^{2_* \frac{1}{s}-1} |\nabla_L u| dxdy \\
&\le \frac{2_*}{k+(\alpha+1)l-s} \left(\int_{\mathbb{R}^{n+m}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha}} \frac{u^{2_*}}{\rho^s} dxdy\right)^{\frac{s-1}{s}} \left(\int_{\mathbb{R}^{n+m}} u^{2^*} dxdy\right)^{\frac{2-s}{2s}} \\
&\quad \times \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{1}{2}}. \qquad (2.10)
\end{aligned}$$

By the Sobolev type inequality (1.3) we have

$$\begin{aligned}
&\int_{\mathbb{R}^{n+m}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha}} \frac{|u|^{2_*}}{\rho^s} dxdy \\
&\qquad \le \left(\frac{2_*}{k+(\alpha+1)l-s}\right)^s \left(\int_{\mathbb{R}^{n+m}} u^{2^*} dxdy\right)^{\frac{2-s}{2}} \left(\int_{R^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{s}{2}} \\
&\qquad \le \left(\frac{2_*}{k+(\alpha+1)l-s}\right)^s S^{\frac{2^*(2-s)}{2}} \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{s}{2}+\frac{2^*}{2}\cdot\frac{2-s}{2}} \\
&\qquad = \left(\frac{2_*}{k+(\alpha+1)l-s}\right)^s S^{\frac{2^*(2-s)}{2}} \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{Q-s}{Q-2}}. \qquad (2.11)
\end{aligned}$$

For $s = 1$, relation (2.10) leads one to

$$\int_{\mathbb{R}^{n+m}} \frac{|x'|^{\alpha}}{\rho^{\alpha}} \frac{|u|^{2_*(1)}}{\rho} dxdy \le \frac{2_*(1)}{k+(\alpha+1)l-1} S^{\frac{2^*}{2}} \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{Q-1}{Q-2}}, \qquad (2.12)$$

where $2_*(1) = \frac{2(Q-1)}{Q-2}$.

If $s=2$, then $2_*=2$ and by (2.10), it follows

$$\int_{\mathbb{R}^{n+m}} \frac{|x'|^{2\alpha}}{\rho^{2\alpha}} \frac{|u|^2}{\rho^2} dxdy \le \frac{2}{k+(\alpha+1)l-2} \int_{\mathbb{R}^{n+m}} \frac{|x'|^{\alpha}}{\rho^{\alpha+1}} u |\nabla_L u| dxdy$$

$$\le \frac{2}{k+(\alpha+1)l-2} \left(\int_{\mathbb{R}^{n+m}} \left(\frac{|x'|^{\alpha} u}{\rho^{\alpha+1}}\right)^2 dxdy\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{1}{2}},$$

namely,

$$\int_{\mathbb{R}^{n+m}} \frac{|x'|^{2\alpha}}{\rho^{2\alpha}} \frac{|u|^2}{\rho^2} dxdy \le \left(\frac{2}{k+(\alpha+1)l-2}\right)^2 \int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy.$$

Case 2: $0<s<1$. Using

$$2_*(s) = \frac{2(Q-s)}{Q-2} = (1-s)2^* + s2_*(1)$$

and (2.12), we have

$$\int_{\mathbb{R}^{n+m}} \frac{|x'|^{s\alpha}}{\rho^{s\alpha}} \frac{|u|^{2_*}}{\rho^s} dxdy = \int_{\mathbb{R}^{n+m}} |u|^{(1-s)2^*} \left(\frac{|x'|^{\alpha} u^{2_*(1)}}{\rho^{\alpha+1}}\right)^s dxdy$$

$$\le \left(\int_{\mathbb{R}^{n+m}} |u|^{2^*} dxdy\right)^{1-s} \left(\int_{\mathbb{R}^{n+m}} \frac{|x'|^{\alpha}}{\rho^{\alpha}} \frac{|u|^{2_*(1)}}{\rho} dxdy\right)^s$$

$$\le S^{2^*(1-s)} \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{2^*(1-s)}{2}} \left(\frac{2_*(1)}{k+(\alpha+1)l-1} S^{\frac{2^*}{2}}\right)^s$$

$$\times \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{(Q-1)s}{Q-2}}$$

$$= S^{\frac{2^*(2-s)}{2}} \left(\frac{2_*(1)}{k+(\alpha+1)l-1}\right)^s \left(\int_{\mathbb{R}^{n+m}} |\nabla_L u|^2 dxdy\right)^{\frac{Q-s}{Q-2}},$$

and the proof is complete.

□

3. Hardy type inequalities on bounded domains

In this section, let $\Omega \subset \mathbb{R}^{n+m}$ denote any bounded domain, $(x,y)=(z_1,z_2)\in\Omega$, $d_1 = d(x',y')$, $B'_R(R)$, $\nabla_{L'}$ as before. Define

$$r_\epsilon := \left(\epsilon^2 + \left|x'\right|^2\right)^{\frac{1}{2}},$$

$$\sigma_\epsilon := \begin{pmatrix} I_k & 0 \\ 0 & r_\epsilon^{\alpha} I_l \end{pmatrix},$$

and set $\sigma_\epsilon \nabla = \nabla^\epsilon_{L'}$. For any vector field $h \in C^1(\Omega_1, \mathbb{R}^{k+l})$, we shall write $\operatorname{div}^\epsilon_{L'} = \operatorname{div}(\sigma_\epsilon h)$.

The following is the main result of this section.

Theorem 2. *Let $p > 1$, $k, l \geq 1$, and $\beta, \gamma \in \mathbb{R}$, be such that $k + (\alpha+1)\,l > \gamma - \beta - p$ and $k > \alpha p - \beta$. Then, for any $u \in D^{1,p}\big(\Omega, |x'|^{\beta-\alpha p} d_1^{(\alpha+1)p-\gamma}\big)$, we get*

$$c^p_{k,l,p,\beta,\gamma} \int_\Omega |u|^p \frac{|x'|^\beta}{d_1^\gamma} dx dy \leq \int_\Omega |\nabla_L u|^p \left|x'\right|^{\beta-\alpha p} d_1^{(\alpha+1)p-\gamma} dx dy, \tag{3.1}$$

where $c_{k,l,p,\beta,\gamma} = \frac{k+(\alpha+1)l+\beta-\gamma}{p}$.

If $(0,0) \in \Omega$, then the constant $c^p_{k,l,p,\beta,\gamma}$ in (3.1) is sharp.

Corollary 1. *If $1 < p < k + (\alpha+1)\,l$, then*

$$c^p_{k,l,p} \int_\Omega \frac{|x'|^{\alpha p}}{d_1^{\alpha p}} \frac{|u|^p}{d_1^p} dx dy \leq \int_\Omega |\nabla_L u|^p \, dx dy \tag{3.2}$$

for $u \in D^{1,p}(\Omega)$;

$$c^p_{k,l,p} \int_\Omega \frac{|u|^p}{d_1^p} dx dy \leq \int_\Omega |\nabla_L u|^p \frac{|x'|^{\alpha p}}{d_1^{\alpha p}} dx dy,$$

for $k > \alpha p$, $u \in D^{1,p}\left(\Omega, \frac{|x'|^{\alpha p}}{d_1^{\alpha p}}\right)$; (3.3)

and

$$c^p_{k,l,p} \int_\Omega \frac{|u|^p}{d_1^p} dx dy \leq \int_\Omega |\nabla_L u|^p \frac{|x'|^{(\alpha+1)p}}{d_1^{(\alpha+1)p}} dx dy \tag{3.4}$$

for $k > (\alpha+1)\,p$, $u \in D^{1,p}\left(\Omega, \frac{|x'|^{(\alpha+1)p}}{d_1^{(\alpha+1)p}}\right)$,

where $c_{k,l,p} = \frac{k+(\alpha+1)l-p}{p}$.

In order to prove our results, we will use the following statement in [3].

Proposition 1. *Let $\epsilon > 0$, and $h \in C^1\big(\Omega_1, \mathbb{R}^{k+l}\big)$ such that $\operatorname{div}^\epsilon_{L'} h > 0$. Then for any $p > 1$ and $u \in C^1_0\left(\Omega, \mathbb{R}^{n+m}\right)$, one has*

$$\int_{\Omega_1} |u|^p \operatorname{div}^\epsilon_{L'} h \, dz_1 \leq p^p \int_{\Omega_1} |h|^p |\operatorname{div}^\epsilon_{L'} h|^{-(p-1)} |\nabla^\epsilon_{L'} u| dz_1. \tag{3.5}$$

The following proposition is also useful. For a similar description in the Euclidean space see [9].

Proposition 2. *If* $\omega \in C_0^\infty\left(\mathbb{R}^{n+m}\right)$*, then*

$$\inf_{\omega \in C_0^\infty(\mathbb{R}^{n+m}),\, \omega \neq 0} \frac{\int_{\mathbb{R}^{n+m}} |\nabla_L \omega|^p \, dxdy}{\int_{\mathbb{R}^{n+m}} |\omega|^p \, dxdy} = 0. \tag{3.6}$$

Proof. Set $\varphi \in C_0^\infty(\mathbb{R}^{n+m})$ to be a nonnegative radial function and satisfy

$$\varphi(d) = \begin{cases} 1 & \text{if } d \leq \frac{1}{2}; \\ 0 & \text{if } d \geq 1, \end{cases}$$

with $\int_{\mathbb{R}^{n+m}} \varphi^p \, dxdy = 1$. Letting $\epsilon < 1$ and putting $\varphi_\epsilon(d) = \epsilon^{-\frac{Q}{p}} \varphi\left(\frac{d}{\epsilon}\right)$, we get $\int_{\mathbb{R}^{n+m}} \varphi_\epsilon(d)^p \, dxdy = 1$. Note that $\nabla_L \varphi_\epsilon(d) = \epsilon^{-\frac{Q}{p}-1} \varphi'\left(\frac{d}{\epsilon}\right) \nabla_L d$. Setting $d = \epsilon\rho$ and using (2.1) in $\mathbb{R}^{n+m}$, we get

$$\begin{aligned} \int_{\mathbb{R}^{n+m}} |\nabla_L \varphi_\epsilon(d)|^p \, dxdy &= \epsilon^{-Q-p} \int_{\mathbb{R}^{n+m}} \psi_{p\alpha} \left|\varphi'\left(\frac{d}{\epsilon}\right)\right|^p dxdy \\ &= \epsilon^{-p} s_{n,m} \int_{\{\rho<1\}} \rho^{Q-1} \left|\varphi'(\rho)\right|^p d\rho \longrightarrow 0 \quad (\epsilon \to \infty) \end{aligned}$$

where

$$s_{n,m} = \left(\frac{1}{\alpha+1}\right)^m \omega_n \omega_m \int_{a_1}^{a_2} |\sin\theta|^{\frac{n+p\alpha}{\alpha+1}-1} |\cos\theta|^{m-1} \, d\theta.$$

This completes the proof. □

Proof of Theorem 2. Without loss of generality, we shall consider a smooth function $u \in C_0^\infty(\Omega)$. The general case will follow by the density argument. For $\epsilon > 0$, define

$$d_{1,\epsilon} = \left(r_\epsilon^{2(\alpha+1)} + (\alpha+1)^2 |y'|^2\right)^{\frac{1}{2(\alpha+1)}}$$

and

$$h_\epsilon^1 = \frac{1}{d_{1,\epsilon}^\gamma} \begin{pmatrix} x' r_\epsilon^\beta \\ (\alpha+1)|x'|^2 y' r_\epsilon^{\beta-\alpha-2} \end{pmatrix}.$$

A simple computation shows that

$$\begin{aligned} \operatorname{div}_{L'}^\epsilon h_\epsilon^1 &= \operatorname{div} \frac{1}{d_{1,\epsilon}^\gamma} \begin{pmatrix} x' r_\epsilon^\beta \\ (\alpha+1)|x'|^2 y' r_\epsilon^{\beta-2} \end{pmatrix} \\ &= \frac{r_\epsilon^\beta}{d_{1,\epsilon}^\gamma} \left(k + ((\alpha+1)l + \beta - \gamma) \frac{|x'|^2}{r_\epsilon^2}\right) \end{aligned} \tag{3.7}$$

and

$$|h_\epsilon^1| = \frac{r_\epsilon^{\beta-\alpha-2}|x'|^2}{d_{1,\epsilon}^\gamma}\left(\frac{r_\epsilon^{2\alpha+4}}{|x'|^2}+(\alpha+1)^2|y'|^2\right)^{\frac{1}{2}}. \tag{3.8}$$

Putting $f_\epsilon(s) = k + ((\alpha+1)l+\beta-\gamma)\frac{|s|^2}{\epsilon^2+s^2}$, $s \geq 0$, it is easy to see that

$$f_\epsilon(s) \geq \begin{cases} k & \text{if } (\alpha+1)l+\beta-\gamma \geq 0, \\ k+(\alpha+1)l+\beta-\gamma & \text{if } (\alpha+1)l+\beta-\gamma < 0, \end{cases} \tag{3.9}$$

for every $\epsilon > 0$ and $s \geq 0$. Since $r_\epsilon \geq \epsilon$, if $k+(\alpha+1)l > \gamma-\beta$, then $\mathrm{div}_{L'}^\epsilon h_\epsilon^1 > 0$.

Now we choose $h = h_\epsilon^1$ in (3.5) and obtain

$$\begin{aligned}
&\int_\Omega |u|^p \frac{r_\epsilon^\beta}{d_{1,\epsilon}^\gamma} f_\epsilon\left(|x'|\right) dxdy = \int_{\Omega_2} dz_2 \int_{\Omega_1} |u|^p \frac{r_\epsilon^\beta}{d_{1,\epsilon}^\gamma} f_\epsilon\left(|x'|\right) dz_1 \\
&\leq p^p \int_{\Omega_2} dz_2 \int_{\Omega_1} |h_\epsilon^1|^p \left|\frac{r_\epsilon^\beta}{d_{1,\epsilon}^\gamma} f_\epsilon\left(|x'|\right)^{-(p-1)}\right| |\nabla_{L'}^\epsilon u|^p dz_1 \\
&= p^p \int_{\Omega_2} dz_2 \int_{\Omega_1} |\nabla_{L'}^\epsilon u|^p \frac{r_\epsilon^{-\beta(p-1)}\left(r_\epsilon^{\beta-\alpha-2}|x'|^2\right)^p\left(\frac{r_\epsilon^{2\alpha+4}}{|x'|^2}+(\alpha+1)^2|y'|^2\right)^{\frac{p}{2}}}{d_{1,\epsilon}^{\gamma p} d_{1,\epsilon}^{-\gamma(p-1)} f_\epsilon(|x'|)^{p-1}} dz_1 \\
&= p^p \int_{\Omega_2} dz_2 \int_{\Omega_1} |\nabla_{L'}^\epsilon u|^p \frac{r_\epsilon^{\beta-(2+\alpha)p}|x'|^{2p}\left(\frac{r_\epsilon^{2\alpha+4}}{|x'|^2}+(\alpha+1)^2|y'|^2\right)^{\frac{p}{2}}}{d_{1,\epsilon}^\gamma f_\epsilon(|x'|)^{p-1}} dz_1.
\end{aligned} \tag{3.10}$$

Let $m_1 = \min\{k, k+(\alpha+1)l+\beta-\gamma\} > 0$. By $r < r_\epsilon$ and (3.9) the right-hand side of (3.10) can be estimated as follows:

$$\begin{aligned}
&|\nabla_{L'}^\epsilon u|^p \frac{r_\epsilon^{\beta-(2+\alpha)p}|x'|^{2p}\left(\frac{r_\epsilon^{2\alpha+4}}{|x'|^2}+(\alpha+1)^2|y'|^2\right)^{\frac{p}{2}}}{d_{1,\epsilon}^\gamma f_\epsilon(|x'|)^{p-1}} \\
&\qquad\qquad \leq \frac{|\nabla_{L'}^\epsilon u|^p}{m_1^{p-1}} r_\epsilon^{\beta-\alpha p} d_{1,\epsilon}^{(\alpha+1)p-\gamma}.
\end{aligned}$$

Therefore, following the assumptions $k+(\alpha+1)l>\gamma-\beta-p$ and $k>\alpha p-\beta$, and applying the Lebesgue dominated convergence theorem to (3.10), we have

$$\begin{aligned}\int_{\Omega}|u|^p\frac{|x'|^{\beta}}{d_1^{\gamma}}f\left(|x'|\right)dxdy&=\lim_{\epsilon\to 0}\int_{\Omega}|u|^p\frac{r_{\epsilon}^{\beta}}{d_{1,\epsilon}^{\gamma}}f_{\epsilon}\left(|x'|\right)dxdy\\&\le\lim_{\epsilon\to 0}p^p\int_{\Omega_2}dz_2\int_{\Omega_1}\frac{|\nabla_{L'}^{\epsilon}u|^p}{m_1^{p-1}}r_{\epsilon}^{\beta-\alpha p}d_{1,\epsilon}^{(\alpha+1)p-\gamma}dz_1\\&\le p^p\int_{\Omega}\frac{|\nabla_L u|^p}{m_1^{p-1}}\left|x'\right|^{\beta-\alpha p}d_1^{(\alpha+1)p-\gamma}dxdy.\end{aligned}$$

This shows the claim of (3.1).

Choosing $(\beta,\gamma)=(\alpha p,(\alpha+1)p)$, $(\beta,\gamma)=(0,p)$ and $(\beta,\gamma)=(-p,0)$ in (3.1), implies the inequalities (3.2), (3.3) and (3.4), respectively.

Next, if $(0,0)\in\Omega$, we prove the constant $c_{k,l,p,\beta,\gamma}^{p}$ is sharp. The approach here comes from that in [9].

Let $C(\Omega)$ be the best constant in (3.1), that is

$$C(\Omega)=\inf_{\phi\in C_0^{\infty}(\Omega),\phi\neq 0}\frac{\int_{\Omega}|\nabla_L\phi|^p|x'|^{\beta-p\alpha}d_1^{(\alpha+1)p-\gamma}dxdy}{\int_{\Omega}|\phi|^p\frac{|x'|^{\beta}}{d_1^{\gamma}}dxdy}.$$

From (3.1), we easily get $C(\Omega)\ge c_{k,l,p,\beta,\gamma}^{p}$. We shall prove the equality sign holds.

Put $\phi\in C_0^{\infty}(\Omega)$ and $\phi=\upsilon(z_1)\omega(z_2)$, where $\upsilon(z_1)\in C_0^{\infty}(\mathbb{R}^{k+l}\setminus\{(0,0)\})$, $\omega(z_2)\in C_0^{\infty}(\mathbb{R}^{n-k+(m-l)})$. By the convexity of the function $(a^2+b^2)^{\frac{p}{2}}$ for $a,b\ge 0$, we have

$$\left(a^2+b^2\right)^{\frac{p}{2}}\le(1-\lambda)^{1-p}a^p+\lambda^{1-p}b^p,$$

where $p>1$, $0<\lambda<1$. Therefore, we find

$$\begin{aligned}|\nabla_L\phi|^p&=|(\nabla_L(\upsilon(z_1)\omega(z_2)))^2|^{\frac{p}{2}}\\&=\left(|\nabla_{L'}\upsilon|^2\omega^2+\upsilon^2|\nabla_{L''}\omega|^2\right)^{\frac{p}{2}}\\&\le(1-\lambda)^{1-p}|\nabla_{L'}\upsilon|^p\omega^p+\lambda^{1-p}\upsilon^p|\nabla_{L''}\omega|^p,\end{aligned}\tag{3.11}$$

where $\nabla_{L''} := (Z_{k+1}, \ldots, Z_n, Z_{n+l+1}, \ldots, Z_m)$. By (3.11), this leads one to

$$\begin{aligned}
C(\Omega) &\le \frac{\int_\Omega |\nabla_L \phi|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dxdy}{\int_\Omega |\phi|^p \frac{|x'|^\beta}{d_1^\gamma} dxdy} \\
&\le (1-\lambda)^{1-p} \frac{\int_\Omega |\nabla_{L'} \upsilon|^p |\omega|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dxdy}{\int_\Omega |\upsilon\omega|^p \frac{|x'|^\beta}{d_1^\gamma} dxdy} \\
&\quad + \lambda^{1-p} \frac{\int_\Omega |\upsilon|^p |\nabla_{L''} \omega|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dxdy}{\int_\Omega |\upsilon\omega|^p \frac{|x'|^\beta}{d_1^\gamma} dxdy} \\
&\le (1-\lambda)^{1-p} \frac{\int_{\mathbb{R}^{n-k+(m-l)}} |\omega|^p dz_2 \int_{\mathbb{R}^{k+l}} |\nabla_{L'} \upsilon|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dz_1}{\int_{\mathbb{R}^{n-k+(m-l)}} |\omega|^p dz_2 \int_{\mathbb{R}^{k+l}} |\upsilon|^p \frac{|x'|^\beta}{d_1^\gamma} dz_1} \\
&\quad + \lambda^{1-p} \frac{\int_{\mathbb{R}^{n-k+(m-l)}} |\nabla_{L''}\omega|^p dz_2 \int_{\mathbb{R}^{k+l}} |\upsilon|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dz_1}{\int_{\mathbb{R}^{n-k+(m-l)}} |\omega|^p dz_2 \int_{\mathbb{R}^{k+l}} |\upsilon|^p \frac{|x'|^\beta}{d_1^\gamma} dz_1} \\
&= (1-\lambda)^{1-p} \frac{\int_{\mathbb{R}^{k+l}} |\nabla_{L'} \upsilon|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dz_1}{\int_{\mathbb{R}^{k+l}} |\upsilon|^p \frac{|x'|^\beta}{d_1^\gamma} dz_1} \\
&\quad + \lambda^{1-p} \frac{\int_{\mathbb{R}^{n-k+(m-l)}} |\nabla_{L''}\omega|^p dz_2}{\int_{\mathbb{R}^{n-k+(m-l)}} |\omega|^p dz_2} \cdot \frac{\int_{\mathbb{R}^{k+l}} |\upsilon|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dz_1}{\int_{\mathbb{R}^{k+l}} |\upsilon|^p \frac{|x'|^\beta}{d_1^\gamma} dz_1}.
\end{aligned}$$

From (3.6), we get

$$\inf_{\omega \in C_0^\infty(\mathbb{R}^{n-k+(m-l)}),\, \omega \neq 0} \frac{\int_{\mathbb{R}^{n-k+(m-l)}} |\nabla_{L''}\omega|^p dz_2}{\int_{\mathbb{R}^{n-k+(m-l)}} |\omega|^p dz_2} = 0.$$

For $0 < \lambda < 1$, we have

$$\begin{aligned}
C(\Omega) &\le \inf_{\phi \in C_0^\infty(\Omega), \phi \neq 0} \frac{\int_\Omega |\nabla_L \phi|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dxdy}{\int_\Omega |\phi|^p \frac{|x'|^\beta}{d_1^\gamma} dxdy} \\
&\le (1-\lambda)^{1-p} \inf_{\upsilon \in C_0^\infty(\mathbb{R}^{k+l}), \upsilon \neq 0} \frac{\int_{\mathbb{R}^{k+l}} |\nabla_{L'} \upsilon|^p |x'|^{\beta - p\alpha} d_1^{(\alpha+1)p-\gamma} dz_1}{\int_{\mathbb{R}^{k+l}} |\upsilon|^p \frac{|x'|^\beta}{d_1^\gamma} dz_1}.
\end{aligned} \tag{3.12}$$

We choose $Q = k + (\alpha+1)l$ in the best constant $c^p_{Q,p,\beta,\gamma}$ of (1.2), and let $\lambda \to 0$ in (3.12), hence

$$C(\Omega) \le c^p_{k,l,p,\beta,\gamma}.$$

The theorem is proved. □

Proof of Corollary 1. Choosing $(\beta,\gamma)=(\alpha p,(\alpha+1)\,p)$, $(\beta,\gamma)=(0,p)$ and taking $(\beta,\gamma)=(-p,0)$ in (3.1), we get inequalities (3.2), (3.3) and (3.4) respectively. The proof of the corollary is complete. □

Remark 3. We can also select $h=d_{1,\epsilon}^{1-p}\left|\nabla_{L'}d_{1,\epsilon}\right|^{p-2}\nabla_{L'}d_{1,\epsilon}$ to prove (3.2), as in the proof of Theorem 2.

Remark 4. In (3.1), by choosing $\beta=\alpha p$, $\gamma=\alpha p-\lambda$, $\lambda\in\mathbb{R}$, we obtain

$$c_{k,l,p,\lambda}^{p}\int_{\Omega}|u|^{p}\left(\frac{|x'|}{d_1}\right)^{\alpha p}d_1^{\lambda}dxdy\le\int_{\Omega}|\nabla_L u|^{p}\,d_1^{p+\lambda}dxdy,$$

where $c_{k,l,p,\lambda}=\frac{|k+(\alpha+1)l+\lambda|}{p}$. If $p=k+(\alpha+1)\,l$, and $\lambda<-1$, we let $\tilde{c}_{k,l,p,\lambda}=\frac{|1+\lambda|}{p}$, $\Omega=\left\{(x',y',x'',y'')\in\mathbb{R}^{k+l}\times\mathbb{R}^{n-k+(m-l)}\,|\,d_1<R\right\}$, and then have

$$\tilde{c}_{k,l,p,\lambda}^{p}\int_{\Omega}|u|^{p}\left(\frac{|x'|}{d_1}\right)^{\alpha p}\left(\ln\left(\frac{R}{d_1}\right)\right)^{\lambda}dxdy\le\int_{\Omega}|\nabla_L u|^{p}\left(\ln\left(\frac{R}{d_1}\right)\right)^{p+\lambda}dxdy.$$

In particular

$$\left(\frac{p-1}{p}\right)^{p}\int_{\Omega}\frac{|u|^{p}}{\left(d_1\ln\left(\frac{R}{d_1}\right)\right)^{p}}\left(\frac{|x'|}{d_1}\right)^{\alpha p}dxdy\le\int_{\Omega}|\nabla_L u|^{p}\,dxdy.$$

This result is more general than D'Ambrosio's in [4].

Remark 5. We note that if $k=n$ and $l=m$, then (3.1) coincides with (1.2).

Theorem 3. *Let $1<p<k$. Then, for every $u\in D^{1,p}(\Omega)$, there exists a constant $b_{k,p}=\frac{k-p}{p}$ such that the following inequalities hold:*

$$b_{k,p}^{p}\int_{\Omega}\frac{|u|^{p}}{|x'|^{p}}dxdy\le\int_{\Omega}|\nabla_L u|^{p}\,dxdy,\tag{3.13}$$

$$b_{k,p}^{p}\int_{\Omega}\frac{|u|^{p}}{|d_1|^{p}}dxdy\le\int_{\Omega}|\nabla_L u|^{p}\,dxdy.\tag{3.14}$$

In particular, if $p=2$ and $k\ge3$, we have

$$\left(\frac{k-2}{2}\right)^{2}\int_{\Omega}\frac{u^2}{d_1^2}dxdy\le\left(\frac{k-2}{2}\right)^{2}\int_{\Omega}\frac{u^2}{|x'|^2}dxdy\le\int_{\Omega}|\nabla_L u|^{2}\,dxdy.$$

Proof. Let us first set

$$h_\epsilon^2=\frac{1}{r_\epsilon^p}\begin{pmatrix}x'\\0\end{pmatrix}.$$

A simple calculation shows that

$$\operatorname{div}_{L'}^{\epsilon}h_\epsilon^2=\frac{1}{r_\epsilon^p}\left(k-p\frac{|x'|^2}{r_\epsilon^2}\right)$$

and

$$|h_\epsilon^2| = \frac{|x'|}{r_\epsilon^p}.$$

Since $p < k$, we get $\mathrm{div}_{L'}^\epsilon h_\epsilon^2 > 0$. From (3.5) we have

$$\int_\Omega \frac{|u|^p}{r_\epsilon^p}\left(k - p\frac{|x'|^2}{r_\epsilon^2}\right) dxdy = \int_{\Omega_2} dz_2 \int_{\Omega_1} \frac{|u|^p}{r_\epsilon^p}\left(k - p\frac{|x'|^2}{r_\epsilon^2}\right) dz_1$$
$$\leq \int_{\Omega_2} dz_2 \int_{\Omega_1} \left(\frac{|x'|}{r_\epsilon^p}\right)^p \left[\frac{1}{r_\epsilon^p}\left(k - p\frac{|x'|^2}{r_\epsilon^2}\right)\right]^{-(p-1)} |\nabla_{L'} u|^p\, dz_1$$
$$\leq \int_\Omega \frac{|x'|^p}{r_\epsilon^p\left(k - p\frac{|x'|^2}{r_\epsilon^2}\right)^{(p-1)}} |\nabla_L u|^p\, dz_1.$$

Using the Lebesgue dominated convergence theorem and putting $\epsilon \to 0$ in the estimate above, we obtain (3.13). Using $|x'| \leq d_1$ and (3.13), we get

$$\int_\Omega \frac{|u|^p}{d_1^p} dxdy \leq \int_\Omega \frac{|u|^p}{|x'|^p} dxdy.$$

Hence (3.14) is obtained. □

Remark 6. From the result above, it follows that the best constants in (3.13) and (3.14) lie in the interval $[(k-p)^p p^{-p},\ (k+(\alpha+1)l-p)^p p^{-p}]$.

As a consequence of (3.5), we give a Poincaré inequality for the vector fields on domains Ω contained in a slab. More precisely, we have

Corollary 2. *Let $\Omega \subset \mathbb{R}^{n+m}$ be an open subset. Suppose that there exist $R > 0, s \in \mathbb{R}$ and an integer $j : 1 \leq j \leq n$, such that for any $(x, y) \in \Omega$, it follows that $|x_j - s| \leq R$. Then, for every $u \in C_0^1(\Omega)$, we have*

$$c\int_\Omega |u|^p\, dxdy \leq \int_\Omega |\nabla_L u|^p\, dxdy.$$

where $c = (pR)^{-p}$.

The proof of the corollary follows from Theorem 3 by using the vector field defined by the formula

$$h := \begin{pmatrix} 0 \\ x_i - s \\ 0 \end{pmatrix}.$$

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